Entropy minimality of $\mathbb{Z}^d$ shifts of finite type

(joint work with Samuel Lightwood)

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Preliminaries

$\mathcal{A}$ some finite alphabet $\qquad d \in \mathbb{N}$ the dimension

$\mathbb{Z}^d$ **full shift on** $\mathcal{A}$: $\qquad \mathcal{A}^{\mathbb{Z}^d} \qquad \sigma : \mathbb{Z}^d \times \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$

$$\forall \vec{\imath}, \vec{\jmath} \in \mathbb{Z}^d, x \in \mathcal{A}^{\mathbb{Z}^d} : \ \sigma(\vec{\imath}, x)_{\vec{\jmath}} := x_{\vec{\imath}+\vec{\jmath}}$$

$\mathbb{Z}^d$ **(sub)shifts**: $\quad X \subseteq \mathcal{A}^{\mathbb{Z}^d} \qquad$ shift invariant, closed subset

given by a family of forbidden patterns $\mathcal{F} \subseteq \bigcup_{F \subsetneq \mathbb{Z}^d \text{ finite}} \mathcal{A}^F$ on finite shapes such that

$$X_{\mathcal{F}} := \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} \,\middle|\, \forall \vec{\imath} \in \mathbb{Z}^d, F \subsetneq \mathbb{Z}^d \text{ finite} : \ x|_{\vec{\imath}+F} \notin \mathcal{F} \right\}$$

$\mathbb{Z}^d$ **shifts of finite type** (SFTs):

X is a $\mathbb{Z}^d$ SFT $:\Longleftrightarrow \ \exists\, \mathcal{F} \subseteq \bigcup_{F \subsetneq \mathbb{Z}^d \text{ finite}} \mathcal{A}^F$ with $|\mathcal{F}| < \infty$ and $X = X_{\mathcal{F}}$ $\qquad$ (local rules)

Topological entropy

$X = X_{\mathcal{F}}$ a $\mathbb{Z}^d$ subshift on $\mathcal{A}$

$\mathcal{L}(X) := \bigcup_{F \subseteq \mathbb{Z}^d \text{ finite}} \{x|_F \mid x \in X\}$ **globally admissible** patterns (language)

d-**dimensional topological entropy** of a $\mathbb{Z}^d$ subshift X:

$$h_{\text{top}}(X) := \limsup_{n \to \infty} \frac{\log \left| \mathcal{L}_{[1,n]^d}(X) \right|}{n^d}$$

For $\mathbb{Z}$ SFTs: **Easy to compute the entropy!** $h_{\text{top}}(X) = \log \lambda_A$

where A is the transition matrix of a digraph representing the $\mathbb{Z}$ SFT X and $\lambda_A \in \mathbb{R}_0^+$ is its Perron value

For $\mathbb{Z}^d$ SFTs: **In general a (very) hard question!**

for $d > 1$ no general algorithm (formula) to compute the entropy, only few examples with known entropy

Entropy minimality

Definition:

A $\mathbb{Z}^d$ subshift X is **entropy minimal** if any (non-empty) proper subshift $Y \subsetneq X$ has strictly less topological entropy, i.e. $h_{\text{top}}(Y) < h_{\text{top}}(X)$.

(very useful in many arguments)

Theorem [folklore]: Every irreducible $\mathbb{Z}$ SFT is entropy minimal.

A $\mathbb{Z}$ SFT (defined by a digraph) is **irreducible** if the graph is **strongly connected**.

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Uniformly mixing $\mathbb{Z}^d$ SFTs

useful to exclude pathologies and to get (stronger) results

Definition: A $\mathbb{Z}^d$ subshift X is called

- **(topologically) mixing** if for any two non-empty finite subsets $V, W \subsetneq \mathbb{Z}^d$ there exists a constant $D_{V,W} \in \mathbb{N}$ so that for any $\vec{\imath} \in \mathbb{Z}^d$ for which V and $\vec{\imath} + W$ have separation at least $D_{V,W}$ and any pair of valid points $x, y \in X$ there exists a valid point $z \in X$ such that $z|_V = x|_V$ and $z|_{\vec{\imath}+W} = y|_{\vec{\imath}+W}$.
  non-uniform mixing condition

- **block gluing** if there exists a constant $g \in \mathbb{N}$ (gap size) such that for any two cuboid blocks $B_1, B_2 \subsetneq \mathbb{Z}^d$ with separation at least g and any pair of valid points $x, y \in X$ there exists a valid point $z \in X$ such that $z|_{B_1} = x|_{B_1}$ and $z|_{B_2} = y|_{B_2}$.

- **uniformly filling** if there exists a constant $l \in \mathbb{N}$ (filling length) such that for any cuboid block $B \subsetneq \mathbb{Z}^d$ and any pair of valid points $x, y \in X$ there exists a valid point $z \in X$ such that $z|_B = x|_B$ and $z|_{\mathbb{Z}^d \setminus (B+[-l,+l]^d)} = y|_{\mathbb{Z}^d \setminus (B+[-l,+l]^d)}$.

Observation [Boyle-Pavlov-S]: For $g \in \mathbb{N}$ and any $\mathbb{Z}^d$ shift X we have:

X uniformly filling with filling length $g \Longrightarrow X$ block gluing with gap $g \Longrightarrow X$ (topol.) mixing

For $\mathbb{Z}$ SFTs the whole hierarchy collapses to a single notion (mixing). In $\mathbb{Z}^d$ $(d > 1)$ the 3 notions are distinct.

Consequences of uniform mixing conditions

Questions: Can we say anything about entropy minimality of $\mathbb{Z}^d$ SFTs?

Even without knowing the exact value of the entropy?

(the $\mathbb{Z}$ proof uses Perron-Frobenius which is not available in $\mathbb{Z}^d$ SFTs)

Theorem [folklore]:

Every uniformly filling $\mathbb{Z}^d$ subshift X (not necessarily SFT) is entropy minimal.

Questions:

What causes (non) entropy minimality in $\mathbb{Z}^d$ SFTs?

Do we really need this very strong uniform mixing condition to assure entropy minimality?

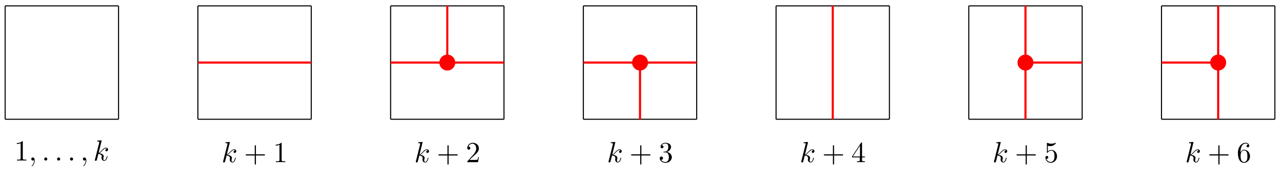
At first it seems **Yes!**, as:

Observation [S]:

There exist block gluing $\mathbb{Z}^d$ SFTs which are not entropy minimal.

Wire shifts

We define the family of nearest neighbor $\mathbb{Z}^2$ SFTs called wire shifts W_k over an alphabet $\mathcal{A}_k$ ($k \in \mathbb{N}_0$) consisting of k **distinct but completely interchangeable blanks** plus 6 **wire symbols** (drawn as unit square Wang tiles):



Obvious rules of **preserving the presence of wires** across edges.

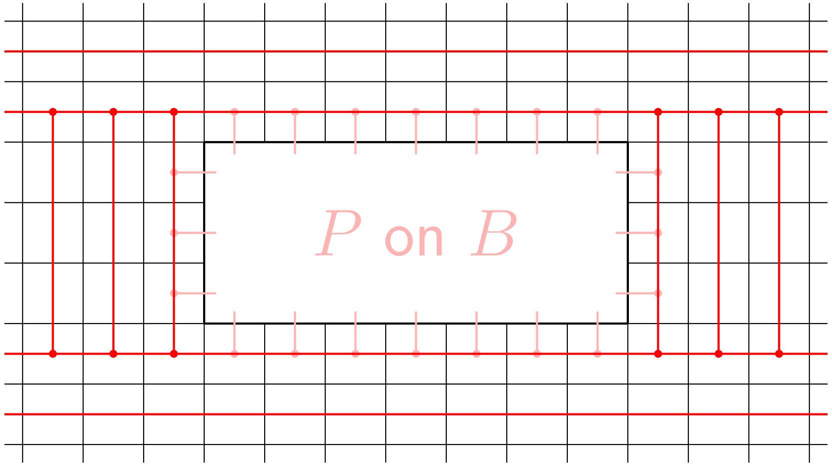
(think of edges as being colored either red or white and apply Wang tiling rules)

Configurations in $\mathbb{Z}^2$ contain **blanks** and possibly a system of **infinite straight wires** which can branch into subwires in T-junctions, but which neither start nor stop. (no pure corners)

(Similarly we can define a family of $\mathbb{Z}^3$ SFTs called Wall shifts etc.)

Observation: Properties of the Wire shifts W_k:

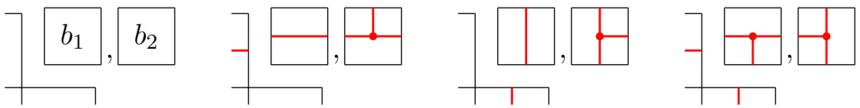
- W_k is **block gluing** for any $k \in \mathbb{N}_0$. (horizontal or vertical separation ≥ 2, build wires as below)



only boundaries matter

- W_k is **not uniformly filling** for $k > 0$ (W_0 is uniformly filling). (wires have to continue)
- The boundary of all blanks is frozen (**non-universal**).
- **Topological entropies** (exactly known for $k > 1$):

$$h_{\text{top}}(W_k) = \log k \quad \text{for } k > 1 \qquad \text{vs.} \qquad \log 1.75 < h_{\text{top}}(W_1) < \log 1.97$$



corner condition

- W_k is **not entropy minimal** for $k > 1$ (contains full shift on k blanks as proper subshift).
  W_0 is entropy minimal (uniformly filling).

Subshifts with signals

Questions: What is the **difference** between block gluing and uniform filling?

Block gluing systems may contain frozen boundaries, signals may escape to infinity.

Definition: A $\mathbb{Z}^d$ subshift has a **signal** if there is a proper subset $\mathcal{S} \subsetneq \mathcal{A}$ of its alphabet and a finite neighborhood $F \subsetneq \mathbb{Z}^d$ such that whenever a symbol from $\mathcal{S}$ occurs at some coordinate it has to be part of an infinite F-connected component formed only by symbols from $\mathcal{S}$.

Recode $\mathbb{Z}^d$ SFTs to nearest neighbor SFTs, then signals are truly connected components of symbols from $\mathcal{S}$.

Examples: Wires in the wire shifts W_k with $k \geq 1$, where $\mathcal{S} = \{$non-blank symbols$\}$.

Signals ($=$ wire symbols) have to escape to infinity $\Longrightarrow$ there are non universal boundaries
(one way to destroy entropy minimality)

Signals may start (or end) at a given coordinate from where they spread (like a rooted tree) or they might come from and go to infinity (like the wires).

Universal boundaries

Question: Is entropy minimality really related to a uniform mixing condition?
What about the wire shift W_1?

Definition:

A pattern $Q \in \mathcal{L}_{\partial C_N}(X)$ on the boundary of the cube $C_N = [1, N]^d$ is M-**universal** if any pattern $P \in \mathcal{L}_{C_M}(X)$ can occur (somewhere) in its interior.

$$\forall\, P \in \mathcal{L}_{C_M}(X) : \ \exists\, x \in X : \ x|_{\partial C_N} = Q \ \wedge \ \exists\, \vec{\imath} \in \mathbb{Z}^d : \ \vec{\imath} + C_M \subseteq C_N \ \wedge \ x|_{\vec{\imath}+C_M} = P.$$

Observations: M-universal boundary patterns are also m-universal for $m < M$.

In uniformly filling shifts all C_N-boundary patterns are $(N-l)$-universal.

Wire shifts W_k $(k > 0)$ have non-universal boundary patterns ($Q =$ all blanks).

But they also have universal boundary patterns.

Rich vs. poor boundaries

Definition: Let $N \in \mathbb{N}$ and $\varepsilon > 0$.

A boundary pattern $Q \in \mathcal{L}_{\partial C_N}(X)$ is ε-**rich**, if

$$\log \left| \left\{ P \in \mathcal{L}_{C_N}(X) \;\middle|\; P|_{\partial C_N} = Q \right\} \right| > (h_{\mathrm{top}}(X) - \varepsilon) \cdot |C_N|$$

Conversely Q is ε-**poor**, if

$$\log \left| \left\{ P \in \mathcal{L}_{C_N}(X) \;\middle|\; P|_{\partial C_N} = Q \right\} \right| \leq (h_{\mathrm{top}}(X) - \varepsilon) \cdot |C_N|$$

Observation [Lightwood-S]:

In the wire shift W_1 every (large enough) boundary pattern is either M-universal or is ε-poor.
(depending on whether or not there are wires near the 4 corners)

In the wire shift W_k $(k > 1)$ there exist (arbitrarily large) ε-rich boundary patterns which are not 1-universal.

A characterization of entropy minimality for $\mathbb{Z}^d$ SFTs

Theorem [Lightwood-S]:
A $\mathbb{Z}^d$ SFT is entropy minimal if and only if the set of all non-universal boundary patterns is poor.

Consequences: The wire shift W_1 is entropy minimal, all other W_k $(k > 1)$ are not.

Maximal measure(s) on W_1 has(ve) full support.

interesting to see what such measure(s) could look like (not obvious, wide open question)

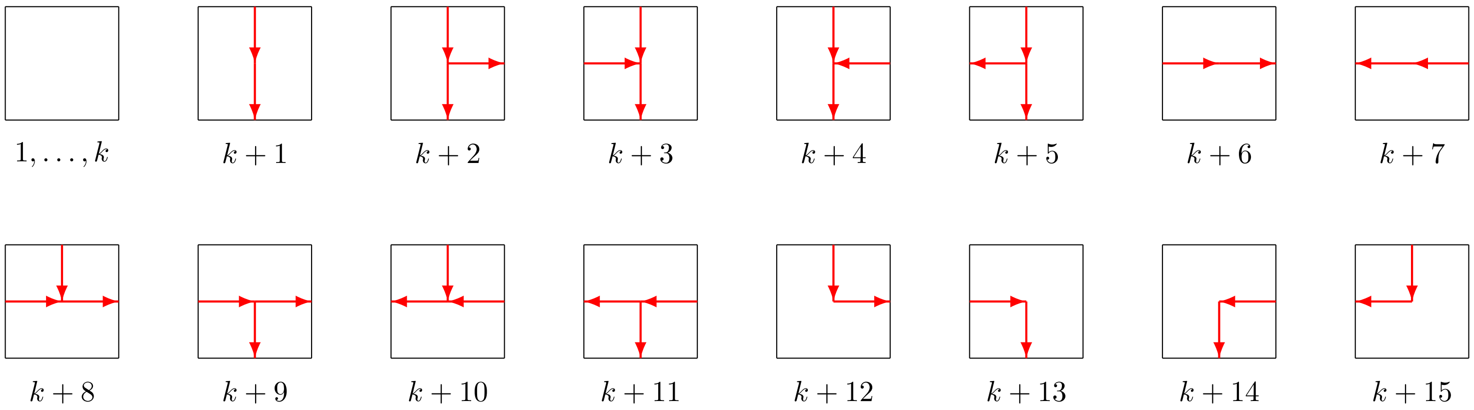
There is a conceptual change of behavior in families like the wire shifts:

uniformly filling with no signals $\dashrightarrow$ entropy minimal block gluing with signals $\dashrightarrow$ non entropy minimal block gluing with signals

Another family of examples

Define the family of corner gluing **meandering streams** $\mathbb{Z}^2$ **SFTs** $X_{\mathsf{MS},k}$ for $k \in \mathbb{N}$ [Boyle-Pavlov-S]:

The alphabet (displayed below) consists of k **blanks** and 15 **stream symbols** (modelling a system of rivers meandering from North to South through the $\mathbb{Z}^2$ plane).



All of $X_{\mathsf{MS},k}$ are corner gluing (not uniformly filling); the first ones are entropy minimal, but for k large they are no longer entropy minimal.