"Confluence" in Ito-Sadahiro number systems

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β-expansions

For $\beta > 1$ we want to write numbers in the form $\sum_{i \leq N} a_i \beta^i$.

Let

$$T_\beta : [0,1) \to [0,1), \quad T_\beta(x) = \beta x - \lfloor \beta x \rfloor$$

Then

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \dots, \quad \text{where } x_i = \lfloor \beta T^{i-1}(x) \rfloor$$

We denote $d_\beta(x) = x_1 x_2 x_3 \cdots \in \{0, 1, \dots, \lceil \beta \rceil - 1\}^{\mathbb{N}}$.

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Admissibility condition

Theorem (W. Parry)

A sequence $x = x_1 x_2 x_3 \dots$ is β-admissible iff for each $i \geq 1$

$$0^\omega \preceq_{\mathrm{lex}} x_i x_{i+1} x_{i+2} \cdots \prec_{\mathrm{lex}} \lim_{\varepsilon \to 0^+} d_\beta(1-\varepsilon).$$

Ordering on $\mathbb{R}$ corresponds to the lexicographic ordering of $d_\beta(x)$.

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$(-\beta)$-expansions

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Let $\mathcal{I} = \left[-\frac{\beta}{\beta+1}, \frac{1}{\beta+1}\right) = [\ell, \ell+1)$ and

$$T_{-\beta} : \mathcal{I} \to \mathcal{I}, \quad T_{-\beta}(x) = -\beta x - \lfloor -\beta x - \ell \rfloor$$

Then

$$x = \frac{x_1}{-\beta} + \frac{x_2}{(-\beta)^2} + \frac{x_3}{(-\beta)^3} + \dots, \quad \text{where } x_i = \lfloor -\beta T^{i-1}(x) - \ell \rfloor.$$

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Admissibility

Theorem (S. Ito, T. Sadahiro)

A string $x_1x_2x_3\dots$ is $(-\beta)$-admissible iff for each $n \geq 1$

$$d_{-\beta}(\ell) \preceq_{\mathrm{alt}} x_i x_{i+1} x_{i+2} \cdots \prec_{\mathrm{alt}} \lim_{\varepsilon \to 0^+} d_{-\beta}(\ell + 1 - \varepsilon)$$

When $x \notin \mathcal{I}$, we divide by a suitable power of $(-\beta)$ and expand $x/(-\beta)^k$.

When $d_{-\beta}(x/(-\beta)^k) = x_1x_2\dots$, we denote

$$\langle x \rangle_{-\beta} = x_1 \dots x_k \bullet x_{k+1} \cdots \approx x_1(-\beta)^{k-1} + \cdots + x_k(-\beta)^0 + \dots$$

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$(-\beta)$-integers

We define $(\pm\beta)$-integers as

$$\mathbb{Z}_\beta = \{x \in \mathbb{R} \mid \langle |x| \rangle = x_1 \dots x_k \bullet 0^\omega\} = \bigcup_{n \geq 0} \beta^n T_\beta^{-n}(0)$$

$$\mathbb{Z}_{-\beta} = \{x \in \mathbb{R} \mid \langle x \rangle = x_1 \dots x_k \bullet 0^\omega\} = \bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^{-n}(0)$$

By coding gaps in $\mathbb{Z}_{-\beta}$ by letters of an alphabet, one gets a bidirectional infinite word u_β, resp. $u_{-\beta}$.

The words u_β and $u_{-\beta}$ are invariant under substitution.

Substitutions are over a finite alphabet for $d_\beta(1)$, resp $d_{-\beta}(\ell)$ eventually periodic

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Some properties of $(\pm\beta)$-integers

Unlike $\mathbb{Z}_\beta$ is $\mathbb{Z}_{-\beta}$ not symmetric around 0.

$\mathbb{Z}_{-\beta} = \{0\}$ iff $\beta < \frac{1+\sqrt{5}}{2}$. This never happens for $\mathbb{Z}_\beta$.

$\mathbb{Z}_\beta$ is relatively dense, i.e. lengths of gaps are $< K$.

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Motivation

Lemma

Let $\beta > 1$ be root of $x^2 - mx - m,\ m \geq 1$. Then

$$\mathbb{Z}_{-\beta} = \{ \sum_{i \geq 0} a_i(-\beta)^i \mid a_i \in \mathcal{A}_{-\beta} \}$$

For β-numeration, we have the following theorem

Theorem (Ch. Frougny)

Let $\beta > 1$ then the following conditions are equivalent:

1. *β is root of $x^k - mx^{k-1} - \cdots - mx - n$ for $m \geq n \geq 1$.*
2. $\mathbb{Z}_\beta = \{ \sum_{i \geq 0} a_i \beta^i \mid a_i \in \{0, 1, \ldots \lfloor \beta \rfloor \} \}$.

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$(-\beta)$-confluence

Theorem (D. Dombek, Z. Masáková, V.)

Let $\beta > 1$. Then three following conditions are equivalent:

1. β *is root of* $x^k - mx^{k-1} - \cdots - mx - n$, *where* $m \geq n \geq 1$ *and* $m = n$ *for* k *even.*
2. $\mathbb{Z}_{-\beta} = \left\{ \sum_{i \geq 0} a_i(-\beta)^i \mid a_i \in \mathcal{A}_{-\beta} \right\}$.
3. *Substitutions fixing* u_β^+ *and* $u_{-\beta}$ *are conjugate.*

Proof

You will see 1) $\Rightarrow$ 2) and consequently 1) $\Rightarrow$ 3) on the blackboard.

Proof continued: 2) $\Rightarrow$ 1)

Confluence property implies spaces in $\mathbb{Z}_{-\beta}$ are ≤ 1.

It follows that $d_{-\beta}(\ell) = m\,0\,m\,0 \ldots m\,0\,a\,b \ldots, \quad ab \neq m0$.

We take the shortest forbidden string $1m0m0\ldots0m$ where m is the maximal digit.

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One can show that admissible transcription is of the form

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Comments

In β-systems, rewriting system associated to β was confluent.

The $(-\beta)$-rewriting system is not confluent, e.g. for $\beta = \frac{1+\sqrt{5}}{2}$ we have

$$1\bullet = 110\bullet = 11010\bullet = \dots$$

Arithmetics of confluent $\pm\beta$?

- If β is $+$confluent then the set of numbers with finite expansion forms a ring.
- If β is $-$confluent then $m+1$ has infinite expansion.

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- Study of optimal representations (K. Dajani et al.)
- Study of Rauzy fractals and reversal invariant language of u_β (J. Bernat)
- Description of spectra of numbers (D. Garth & K. Hare)

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Comparison of β- and $(-\beta)$- numeration

Ch. Kalle: Let $\beta \in (1, 2)$ then T_β and $T_{-\beta}$ are measurably isomorphic iff β is root of $x^k - x^{k-1} - \cdots - x - 1$.

Conjecture: This holds also for roots of

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