Symbolic covers of toral automorphisms

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Let $n \geq 2$. Every $A \in \mathrm{GL}(n, \mathbb{Z})$ defines a 'linear' automorphism of the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Assume for the moment that A is hyperbolic (no eigenvalues of absolute value 1). Then A — acting linearly on $\mathbb{R}^n$ — has an expanding (or unstable) eigenspace W^u and a contracting (or stable) eigenspace W^s. Under the quotient map $\pi\colon \mathbb{R}^n \longrightarrow \mathbb{T}^n$ these two spaces get mapped to dense subgroups of $\mathbb{T}^n$ which will be denoted by X^u and X^s.

Basic example: Take the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. It has an expanding eigenspace v_1 and a contracting eigenspace v_2.

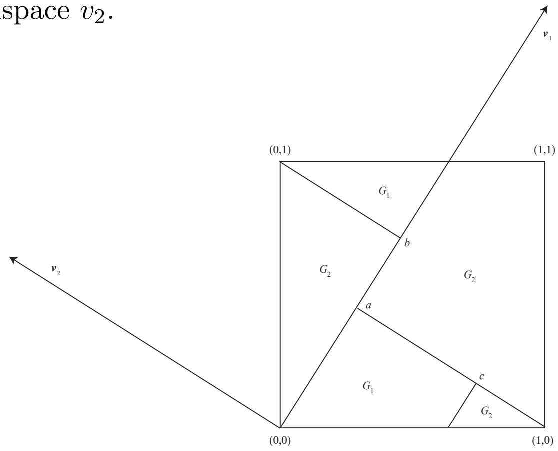

Figure 1

When mapping these eigenspaces to $\mathbb{T}^2$ they intersect in infinitely many points, e.g. the points a, b, c in te drawing. The rectangles $G_1, G_2 \subset \mathbb{T}^2$ in the drawing, whose boundaries are pieces of v_1 and v_2, form a *Markov partition* for A: if we assign, to every $x \in \mathbb{T}^2$, the sequence $(w_k) \in \{0,1\}^{\mathbb{Z}}$ with

$$w_k = j \pmod 2 \text{ if } A^k x \in G_j, \; j = 1, 2, \; k \in \mathbb{Z},$$

we obtain an almost one-to-one map from $\mathbb{T}^2$ to the *golden mean* shift of finite type $V = \{(y_k) \in \{0,1\}^{\mathbb{Z}} : y_k y_{k+1} = 0 \text{ for all } k \in \mathbb{Z}\}$, which we can reverse to obtain a continuous, surjective, almost one-to-one map $\phi\colon V \longrightarrow \mathbb{T}^2$ and a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\ \sigma\ } & V \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ \mathbb{T}^2 & \xrightarrow[A]{} & \mathbb{T}^2 \end{array}$$

where $\sigma\colon V \longrightarrow V$ is the shift $(\sigma y)_k = y_{k+1}$.

A similar construction can be used to obtain Markov partitions for arbitrary hyperbolic automorphisms A of $\mathbb{T}^n$, in which the elements of the Markov partitions are again obtained from pieces of the stable and unstable subgroups of A through a more complicated process (resulting in fractal boundaries of these sets).

In the early 1990's Vershik proposed a different method for constructing symbolic covers for hyperbolic toral automorphisms which I'll describe from a more general viewpoint. The points a, b, c in Figure 1 all lie on an intersection of X^u and X^s in $\mathbb{T}^2$, and are thus *homoclinic*: their forward and backward orbits under A converge to 0 — and they do so exponentially fast. We focus on the point a, which is the image under the quotient map $\pi\colon \mathbb{R}^2 \longrightarrow \mathbb{T}^2$ of the unique point in the intersection of W^u with $W^s + (1,0)$. For every $v = (v_k) \in \ell^\infty(\mathbb{Z},\mathbb{Z})$ (the set of bounded two-sided integer sequences), the element $\xi(v) = \sum_{k\in\mathbb{Z}} v_k A^{-k} a$ is well-defined, and the resulting map $\xi\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow \mathbb{T}^2$ is equivariant: the diagram

$$\begin{array}{ccc} \ell^\infty(\mathbb{Z},\mathbb{Z}) & \xrightarrow{\bar{\sigma}} & \ell^\infty(\mathbb{Z},\mathbb{Z}) \\ \xi\downarrow & & \downarrow\xi \\ \mathbb{T}^2 & \xrightarrow[A]{} & \mathbb{T}^2 \end{array}$$

commutes, where $\bar{\sigma}$ is the shift on $\ell^\infty(\mathbb{Z},\mathbb{Z})$.

Claim 1: *ξ is surjective.*

In order to prove this we assume that $A \in \mathrm{GL}(n,\mathbb{Z})$ is a companion matrix of the form

$$A_f = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -f_0 & -f_1 & -f_2 & \dots & -f_{n-2} & -f_{n-1} \end{bmatrix},$$

where $f = f_0 + \dots + f_{n-1}z^{n-1} + f_n z^n$ is the characteristic polynomial of A_f (note that $f_n = |f_0| = 1$). *We also assume — to simplify things a little — that f is irreducible.*

Denote by $\sigma\colon \mathbb{T}^\mathbb{Z} \longrightarrow \mathbb{T}^\mathbb{Z}$ the shift, and consider the continuous, surjective group homomorphism $f(\sigma) = f_0 + f_1\sigma + \dots + f_n\sigma^n\colon \mathbb{T}^\mathbb{Z} \longrightarrow \mathbb{T}^\mathbb{Z}$. We set

$$X_f = \ker f(\sigma) = \{x = (x_k) \in \mathbb{T}^\mathbb{Z} : f_0x_k + f_1x_{k+1} + \dots + f_nx_{k+n} = 0 \text{ for every } k \in \mathbb{Z}\}, \quad (1)$$

and denote by σ_f the shift on X_f. Since $f_n = |f_0| = 1$, the map $\psi\colon X_f \longrightarrow \mathbb{T}^n$, defined by

$$\psi(x) = \begin{bmatrix} x_0 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

for every $x \in X_f$, is an algebraic conjugacy between σ_f and A_f: ψ is a group isomorphism which makes the diagram

$$\begin{array}{ccc} X_f & \xrightarrow{\sigma_f} & X_f \\ \psi\downarrow & & \downarrow\psi \\ \mathbb{T}^n & \xrightarrow[A_f]{} & \mathbb{T}^n \end{array}$$

commute. We linearize (X_f, σ_f) by setting

$$W_f = \{v \in \ell^\infty(\mathbb{Z},\mathbb{R}) : \pi(v) \in X_f\} = \{v \in \ell^\infty(\mathbb{Z},\mathbb{R}) : f(\bar{\sigma})v \in \ell^\infty(\mathbb{Z},\mathbb{Z})\}, \quad (2)$$

where $\pi\colon \ell^\infty(\mathbb{Z},\mathbb{R}) \longrightarrow \mathbb{T}^{\mathbb{Z}}$ is component-wise reduction $(\bmod 1)$ and $\bar{\sigma}$ is the shift on $\ell^\infty(\mathbb{Z},\mathbb{R})$. In order to prove Claim 1 we consider the group ring $\ell^1(\mathbb{Z},\mathbb{R})$ and identify each $v = (v_k) \in \ell^1(\mathbb{Z},\mathbb{R})$ with the two-sided power series $\sum_{k\in\mathbb{Z}} v_k z^k$. For $v, w \in \ell^1(\mathbb{Z},\mathbb{R}) \subset \ell^\infty(\mathbb{Z},\mathbb{R})$, the product of these power series corresponds to the usual convolution of v and w in $\ell^1(\mathbb{Z},\mathbb{R})$. Then our polynomial f, viewed as an element of $\ell^1(\mathbb{Z},\mathbb{R})$, is invertible. This follows either from Wiener's theorem, or by using a partial fraction decomposition

$$\frac{1}{f} = \frac{1}{f_n} \cdot \sum_{\gamma} \frac{c_\gamma}{z-\gamma},$$

where the sum is taken over the roots of f, and by expressing each term $\frac{1}{z-\gamma}$ separately as a summable two-sided power series:

$$\frac{1}{z-\gamma} = \begin{cases} z^{-1}\sum_{k\geq 0} \gamma^k z^{-k} & \text{if } |\gamma| < 1, \\ -\gamma^{-1}\sum_{k\geq 0} \gamma^{-k} z^k & \text{if } |\gamma| > 1. \end{cases}$$

Since the point $y = f^{-1} = \sum_{k\in\mathbb{Z}} y_k z^k$ obtained in this manner has summable coefficients, we can form the group homomorphism $\bar{\xi} = \sum_{k\in\mathbb{Z}} y_k \bar{\sigma}^k \colon \ell^\infty(\mathbb{Z},\mathbb{R}) \longrightarrow \ell^\infty(\mathbb{Z},\mathbb{R})$, which satisfies that $\bar{\xi} = f(\bar{\sigma})^{-1}$. From the definition of W_f it is now clear that $W_f = \bar{\xi}(\ell^\infty(\mathbb{Z},\mathbb{Z}))$ and $f(\bar{\sigma})W_f = \ell^\infty(\mathbb{Z},\mathbb{Z})$. Hence $\xi \coloneqq \pi \circ \bar{\xi}\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow X_f$ is surjective. In order to verify that ξ is really the map appearing in the statement of Claim 1 one can check that the point $\mathbf{v}^{(0)} \in \ell^\infty(\mathbb{Z},\mathbb{Z})$, defined by

$$\mathbf{v}^{(0)}_k = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

gets mapped by ξ to the homoclinic point a in Figure 1.

We obtain the following result:

Theorem 1. *If $f = f_0 + \cdots + f_n z^n$ is an irreducible polynomial with integer coefficients and no roots of absolute value* 1 (*we call such a polynomial hyperbolic*), *and if $X_f = \ker f(\sigma) \subset \mathbb{T}^{\mathbb{Z}}$ is the closed, shift-invariant subgroup defined in* (1), *then the map $\xi\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow X_f$ defined in the last paragraph is a shift-equivariant surjective group homomorphism with kernel $f(\bar{\sigma})(\ell^\infty(\mathbb{Z},\mathbb{Z}))$.*

Theorem 1 is obviously not restricted to irreducible toral automorphisms. We could take, for example, $f = z - 2$ or $f = 3 - 2z$, in which case the space X_f would be a solenoid rather than a torus.

Theorem 1 yields the diagram

$$\begin{array}{ccc} f(\sigma)(\ell^\infty(\mathbb{Z},\mathbb{Z})) & \xrightarrow{\bar{\sigma}} & f(\sigma)(\ell^\infty(\mathbb{Z},\mathbb{Z})) \\ \big\downarrow & & \big\downarrow \\ \ell^\infty(\mathbb{Z},\mathbb{Z}) & \xrightarrow{\bar{\sigma}} & \ell^\infty(\mathbb{Z},\mathbb{Z}) \\ {\scriptstyle\xi}\big\downarrow & & \big\downarrow{\scriptstyle\xi} \\ X_f & \xrightarrow{\sigma_f} & X_f \end{array}$$

and allows us to identify X_f equivariantly with $\ell^\infty(\mathbb{Z},\mathbb{Z})/f(\bar{\sigma})(\ell^\infty(\mathbb{Z},\mathbb{Z}))$. In order to use this result to obtain symbolic covers or representations of X_f one has to find closed, bounded, shift-invariant, subsets $V \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ which meet every coset of $f(\bar{\sigma})(\ell^\infty(\mathbb{Z},\mathbb{Z}))$

in $\ell^\infty(\mathbb{Z},\mathbb{Z})$ in at least one point (i.e., such that the restriction $\xi|_V : V \longrightarrow X_f$ is surjective), but whose intersection with each of these cosets is as small as possible. If f is a Pisot polynomial (i.e., if f has one large root $\beta > 1$ and all other roots have absolute value < 1), then the two-sided β-shift V_β is sofic and satisfies that $\xi(V_\beta) = X_f$; it is conjectured (but proved only in some special cases) that the restriction $\xi|_{V_\beta}$ is almost one-to-one (in which case we say that V_β is a sofic representation of X_f). For $f = z^2 - z - 1$ this example was the starting point for Vershik's original construction. I should also mention the following general result.

Theorem 2 (S, 2000)**.** *Let f be an irreducible hyperbolic polynomial with integer coefficients, and let $X_f = \ker f(\sigma) \subset \mathbb{T}^{\mathbb{Z}}$ be the closed, shift-invariant subgroup defined in* (1)*. Then there exists a sofic shift $V \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ such that the restriction $\xi|_V : V \longrightarrow X_f$ is surjective and almost one-to-one. In other words, V is a sofic representation of X_f.*

Nonhyperbolic polynomials. The last part of the talk (which I didn't get to) was supposed to discuss irreducible nonhyperbolic polynomials, i.e., irreducible noncyclotomic polynomials with some roots of absolute value 1. Examples are $f = 1 - z - z^2 - z^3 + z^4$ (a *Salem polynomial* with one root $\beta > 1$, two roots of absolute value 1, and the root $1/\beta$), or $f = 5 - 6z + 5z^2$ (with two noncyclotomic roots of absolute value 1). One can define $X_f \subset \mathbb{T}^{\mathbb{Z}}$ and $W_f \subset \ell^\infty(\mathbb{Z},\mathbb{R})$ exactly as before; for $f = 1 - z - z^2 - z^3 + z^4$, σ_f is algebraically conjugate to the toral automorphism

$$A_f = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}, \tag{3}$$

but in general one again obtains solenoids. For any such f, the automorphism σ_f of the group X_f is ergodic, but nonexpansive, and has no homoclinic points and no Markov partitions. The map $f(\bar{\sigma}) : W_f \longrightarrow \ell^\infty(\mathbb{Z},\mathbb{Z})$ is neither injective nor surjective, and the space $f(\bar{\sigma})(W_f) \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ is a bit of a mystery. The search for symbolic representations (which was originally motivated by the question whether the two-sided β-shift of a Salem number β could be regarded as a symbolic representation of the corresponding nonhyperbolic toral automorphism A_f — or the shift space X_f — defined by the minimal polynomial f of β).

Although the following discussion is quite general, I'll keep referring to the toral automorphism A_f in (3). The matrix A_f has one-dimensional expanding and contracting subspaces $W^u, W^s \subset \mathbb{R}^4$, and a two-dimensional invariant subspace $W^{(0)}$ on which A_f acts isometrically by rotation. Under the quotient map $\pi : \mathbb{R}^4 \longrightarrow \mathbb{T}^4 \cong X_f$ these three spaces get mapped to dense subgroups of $X_f \cong \mathbb{T}^4$ which will be denoted by X^u, X^s and $X^{(0)}$, the *unstable*, *stable* and *central* subgroups. Although the intersection $X^u \cap X^s$ is empty, the intersections $(X^{(u)} + X^{(0)}) \cap X^s$ and $X^{(u)} \cap (X^{(0)} + X^s)$ contain nonzero points which are *forward* and *backward homoclinic*, respectively. We denote by a^+ and a^- the images under π of the unique points in $(W^{(u)} + W^{(0)}) \cap W^s$ and $W^{(u)} \cap (W^{(0)} + W^s)$, respectively, and set, for every $v \in \ell^\infty(\mathbb{Z},\mathbb{Z})$,

$$\bar{\xi}^*(v) = \sum_{n \geq 0} v_n \bar{\sigma}^{-n}(a^-) + \sum_{n < 0} v_n \bar{\sigma}^{-n}(a^+). \tag{4}$$

Since the coordinates a_n^+ and a_{-n}^- decay exponentially as $n \to \infty$ and $W_h \subset \ell^\infty(\mathbb{Z},\mathbb{R})$ is closed, $\bar{\xi}^*(v)$ is well-defined, but it will in general not lie in $\ell^\infty(\mathbb{Z},\mathbb{R})$, but in

$$\ell^*(\mathbb{Z},\mathbb{R}) = \left\{ w = (w_n) \in \mathbb{R}^{\mathbb{Z}} : \sup_{n \in \mathbb{Z}} \tfrac{|w_n|}{|n|+1} < \infty \right\}.$$

It is not difficult to check that

$$f(\bar{\sigma}) \circ \bar{\xi}^*(v) = v$$

for every $v \in \ell^\infty(\mathbb{Z}, \mathbb{Z})$, and that

$$f(\bar{\sigma})(W_f) = \{v \in \ell^\infty(\mathbb{Z}, \mathbb{Z}) : \bar{\xi}^*(v) \in \ell^\infty(\mathbb{Z}, \mathbb{Z})\}$$

We again write $\pi\colon \ell^*(\mathbb{Z}, \mathbb{R}) \longrightarrow \mathbb{T}^{\mathbb{Z}}$ for coordinate-wise reduction (mod 1) and set

$$W_f^* = \{w \in \ell^*(\mathbb{Z}, \mathbb{Z}) : \pi(w) \in X_f\}.$$

Then $\bar{\xi}^*(\ell^\infty(\mathbb{Z}, \mathbb{Z})) \subset W_f^*$, but the maps $\bar{\xi}^*\colon \ell^\infty(\mathbb{Z}, \mathbb{Z}) \longrightarrow \ell^*(\mathbb{Z}, \mathbb{Z})$ and

$$\xi^* = \pi \circ \bar{\xi}^*\colon \ell^\infty(\mathbb{Z}, \mathbb{Z}) \longrightarrow X_f \tag{5}$$

are not shift-equivariant: for every $v \in \ell^\infty(\mathbb{Z}, \mathbb{Z})$,

$$\sigma_f \circ \xi^*(v) - \xi^* \circ \bar{\sigma}(v) \in \pi(\ker f(\bar{\sigma})) \ (= X^{(0)} \text{ in our special case}).$$

If $V \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ is a closed, shift-invariant subset, then the non-equivariance of ξ suggests that we should not look at $\xi(V)$, but at the σ_f-invariant set $\xi(V) + \ker f(\sigma)$ (or, in our special case, the A_f-invariant set $\xi(V) + X^{(0)}$).

Definition. A closed, bounded, shift-invariant subset $V \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ is a (symbolic) *pseudo-cover* of X_f if $\xi(V) + \ker f(\sigma) = X_f$.

Problem: Let $f = 1 - z - z^2 - z^3 + z^4$, and let A_f be the matrix (3). Is the two-sided beta-shift V_β determined by the root $\beta > 1$ of f a symbolic pseudo-cover of $X_f = \mathbb{T}^4$? This still unresolved problem provided much of the initial motivation for the work on nonhyperbolic polynomials described here.

Although I cannot say much about the automorphism (3) or, more generally, about β-shifts arising from Salem numbers, I'll finish by stating a recent general result.

Theorem 3 (S, 2013)**.** *Let f be a noncyclotomic irreducible nonhyperbolic polynomial with integer coefficients, and let σ_f be the shift on the group $X_f \subset \mathbb{T}^{\mathbb{Z}}$ defined in* (1)*. Then there exists a symbolic pseudo-cover $V \subset f(\bar{\sigma})(W_f) \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ whose entropy coincides with that of the automorphism σ_f.*

Problem: Under the hypotheses of Theorem 3, does there always exist an equal entropy *sofic* pseudocover $V \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ of X_f?

Remark. For background, details and references concerning most of the results mentioned here see the brief survey:

K. Schmidt, *Quotients of $\ell^\infty(\mathbf{Z}, \mathbf{Z})$ and symbolic covers of toral automorphisms.*, Amer. Math. Soc. Transl. **217** (2006), 223–246
or
`http://www.mat.univie.ac.at/~kschmidt/Publications/vershik.pdf`.