Local-global principles for quadratic forms

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The p-adics

- Let $\mathbb{Q}$ denote the field of rational numbers
- For every prime p, $\|.\|_p$ denotes the p-adic norm on $\mathbb{Q}$ defined by

$$\left\| p^m \frac{r}{s} \right\| = \left(\frac{1}{p} \right)^m, \ p \nmid r, \ p \nmid s$$

- $\mathbb{Q}_p$ denotes the completion of $\mathbb{Q}$ for the metric induced by $\|.\|_p$
- $\mathbb{Q}_p$ is also the field of fractions of the inverse limit

$$\mathbb{Z}_p = \lim_{\leftarrow_m} \mathbb{Z}/p^m\mathbb{Z}$$

Local fields

- $\mathbb{Q}_\infty$, the completion of $\mathbb{Q}$ for the usual metric $|.|$ is $\mathbb{R}$
- The fields $\mathbb{Q}_p$ for each prime p and $\mathbb{Q}_\infty = \mathbb{R}$ form a set of overfields of $\mathbb{Q}$
- Study of algebraic structures like *quadratic forms* or *division algebras* is simple over these fields

An example

Let $q = \sum_{1 \leq i \leq m} a_i X_i^2$ where $a_i \in \mathbb{Z}$ with $\gcd_i(a_i) = 1$ be a quadratic form over $\mathbb{Q}$

Question

When is q isotropic ? i.e. when do there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{Q}$ not all zero with

$$\sum a_i \lambda_i^2 = 0 \ ?$$

A local answer

- It is easy to look for congruence solutions modulo p^n because $\mathbb{Z}/p^n\mathbb{Z}$ is a finite ring !

- One can pass from existence of solutions modulo p^n, $n \geq 1$ to existence of solutions in $\mathbb{Q}_p$

  ($\mathbb{Q}_p$ is *complete* and *solutions can be lifted*)

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  ($\mathbb{Q}_p$ is *complete* and *solutions can be lifted*)

- q is isotropic over $\mathbb{R}$ if and only if the a_i s don't all have the same sign, i.e. q is *indefinite*

From local to global

Theorem (Hasse-Minkowski)

If q is isotropic over $\mathbb{R}$ and $\mathbb{Q}_p$ for each prime p, then q is isotropic over $\mathbb{Q}$

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In geometric terms, if $X_q : q = 0$ in $\mathbb{P}^{n-1}$ is the quadric associated with q, then

$$X_q(\mathbb{Q}_p) \neq \emptyset \; \forall \; p \;, X_q(\mathbb{R}) \neq \emptyset \implies X_q(\mathbb{Q}) \neq \emptyset$$

Hasse principle

- We may replace $\mathbb{Q}$ by a *number field* k which is any finite extension extension of $\mathbb{Q}$
- Let Ω_k be the set of all places of k (which are extensions of the places on $\mathbb{Q}$ associated to primes and ∞)
- For each $v \in \Omega_k$, let k_v denote the completion of k at v

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- For each $v \in \Omega_k$, let k_v denote the completion of k at v
- Let X be a variety defined over k (common zeroes of polynomials over k)
- We say that X satisfies *Hasse principle* if

$$X(k_v) \neq \emptyset \ \forall \ v \in \Omega_k \implies X(k) \neq \emptyset$$

Hasse principle

Question

Which classes of varieties over number fields satisfy Hasse principle?

Examples

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- We restrict our attention to *homogeneous spaces* under connected linear algebraic groups

Groups and actions

- Let G be a connected linear algebraic group over k ($G \hookrightarrow \mathrm{GL}_n(k)$ is a subgroup defined by polynomial equations)
- Examples are SL_n, SO_n, Sp_{2n} etc

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  ($G \hookrightarrow \mathrm{GL}_n(k)$ is a subgroup defined by polynomial equations)
- Examples are SL_n, SO_n, Sp_{2n} etc
- Let X be a variety defined over k
- Let $\overline{k}$ be an algebraic closure of k
- Suppose there is a (polynomial) action of G on X. This yields a group action

$$G\left(\overline{k}\right) \times X\left(\overline{k}\right) \to X\left(\overline{k}\right)$$

Homogeneous spaces

- X is a *homogeneous* space under G if the action of $G\left(\overline{k}\right)$ on $X\left(\overline{k}\right)$ is *transitive*
- X is a *principal homogeneous* space under G if the action of $G\left(\overline{k}\right)$ on $X\left(\overline{k}\right)$ is *simply transitive*
- X is a *projective homogeneous* space under G if X is a projective variety and a homogeneous space under G

Examples

Example (Principal homogenous space)

Let $\lambda \in k \setminus \{0\}$*. Define* X *by*

$$X(k) = \{z \in \mathrm{M}_n(k) | \det(z) = \lambda\}$$

Then X *is a principal homogeneous space under* SL_n

Example (Projective homogenous space)

Let q *be a non-degenerate quadratic form over* k *of* $\dim \geq 3$*.*
Define X_q *by*

$$X_q(k) = \{z \in \mathbb{P}^{n-1}(k) | q(z) = 0\}$$

Then X_q *is a projective homogeneous space under* $\mathrm{SO}(q)$

Hasse principle for principal homogeneous spaces

Theorem I

Let k be a number field and G, a semi-simple simply connected linear algebraic group defined over k. Then Hasse principle holds for principal homogenous spaces under G

- Examples of simple simply connected groups :

  $\mathrm{SL}_n, \mathrm{Sp}_{2n}, \mathrm{Spin}(q)$ for $\dim(q) \geq 3$, $\mathrm{SU}(h)$ where h is a hermitian form over L where $[L : k] = 2$ and twisted forms of these groups.

- Theorem is due to Kneser for classical groups, Harder for exceptional groups except type E_8 and Chernousov for type E_8

Hasse principle for projective homogeneous spaces

Theorem II (Harder)

Let k be a number field and G, a connected linear algebraic group defined over k. Then Hasse principle holds for projective homogenous spaces under G

Brauer-Manin obstruction

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- Let X/k be a smooth geometrically integral variety over number field k
- The *Brauer-Manin set* BM is a subset of the product of local points defined using the Brauer group of X :

$$X(k) \subseteq BM \subseteq \left(\prod_v X(k_v) \right)$$

Brauer-Manin obstruction

We say that Brauer-Manin obstruction is the *only obstruction to Hasse Principle* for a class of varieties $\mathscr{X}$, if $\forall\ X \in \mathscr{X}$,

$$\mathit{BM\ of\ X} \text{ is not empty } \implies X(k) \text{ is not empty}$$

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Theorem (Sansuc, Voskresenskii)

Brauer-Manin obstruction is the only obstruction to Hasse principle for principal homogeneous spaces.

Hasse principle for PHS under k-rational groups

A k-group G is k-rational if $k(G)/k$ is a purely transcendental extension

Corollary (Sansuc, Voskresenskii)

Suppose G is a k-rational group. Then Hasse principle holds for principal homogeneous spaces under G over k

Proof.

- Let X be a principal homogeneous space under G over k
- Let $\overline{X}$ be a smooth compactification
- FACT : $\mathrm{Br}(k) \twoheadrightarrow \mathrm{Br}\left(\overline{X}\right)$ is surjective
- Thus the BM set of X is $\prod_{v \in \Omega_k} X\left(k_v\right)$

□

Discrete valuations

- To discuss Hasse principle in a more general setting of function fields, we first define *valuation* rings and fields
- A *discrete valuation* on a field F is a homomorphism $v : F^* \to \mathbb{Z}$ such that for each $a, b \in F^*$ with $a, b, a+b$ non-zero
$$v(a+b) \geq \min\left(v(a), v(b)\right)$$
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- One sets $v(0) = \infty$
- $\mathcal{O}_v = \{a \in F | v(a) \geq 0\}$ is the *valuation ring*
- $\mathcal{O}_v$ has a *unique* maximal ideal $\mathcal{M}_v = \{a \in F | v(a) > 0\}$ generated by any π (*parameter*) with $v(\pi) = 1$
- $\kappa(\pi) := \mathcal{O}_v / \langle \pi \rangle$ is the *residue field* for v

Completions

- Let $v : F^* \to \mathbb{Z}$ be a discrete valuation on a field F and let π be a parameter of F
- Let $\lambda > 1$. Then v defines a norm on F by

$$|a|_v = \left(\frac{1}{\lambda}\right)^{v(a)} \forall a \in \mathbb{F} \setminus \{0\}$$

- F_v is defined to be the field obtained by completing F with respect to $|.|_v$
- F_v is a discrete valued field with valuation ring $\hat{\mathcal{O}}_v$
- π is a parameter for F_v also and the residue field remains the same

Example I

- Let $v_p : \mathbb{Q}^* \to \mathbb{Z}$ with $v\left(p^m \frac{r}{s}\right) = m$ where $p \not| r, s$
- v_p defines a discrete valuation on $\mathbb{Q}$
- $|a|_{v_p} = \left(\frac{1}{p}\right)^{v(a)}$ for $a \in \mathbb{Q} \setminus \{0\}$ defines the *p-adic* norm on $\mathbb{Q}$
- The completion of $\mathbb{Q}$ with respect to $|.|_{v_p}$ adic norm is the *p-adic* field $\mathbb{Q}_p$ defined earlier
- $\mathcal{O}_{v_p} = \mathbb{Z}_p$ and a parameter $\pi = p$

Example II

- Let $K(t)$ denote the rational function field in one variable over a field K
- Let $v_t : K(t)^* \to \mathbb{Z}$ with $v\left(t^m \frac{f}{g}\right) = m$ where $f, g \in K[t]$ and $t \not| f, g$
- v_t defines a discrete valuation on $K(t)$
- The completion of $K(t)$ with respect to v_t is the field of *Laurent series* $K((t))$
- $\mathcal{O}_{v_t} = K[[t]]$, the ring of *formal power series* in t and a parameter $\pi = t$

Function fields

- Let K be a number field without ordering or a p-adic field
- Let $\mathcal{O}$ be the ring of integers in K
- Let X/K be a smooth projective geometrically integral curve over K
- Let $F = K(X)$

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- Let Ω_0 denote the set of discrete valuations of F trivial on K (corresponds to closed points of X)
- However Ω_0 misses all discrete valuations of F restricting to a p-adic valuation of K !

Divisorial discrete valuations

- Let $\mathscr{X} \to \mathcal{O}$ be a regular proper model of X/K
- Let $\Omega_{\mathscr{X}}$ be the set of all discrete valuations of F centered on codimension one points of $\mathscr{X}$
- Let $\Omega = \bigcup_{\mathscr{X}} \Omega_{\mathscr{X}}$ consist of *divisorial discrete valuations* of F

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- Let $\Omega = \bigcup_{\mathscr{X}} \Omega_{\mathscr{X}}$ consist of *divisorial discrete valuations* of F
- One could look for Hasse principle for varieties over F with reference to Ω
- For $v \in \Omega$, F_v denotes the completion of F at v with residue field $\kappa(v)$ which is either a p-adic field or a global field of positive characteristic
- One understands the arithmetic of $\kappa(v)$ from class field theory

Isotropy over F_v

- Let q be a quadratic form over F
- Over F_v, $q \simeq u_1x_1^2 + \ldots + u_rx_r^2 + \pi v_1y_1^2 + \ldots + \pi v_sy_s^2$ with π a parameter at v and u_i, v_i units in $\mathcal{O}_v$ (i.e $v(u_i) = v(v_j) = 0$)

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- This happens if and only if $\overline{u_1}x_1^2 + \ldots + \overline{u_r}x_r^2$ or $\overline{v_1}y_1^2 + \ldots + \overline{v_s}y_s^2$ is isotropic over $\kappa(v)$

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- Thus if $r \geq 5$ or $s \geq 5$, then q is isotropic over F_v
- Hence if $\dim q \geq 9$, then q is isotropic over F_v

Hasse principle for quadrics

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Open question

Let k be a number field without ordering and let $F = k(X)$ where X is a curve over k. Does there exist $N > 0$ such that every quadratic form in more than N variables over F has a non-trivial zero ? (i.e is $u(F) < \infty$?)

Failure of Hasse principle

- Caution : Hasse principle fails for quadratic forms of dimension 3 or 4 over $k(X)$ where k is a number field

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- Non-trivial elements $\alpha \in {}_2Ш(J(X))$ for an elliptic curve X yield quaternion *division* algebras over $F = k(X)$ which are locally split for all v. Thus, their norm forms are rank 4 anisotropic quadratic forms which are locally split

Failure of Hasse principle

- Caution : Hasse principle fails for quadratic forms of dimension 3 or 4 over $k(X)$ where k is a number field
- Non-trivial elements $\alpha \in {}_2\text{Ш}(J(X))$ for an elliptic curve X yield quaternion *division* algebras over $F = k(X)$ which are locally split for all v. Thus, their norm forms are rank 4 anisotropic quadratic forms which are locally split
- However the question whether Hasse principle holds for quadratic forms over $k(X)$ of dimension at least 5 is open

Finiteness of the u-invariant

- If k is a p-adic field, indeed every 9 dimensional quadratic form over $F = k(X)$ has a non-trivial zero
- Parimala-Suresh for $p \neq 2$
- Heath-Brown/Leep for all p

Hasse principle for quadrics over function fields

Theorem (Colliot-Thélène–Parimala–Suresh)

Let k be a p-adic field where $p \neq 2$ and let $F = k(X)$ for a curve X/k. Then Hasse principle holds for quadrics over F with respect to Ω

Proof relies on *patching techniques* developed by Harbater-Hartmann-Krashen

HHK patching

- Let K be a p-adic field
- Let $\mathcal{O}$ be the valuation ring of K and κ its residue field
- Let X be a smooth projective geometrically integral curve over K
- Let $F = K(X)$
- $\mathscr{X} \xrightarrow{\eta} Spec\,(\mathcal{O})$ be a regular proper model of X/K
- Let $\mathscr{X}_0 \to Spec(\kappa)$ be the reduced special fiber of $\mathscr{X}$
- Assume that $\mathscr{X}_0$ is a union of regular curves with normal crossings

HHK patching

- For each $x \in \mathscr{X}_0$, let $F_x = ff(\hat{\mathbb{O}}_{\mathscr{X},x})$
- If x is codimension one on $\mathscr{X}$, F_x is the complete discretely valued field with respect to ν_x
- If x is codimension two on $\mathscr{X}$, F_x is the field of fractions of the completion of the two-dimensional regular local ring at x

HHK patching

Theorem (HHK)

Let G be a connected linear algebraic group over F which is F-rational. Then Hasse principle holds with reference to $\{F_x | x \in \mathscr{X}_0\}$ for

1. *Principal homogeneous spaces under G*
2. *Projective homogeneous space under G*

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Remark

If Z is an F-variety, then $Z(F_x) \neq \emptyset \ \forall \ x \in \mathscr{X}_0$ implies $Z(F_v) \neq \emptyset \ \forall \ v \in \Omega$

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Proof of CT-P-S relies on showing the converse, namely that $Z(F_v) \neq \emptyset \ \forall \ v \in \Omega$ implies $Z(F_x) \neq \emptyset \ \forall \ x \in \mathscr{X}_0$ for quadrics Z

Conjectures for Hasse principle over function fields

Conjecture I (Colliot-Thélène–Parimala–Suresh)

Let K be a p-adic field and $F = K(X)$, a function field in one variable over K. Let G be a semi-simple simply connected linear algebraic group over F. Then Hasse principle holds for principal homogeneous spaces under G over F

Conjecture II (Colliot-Thélène–Parimala–Suresh)

Let K and F be as in Conjecture I. Let G be a connected linear algebraic group over F. Then Hasse principle holds for projective homogeneous spaces under G over F

These conjectures run parallel to the theorems in the number field case

Conjecture II

- (Grothendieck) : Conjecture II holds for Severi-Brauer varieties

  If $X = SB(A)$ is the Severi-Brauer variety defined by A, then $X(L) \neq \emptyset \iff A \otimes_F L \simeq \mathrm{M}_n(L)$

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  Grothendieck shows A_{F_v} is split for all $v \in \Omega$ implies A is split

- (S.Reddy and V.Suresh) : Conjecture II holds for Generalized Severi-Brauer varieties. They show

$$\mathrm{index}(A) = \mathrm{lcm}_{v \in \Omega}\, \mathrm{index}\,(A_v)$$

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- (Z.Wu) : Conjecture II holds for projective homogeneous spaces under unitary groups with some restrictions for groups of type ${}^2A_n$. Method of proof is via reducing to HHK patches

Conjecture I

Theorem (Preeti, Y.Hu)

Conjecture I holds for principal homogenous spaces under groups G of type B_n, C_n, D_n or ${}^2A_n$ with constraints, namely

$G = \mathrm{SU}(A, \sigma)$ *where* A *is a central simple algebra with unitary* L/F *involution and* $\mathrm{index}(A)$ *is square-free*

Conjecture I for ${}^1A_n$

- Let A/F is a central simple algebra – A is a form for the matrix algebra
- The determinant on matrices yields a function $Nrd : A \to F$
- Let $G = \mathrm{SL}_1(A)$ denote the group of reduced norm one elements in A
- Principal homogeneous spaces under G upto isomorphism are given by $F^*/\operatorname{Nrd}(A^*)$.
- That is for $\lambda \in F^*$, $\mathrm{Nrd}(z) = \lambda$ is the associated PHS
- Conjecture I for $\mathrm{SL}_1(A)$ asserts that local reduced norms are reduced norms

Invariants for $\mathrm{SL}_1(A)$

- For a field F with characteristic not dividing n, $\mathrm{H}^3\left(F, \mu_n^{\otimes 2}\right)$ denotes the *Galois cohomology* group of F with values in $\mu_n^{\otimes 2}$

- There are invariants for principal homogeneous spaces under simple simply connected linear algebraic groups with values in Galois cohomology, due to Rost

- The invariant for $\mathrm{SL}_1(A)$ with index $A = n$

$$F^* / \operatorname{Nrd}(A^*) \to \mathrm{H}^3\left(F, \mu_n^{\otimes 2}\right), \quad [\lambda] \rightsquigarrow (\lambda) \bullet [A]$$

goes back to Suslin.

Conjecture I for ${}^1A_n$ for index(A) square-free

For $G = \mathrm{SL}_1(A)$ with index(A) square-free, Conjecture I follows from the following theorems of Merkurjev-Suslin and Kato

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Theorem (Merkurjev-Suslin)

Let A be a central simple algebra of square-free index n. Then the following map has trivial kernel

$$F^* / \mathrm{Nrd}\left(A^*\right) \to \mathrm{H}^3\left(F, \mu_n^{\otimes 2}\right), \quad [\lambda] \rightsquigarrow (\lambda) \bullet [A]$$

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Theorem (Kato)

The following map has trivial kernel

$$\mathrm{H}^3\left(F, \mu_n^{\otimes 2}\right) \to \prod_{v \in \Omega_{\mathscr{X}}} \mathrm{H}^3\left(F_v, \mu_n^{\otimes 2}\right)$$

Conjecture I for ${}^1A_n$

Theorem (Parimala-Preeti-Suresh)

Conjecture I holds for $G = \mathrm{SL}_1(A)$ *where* A *is a central simple algebra with* $(\mathrm{index}(A), p) = 1$

Obstructions

- One would like to understand the failure of Hasse principle in general via obstructions similar to the Brauer-Manin obstructions
- Indeed such *reciprocity* obstructions were constructed using the Brauer group in this setting
- Using these obstructions, examples of principal homogeneous spaces under non-rational tori were constructed by Colliot-Thélène–Parimala–Suresh which fail Hasse principle

A question

The following question remains open :

Let K be a p-adic field and $F = K(X)$, a function field in one variable over K. Let G be a connected F-rational linear algebraic group under F. Does Hasse principle hold for principal homogeneous spaces under G with respect to Ω ?

Hasse principle for rational tori

Theorem (Harari, Szamuely)

Let K be a p-adic field and $F = K(X)$, a function field in one variable over K. Let T be a connected F-rational torus. Then Hasse principle holds for principal homogeneous spaces under T with respect to Ω_0