Period integrals and their differential systems

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CRG Geometry and Physics Seminar
University of British Columbia
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Based on joint works with
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2. Outline

- Brief overview: classical theory of hypergeometric functions and elliptic integrals.
- Riemann-Hilbert problem for period integrals.
- Introduction to tautological systems.
- D-module description of tautological systems.
- Some applications.

A study on the interplay between

SPECIAL FUNCTIONS $\leftrightarrow$ **COMPLEX GEOMETRY**

4. What is a special function?

Loosely defined, a special function is a (multi-valued) analytic function that is governed by a system of linear PDEs with **polynomial** coefficients in $\mathbf{C}^n$.

E.g. $\sin(z)$, $\cos(z)$, e^z, z^α, $\log(z)$,...

But without further restrictions, there does not appear to be a coherent theory...

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5. Let's look to the ancient masters ...

Figure: Leonhard Euler 1707-1783

Carl F. Gauss 1777-1855

6. Euler-Gauss hypergeometric functions

The EG hypergeometric equation is the ODE defined on $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$:

$$z(1-z)\frac{d^2}{dz^2} + [c - (a+b+1)z]\,\frac{d}{dz} - ab = 0$$

where $a, b, c \in \mathbf{C}$ are fixed parameters.

Every second-order linear ODE on $\mathbf{P}^1$ with three regular singular points can be transformed into this equation.

A EG hypergeometric function is a local solution to this equation. For $c \notin \mathbf{Z}_{\leq 0}$, around $z = 0$, it has a power series solution of the form

$${}_2F_1(a, b, c; z) := \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

with radius of convergence 1. Here $(\alpha)_n = \prod_{k=0}^{n-1} (\alpha + k) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$.

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7. From complex geometry to EG functions

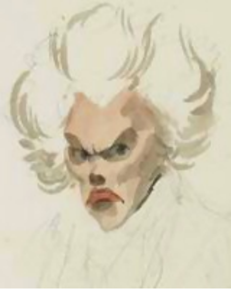

Figure: Portrait of Adrien-Marie Legendre (1752-1833) by Julien-Leopold Boilly

The first connection to complex geometry of the hypergeometric functions is attributed to Legendre, through the theory of elliptic integrals.

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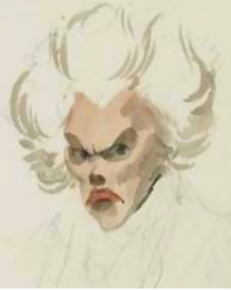

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8. From complex geometry to EG functions

The Legendre family of elliptic curves:

$$Y_\lambda : \quad y^2 = x(x-1)(x-\lambda), \quad (x,y) \equiv [x,y,1] \in \mathbf{P}^2$$

parameterized by $\lambda \in B := \mathbf{C} - \{0,1\}$.

For $\lambda \in B$,

$$Y_\lambda \simeq^{homeo.} T^2.$$

For a given $\lambda_0 \in B$, we also have canonical identification

$$H^1(Y_\lambda, \mathbf{C}) \equiv H^1(Y_{\lambda_0}, \mathbf{C}) \equiv H^1(T, \mathbf{C}) \cong \mathbf{C}^2$$

if λ varies in any contractible neighborhood U of λ_0.

The 1-form

$$\omega_\lambda := \frac{dx}{y}$$

is holomorphic on Y_λ, so it is d-closed and defines a cohomology class on $[\omega_\lambda] \in H^1(T, \mathbf{C}) \equiv \mathbf{C}^2$. This vector varies holomorphically with $\lambda \in U$.

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9. Period integrals

Fix a basis $\gamma_1, \gamma_2 \in H_1(T, \mathbf{Z}) = H^1(T, \mathbf{Z})^*$. Then

$$[\omega_\lambda] = \gamma_1^* \langle \gamma_1^*, \omega_\lambda \rangle + \gamma_2^* \langle \gamma_2^*, \omega_\lambda \rangle = \gamma_1^* \int_{\gamma_1} \omega_\lambda + \gamma_2^* \int_{\gamma_2} \omega_\lambda.$$

The coefficient functions $\int_{\gamma_i} \omega_\lambda \in \mathcal{O}_B(U)$ are called period integrals of the family Y_λ.

Remark: Even though they are defined locally, these period integrals admit (multi-valued) analytic continuations along any path in B. Therefore the period integrals generate a **local system** on B.

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10. Differential equations for period integrals

Proposition: The period integrals are precisely the solutions to the EG equation (for $a = b = \frac{1}{2}$, $c = 1$):

$$\mathcal{L}\varphi := \lambda(1-\lambda)\frac{d^2}{d\lambda^2}\varphi + (1-2\lambda)\frac{d}{d\lambda}\varphi - \frac{1}{4}\varphi.$$

Proof. Check that

$$\mathcal{L}\omega_\lambda = \left(\frac{\partial}{\partial x}\frac{(x-1)^2x^2}{2y^3}\right)dx$$

Right side is an exact 1-form on Y_λ-finite set.

It follows that

$$\mathcal{L}\int_{\gamma_i}\omega_\lambda = \int_{\gamma_i}\mathcal{L}\omega_\lambda = 0$$

by Stoke's theorem. □

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11. **Computing period integrals**

Remarks: This effectively reduces the task of computing each integral $\int_{\gamma_i} \omega_\lambda$ to one of determining two constants in the general solution to an ODE.

For example, at $\lambda = 0$, the curve Y_λ develops a node. With a little more work – basically by studying how the form ω_λ develops a pole when $\lambda = 0$, we can determine those constants.

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12. **Computing period integrals**

If γ_1 is the basic 1-cycle on Y_0 that avoids the node, then

$$\int_{\gamma_1} \omega_\lambda = {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1, \lambda).$$

If γ_2 is the basic 1-cycle that runs through the node, then

$$\int_{\gamma_2} \omega_\lambda = {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1, \lambda) \log \lambda + g_1(\lambda)$$

where $g_1(\lambda)$ is a unique power series determined by the EG equation.

Thus we have effectively solved an integration problem – elliptic integrals – by relating it to the geometry of curves.

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13. Remarks

- There is a similar story for hyper-elliptic integrals (Euler)

$$\int_\gamma \frac{x^k dx}{\sqrt{Q(x)}}$$

where $Q(x)$ is square free polynomial.

- This interplay between special integrals and geometry will be the spirit in which we proceed to study higher dimensional analogues of elliptic integrals.

14. Remarks

- Consideration of other special functions (often with physics motivations) have led to development of more general hypergeometric functions: Kummer, Legendre, Hermit, Bessel, H. Schwarz, Pochammer, Appell,...
- Modern theory (1990's): Gel'fand school initiated a systematic study of hypergeometric functions of several variables.
- In parallel, consideration of period integrals have also led to development of modern Hodge theory: Riemann, Hodge, Griffiths, Schmid, Simpson,...

15. Higher dimensional analogues: Period sheaves

Let B connected complex manifold (parameter space).

Let $E \to B$ be a vector bundle equipped with a flat connection

$$\nabla : \mathcal{O}(E) \to \mathcal{O}(E) \otimes \Omega^1_B.$$

Let

$$\langle \, , \, \rangle : \mathcal{O}(E) \otimes \mathcal{O}(E^*) \to \mathcal{O}_B$$

be the usual pairing.

Fix global section $s^* \in \Gamma(B, E^*)$.

Definition: The **period sheaf**

$$\mathbf{\Pi} \equiv \mathbf{\Pi}(E, s^*) \subset \mathcal{O}_B$$

is the image of the map

$$\mathcal{O}(E) \supset \ker \nabla \to \mathcal{O}_B, \quad \gamma \mapsto \langle \gamma, s^* \rangle.$$

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16. Period sheaves from Complex Geometry

Let $\pi : \mathcal{Y} \to B$ be a family of d-dimensional compact complex manifolds, with $Y_b := \pi^{-1}(b)$.

From topology: cohomology groups of fibers $H^k(Y_b, \mathbf{C})$ form a vector bundle $E^* := R^k\pi_*\mathbf{C}$ over B; dual bundle $E = E^{**}$ has fibers $H_k(Y_b, \mathbf{C})$, and

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is the Poincaré pairing; E is equipped with a canonical flat (Gauss-Manin) connection ∇.

Fix $s^* \in \Gamma(B, E^*)$, and represent $s^*(b) \in H^k(Y_b, \mathbf{C})$ by a closed form on Y_b. Represent section $\gamma \in \ker \nabla$ by cycle on Y_b. So, a local section $f \in \mathbf{\Pi}(U)$ becomes an integral

$$f(b) = \langle \gamma, s^*(b) \rangle = \int_\gamma s^*(b).$$

We call this a **period integral** of $\mathcal{Y}$ with respect to s^*.

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17. **Problem**

Fix a compact Kähler manifold X^{d+1}, and assume

$$\pi : \mathcal{Y} \to B$$

is a family of smooth Calabi-Yau hypersurfaces (complete intersections) in X. Consider the associated flat bundle $E^* = R^d \pi_* \mathbf{C}$.

The subspaces
$$\Gamma(Y_b, K_{Y_b}) \subset H^d(Y_b, \mathbf{C}).$$

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Remark: For simplicity, we restricted ourselves to the case of Calabi-Yau families. (Almost all results here will generalize to families of general type, i.e. the canonical bundle is ample.)

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• Physics: compute Yukawa coupling in Type IIB string theory (Candelas-de la Ossa-Green-Parkes, 1990.)
and counting instantons ("Gromov-Witten" invariants) in Type IIA string theory, by Mirror Symmetry.

• Hodge theory: study of period mapping, when the Y_b are projective and B simply-connected:

$$P : B \to \mathbf{P}^m, \quad b \mapsto [\int_{\gamma_0} \omega(b), ..., \int_{\gamma_m} \omega(b)].$$

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21. What's known: hypersurfaces in $X = \mathbf{P}^{d+1}$

Dwork-Griffiths' reduction-of-pole method can (in principle) be used to derive differential equations; often works for **one-parameter** families only.

Example. For the Legendre family, this method yields precisely the EG equation

$$\lambda(1-\lambda)\frac{d^2}{d\lambda^2}\varphi + (1-2\lambda)\frac{d}{d\lambda}\varphi - \frac{1}{4}\varphi = 0.$$

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A toric manifold is, roughly speaking, a manifold containing a torus $(\mathbf{C}^\times)^n$ as an open dense subset, such that the action of the torus on itself, extends to the whole manifold.

Let X^{d+1} be a toric manifold with respect to torus T, Assume $c_1(X) \geq 0$, and assume that generic CY hypersurface in X is smooth. Consider the family $\pi : \mathcal{Y} \to B$ of all such hypersurfaces.

Let $\hat{\mathfrak{t}}$ be the Lie algebra of $T \times \mathbf{C}^\times$. Then T induces a linear action on $H^0(-K_X)$, and $\mathbf{C}^\times$ acts by scaling. So, we have a Lie algebra action

$$\hat{\mathfrak{t}} \to End\ H^0(-K_X), \quad y \mapsto Z_y.$$

Let $\beta : \hat{\mathfrak{t}} \to \mathbf{C}$ be a character which takes zero on T, and takes 1 on the Euler operator, as a generator of the Lie algebra of $\mathbf{C}^\times$.

Each section $f \in H^0(-K_X)$ restricted to $T \subset X$ is a Laurent polynomial. In fact, the restriction of $H^0(-K_X)$ has a basis of Laurent monomials x^{μ_i} in $x_0, .., x_d$ – coordinates on $T = (\mathbf{C}^\times)^{d+1}$.

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Proposition: *The period integrals of the family $\mathcal{Y}$ of CY hypersurfaces in X satisfy the PDE system*

$$\Box_l \varphi = 0, \quad (Z_y + \beta(y))\varphi = 0, \qquad y \in \hat{\mathfrak{t}}$$

where the l are integral vectors such that $\sum_i l_i \mu_i = 0$, $\sum_i l_i = 0$, and

$$\Box_l := \prod_{l_i > 0} \left(\frac{\partial}{\partial a_i}\right)^{l_i} - \prod_{l_i < 0} \left(\frac{\partial}{\partial a_i}\right)^{-l_i}$$

This system is called a **GKZ hypergeometric system.**

Remark: A theorem of GKZ says that solution space of this system is finite dim. However, this system is never complete – there are always more solutions than period integrals. But there are two conjectural ways to pick out the period integrals among solutions.

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There were a few more isolated examples on the RH problem for period integrals beyond toric hypersurfaces between 1996-2010.

For example, the problem was open even for the case of hypersurfaces in a flag variety (i.e. GL_n/P).

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25. Tautological Systems

Consider the case of a general projective manifold X.

Data & notations:
X: projective manifold
G: complex algebraic group, with Lie algebra $\mathfrak{g}$
$G \times X \to X, \quad (g, x) \mapsto gx$, a group action
L: an equivariant base-point-free line bundle on X
$V := H^0(X, L)^*$

$\phi : X \to \mathbf{P}V$ the corresp. equivariant map
I_ϕ: the ideal of $\phi(X)$
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26. Example to keep in mind

$X = \mathbf{P}^2$
$G = PSL_3$
$L = O(3)$
$V^* = Sym^3\ \mathbf{C}^3$

$\phi : X \hookrightarrow \mathbf{P}V$ is the Segre embedding,
$[z_0, z_1, z_2] \mapsto [z_0^3, z_0^2 z_1, z_0^2 z_2, .., z_2^3]$.
I_ϕ=the quadratic ideal generated by the Veronese binomials.
D_{V^*}= the Weyl algebra $\mathbf{C}[a_0, ..., a_9, \frac{\partial}{\partial a_0}, .., \frac{\partial}{\partial a_9}]$.

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27. **Group actions**

Define a Lie algebra map (Fourier transform):

$$V^* \to Der\ Sym(V), \quad \zeta \mapsto \partial_\zeta, \quad \partial_\zeta a := \langle a, \zeta \rangle.$$

The linear action $G \to Aut\ V$ induces Lie algebra map

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Let a_i and ζ_i be any dual bases of V, V^*. Then $\partial_{\zeta_i} = \frac{\partial}{\partial a_i}$.

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28. Tautological systems

Definition: Fix $\beta \in \mathbf{C}$. Let $\tau(X, L, G, \beta)$ be the left ideal in D_{V^*} generated by the following differential operators:
$\{p(\partial_\zeta) | p(\zeta) \in I_\phi\}$, (polynomial operators)
$\{Z_x | x \in \mathfrak{g}\}$, ($G$ operators)
$\varepsilon_\beta := \sum_i a_i \frac{\partial}{\partial a_i} + \beta$, (Euler operator.)
We call this system of differential operators a **tautological system**.

29. **Regularity & Holonomicity**

Theorem: [Lian-Song-Yau] Suppose X has only finite number of G orbits. Then the tautological system $\tau(X, L, G, \beta)$ is **regular holonomic**. Moreover, the solution rank is bounded above by the degree of $X \mapsto \mathbf{P}V$ if the $\mathbf{C}[X]$ is Cohen-Macaulay.

Corollary: Any formal power series solution is **analytic**; the sheaf of solutions is a locally constant sheaf of **finite rank** on some open $V^*_{gen} \subset V^*$.

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30. **From complex geometry to special functions**

Let X be a compact complex G-manifold such that $-K_X$ is base point free. Consider the family $\mathcal{Y}$ of all CY hypersurfaces in X.

Theorem: [Lian-Yau] The period integrals of the family $\mathcal{Y}$

$$\int_\gamma \omega$$

are solutions to the tautological system $\tau(X, -K_X, G, 1)$.

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31. Solution rank of τ – special case

Consider the family of CY hypersurfaces Y_σ in X, and write $\tau \equiv \tau(X, -K_X, G, 1)$ for the corresponding tautological system.

Theorem: [Bloch-H-Lian-Srinivas-Yau] Let G be a semisimple group and X^n a projective homogeneous G-space (i.e. G/P), such that $\mathfrak{g} \otimes \Gamma(X, K_X^{-r}) \twoheadrightarrow \Gamma(X, TX \otimes K_X^{-r})$. Then the solution rank of τ at any point σ is $\dim H^n(X - Y_\sigma)$.

Remark: (1) It was conjectured that the statement is true without the surjectivity assumption. The latter seems difficult to check in general.

(2) The proof uses a method of Dimca to interpret the de Rham cohomology of the complement and the Lie algebra homology group of certain $\mathfrak{g}$-module.

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Theorem: [Bloch-H-Lian-Srinivas-Yau] Let G be a semisimple group and X^n a projective homogeneous G-space (i.e. G/P), such that $\mathfrak{g} \otimes \Gamma(X, K_X^{-r}) \twoheadrightarrow \Gamma(X, TX \otimes K_X^{-r})$. Then the solution rank of τ at any point σ is $\dim H^n(X - Y_\sigma)$.

Remark: (1) It was conjectured that the statement is true without the surjectivity assumption. The latter seems difficult to check in general.

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Recall that rk $\Pi(E,\omega) \leq$ solution rk of τ. When is this an equality, i.e. when is τ **complete**?

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34. Algebraic rank formula

Introduce notations:

Fix a very ample line bundle L over a projective G-variety X, and put

$$R = \oplus_{j=0}^{\infty} \Gamma(X, L^j)$$

the coordinate ring of X with respect to the tautological embedding $X \hookrightarrow \mathbf{P}V$, $V := \Gamma(X, L)^*$.

Let

$$Z^* : \hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbf{C} \to EndV^*$$

be the dual representation of V.

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Observation:

(1) The space $R[V]e^f$ has a natural $D_{V^*} = \mathbf{C}[a, \partial]$-module structure given by (the "$\frac{1}{2}$-Fourier transform"):

$$a_i \mapsto a_i, \quad \partial_i \mapsto a_i^* + \partial_i.$$

(Note that $a_i \in V \subset R$ and $a_i^* \in V^*$ acts by left multiplications on $R[V]e^f$.)

(2) The operators $Z^*(\hat{\mathfrak{g}})$ commute with D_{V^*}, hence $Z^*(\hat{\mathfrak{g}})R[V]e^f$ is a D_{V^*}-submodule of $R[V]e^f$.

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36. Applications

The theorem has many interesting applications.

- **Corollary**(BHLSY,HLZ): Let X be a projective homogeneous G-space. Then the space of solutions of the differential system τ at any point $b \in V^*$ is canonically isomorphic to

$$H_0^{Lie}(\hat{\mathfrak{g}}, Re^{f_b})^*.$$

- **Example:** $G = PSL_n$, and $X = \mathbf{P}^{n-1}$. Then $L = K_X^{-1} = \mathcal{O}(n)$. Let $x_1, ..., x_n$ be the homogeneous coordinates of X. Then for generic $b \in V^*$, the monomials

$$x_1^{k_1} \cdots x_n^{k_n} e^{f_b}, \quad n | \sum k_i, \quad 0 \leq k_i \leq n-2$$

form a basis of $H_0^{Lie}(\hat{\mathfrak{g}}, Re^{f_b})$.

37. Applications

- **Completeness.** Counting the monomials, we find that generically there are exactly

$$\frac{n-1}{n}((n-1)^{n-1}-(-1)^{n-1})$$

  solutions to the tautological system τ for the universal CY family in $\mathbf{P}^{n-1}$ above. This proves τ is complete, because the period sheaf of this family has this rank.
- **Explicit solutions** (M. Zhu): The result on solution rank has recently led to proof of the so-called 'Hyperplane Conjecture' for $X = \mathbf{P}^{n-1}$. Namely, the period integrals of the universal CY family are precisely given by the combinatorial solution formula of Hosono-Lian-Yau (1995), to the extended GKZ system.

38. Applications: mirror symmetry

- **Constructing LCSL degenerations.** Recall that a LCSL degenerate CY Y_{b_∞} corresponds to $b_\infty \in V^*$, where the local monodromy is maximally unipotent, hence there is just one analytic solution at b_∞. By the rank formula, we have

$$\dim H_0^{Lie}(\hat{\mathfrak{g}}, Re^{f_{b_\infty}}) = 1.$$

- **Example.** Consider the degenerate CY $b_\infty = x_1 \cdots x_n = 0$. Then one finds that

$$H_0^{Lie}(\hat{\mathfrak{g}}, Re^{f_{b_\infty}}) = \mathbf{C}e^{f_{b_\infty}}.$$

  This is the famous LCSL degeneration for the CY family in $\mathbf{P}^{n-1}$, where instanton counting can be done by Mirror Symmetry.

39. Applications: constructing LCSL degenerations

- More generally, for X^n any projective homogenous G-variety and $L = K_X^{-1}$, we have

$$H_n(X - Y_b) \simeq Hom_D(\tau, \mathcal{O}_b^{an}) \simeq H_0^{Lie}(\hat{\mathfrak{g}}, Re^{f_b})^*$$

  for any $b \in V^*$. So, we can construct LCSL candidates by either geometric methods (lhs) or representation theoretic methods (rhs): *look for points $b \in V^*$ where either side is 1 dim.*

- **Detecting rank 1 fibers.** We say that a fiber Y_b has rank 1 if $\dim H_n(X - Y_b) = 1$. Thus to look for LCSL CY, we can look for divisor Y_b in X whose complement has a particular homotopy type.
  Example. For $b_\infty = x_1 \cdots x_n = 0$ in $\mathbf{P}^{n-1}$, the complement is homotopic to n-torus.

40. Applications: constructing LCSL degenerations

- (BHLSY) For the Grassmannian $X = G(k, n)$, we consider the degenerate CY

$$b_\infty = x_{1\cdots k} x_{2\cdots(k+1)} \cdots x_{n1\cdots(k-1)} = 0$$

where the x_I are the Plücker coordinates. We can compute directly the sl_n coinvariants on the module $Re^{f_{b_\infty}}$:

$$H_0^{Lie}(\hat{\mathfrak{g}}, Re^{f_{b_\infty}}) = \mathbf{C}e^{f_{b_\infty}}.$$

Or, we can also compute $H_n(X - Y_{b_\infty})$ topologically by induction on the n components of the divisor Y_{b_∞}, starting from

$$x_{1\cdots k} = 0.$$

41. Applications: constructing LCSL degenerations

(HLZ): Next, we generalize in two ways.

- First, we can "glue" together lower dimensional rank 1 fibers in smaller Grassmannians to yield rank 1 fibers in an arbitrary (type A) partial flag variety.
- Second, we can construct directly a canonical rank 1 fiber in every projective homogenous variety $X = G/P$ as follows.

42. Applications: constructing LCSL degenerations

- There is a natural stratification of the flag variety G/B, called the Richardson stratification. It induces a similar stratification under the projection $G/B \to G/P$. Then

$$Y_{b_\infty} := \text{union of closures of codimension 1 strata.}$$

  is Y_{b_∞} an anticanonical divisor.
- Moreover, Y_{b_∞} is a rank 1 fiber of the universal CY family in $X = G/P$. This is a consequence the solution rank formula, together with the classical BGG multiplicity theorem for Verma modules (or the Kazhdan-Lusztig conjecture).
- **Remark:** Taking $X = G(k, n)$ recovers the rank 1 fiber

$$Y_{b_\infty} = \{x_{1\cdots k}x_{2\cdots(k+1)}\cdots = 0\}.$$

43. LCSL degeneration for toric hypersurfaces

- (HLY): Consider the case a projective toric manifold X^n. Then

$$Y_{b_\infty} := \text{union of T-invariant divisors in } X$$

is anticanonical in X.

- Once again, we find

$$H_0^{Lie}(\hat{\mathfrak{t}}, Re^{f_{b_\infty}}) = \mathbf{C}e^{f_{b_\infty}}$$

hence Y_{b_∞} is a rank 1 fiber. This is also a LCSL degeneration.

44. Applications: injectivity of parallel transport

- (HLZ): For arbitrary finite-orbit G-variety X^n, one of the isomorphisms generalizes to an injective map of local systems:

$$H_n(X - Y_b) \quad \to Hom_D(\tau, \mathcal{O}_b^{an}) \simeq H_0^{Lie}(\hat{\mathfrak{t}}, Re^{f_b})$$
$$\Gamma \to \int_\Gamma \frac{\omega}{f_b}.$$

  Here ω is the unique (up to scalar) holomorphic top form on the complement of the zero section in K_X.

- Note that under the map, parallel transport on the local system $H_n(X - Y_*)$ coincides with analytic continuation on $Hom_D(\tau, \mathcal{O}^{an})$. It follows that for a given point $a \in V^*$, and b any nearby point, the parallel transport map

$$H_n(X - Y_a) \to H_n(X - Y_b)$$

  is also injective.

- **Remark**: This answers a question posed by of Bloch.

45. **Mirror of** G/P (work in progress)

These special points allow us to propose a mirror construction of G/P. In addition, the algebraic rank formula, combined with a mixed Hodge structure resulting in the geometric rank formula, give rise to a Frobenius ring structure near the "Fermat point", which is likely to be identified with the small quantum cohomology ring on the mirror A-side– these constructions will hopefully help clarify many issues regarding mirror of G/P, as well as the "hyperplane conjecture".

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46. **Chain integral solutions to GKZ**

A much more general formula is proved that gives the rank as the (compactly supported) middle cohomology of a certain perverse sheaf, for an arbitrary G-manifold X with a finite number of G-orbits.

Remark: Before this result, the rank was only known for GKZ (toric) case, at a generic point.

The general rank formula actually says much more about τ: As an example, for $X = \mathbf{P}^n$, $G = (\mathbf{C}^*)^n$: the maximal torus of SL_{n+1}, τ reduces to a GKZ system, for which now we can explicitly construct all solutions, as integrals of the holomorphic top form, over certain cycles and chains. This can be done in general for a toric variety, and there are clear evidence that all these solutions are in fact relevant in mirror symmetry.

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These chains are canonically constructed by a spectral sequence, converging to a generic stalk of the solution sheaf of the GKZ system, given in the general formula as a compactly supported middle cohomology of a perverse sheaf.

In fact, these chain integrals were called "semi-periods", and are also relevant in the arithmetic of Calabi-Yau over finite fields, as was shown by Candelas, Ossa, and Rodriguez-Villegas. Some examples of semi-periods were also studied by physicists Avram et al.

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• (Lian-Yau, H-Lian-Zhu, Chen-H-Lian) Most of the results discussed here carry over to general type complete intersections, and to the full period mapping, with some slight modifications.

• Tautological systems provide a new approach to study period integrals for manifolds of general type – **higher dimension analogues** of the classical hyper-elliptic integrals

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• If X is a toric manifold and $G = Aut\ X$, then a tautological system for X specializes to an extended GKZ system, introduced in a series of papers (∼1994) by Hosono-Klemm-Theisen-Yau and Hosono-Lian-Yau on Mirror Symmetry.

• If X is a spherical variety, (a G-variety with an open dense B-orbit) and G is a reductive algebraic group, then a tautological system for X specializes to a Kapranov's system (1997.)

• Therefore, tautological systems **unify and generalize all of the above classes of special functions.** And thanks to the powerful tools of the theory of D-modules, we also have a good control of this differential system.

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• If X is a toric manifold and $G = T$ the usual torus, then a tautological system for X specializes to a GKZ hypergeometric system (1989). Explicit formulas for general solutions are also known in this case (G-K-Z, H-L-Y).

• If X is a toric manifold and $G = Aut\ X$, then a tautological system for X specializes to an extended GKZ system, introduced in a series of papers (∼1994) by Hosono-Klemm-Theisen-Yau and Hosono-Lian-Yau on Mirror Symmetry.

• If X is a spherical variety, (a G-variety with an open dense B-orbit) and G is a reductive algebraic group, then a tautological system for X specializes to a Kapranov's system (1997.)

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Thank you for your attention!