Conjugacy and equivalence of weighted automata

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The results presented in this talk are based on a joint work with

Marie-Pierre Béal (Univ. Paris-Est)

and

Sylvain Lombardy (Univ. Bordeaux)

published in *Proc. of CSR 2006*.
The complete journal version is still in preparation.

Some of the results have been included in the chapter

Rational and recognizable series

of the *Handbook of Weighted Automata*, Springer, 2009.

Part I

An introductory result

The Rational Bijection Theorem

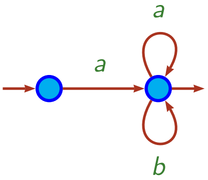
Theorem

If two rational languages have the same growth function, then there exists a letter-to-letter rational bijection that maps one language onto the other.

An example: a first language

$$L = a\,(a+b)^*$$

$\mathcal{A}$

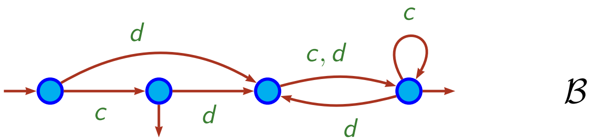


a	aaa	$aaaa$	$abaa$
	aab	$aaab$	$abab$
aa	aba	$aaba$	$abba$
ab	abb	$aabb$	$abbb$

$$\forall n \in \mathbb{N}\,,\ n > 0 \qquad \mathrm{g}_L\,(n) \;=\; \mathrm{Card}\,(L \cap \{a,b\}^n) \;=\; 2^{n-1}$$

An example: a second language

$$K = (c + d\,c + d\,d)^* \setminus \{c\,c\,(c+d)^* \cup 1_{B^*}\}$$

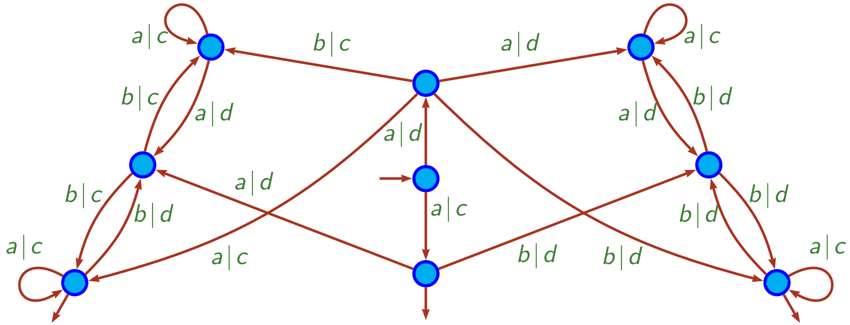


c	*cdc*	*cdcc*	*dcdd*
	cdd	*cddc*	*ddcc*
dc	*dcc*	*dccc*	*dddc*
dd	*ddc*	*dcdc*	*dddd*

$$\forall n \in \mathbb{N},\ n > 0 \qquad \mathrm{g}_K(n) = \mathrm{Card}\,(K \cap \{c,d\}^n) = 2^{n-1}$$

An example: the rational bijection

$L = a(a+b)^*$ $\qquad$ $K = (c + d\,c + d\,d)^* \setminus \{c\,c\,(c+d)^* \cup 1_{B^*}\}$



a	aaa	aaaa	abaa	c	cdc	cdcc	dcdd
	aab	aaab	abab		cdd	cddc	ddcc
aa	aba	aaba	abba	dc	dcc	dccc	dddc
ab	abb	aabb	abbb	dd	ddc	dcdc	dddd

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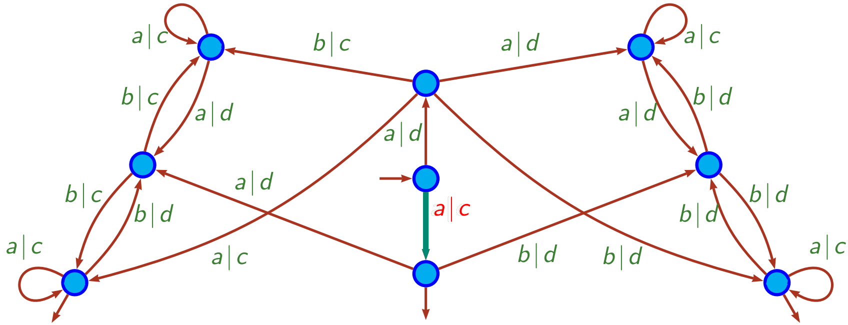

a	*aaa*	*aaaa*	*abaa*	*c*	*cdc*	*cdcc*	*dcdd*
	aab	*aaab*	*abab*		*cdd*	*cddc*	*ddcc*
aa	*aba*	*aaba*	*abba*	*dc*	*dcc*	*dccc*	*dddc*
ab	*abb*	*aabb*	*abbb*	*dd*	*ddc*	*dcdc*	*dddd*

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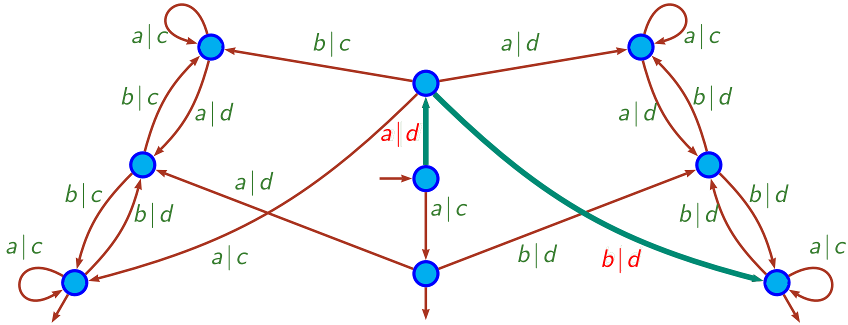

a	*aaa*	*aaaa*	*abaa*	*c*	*cdc*	*cdcc*	*dcdd*
	aab	*aaab*	*abab*		*cdd*	*cddc*	*ddcc*
aa	*aba*	*aaba*	*abba*	*dc*	*dcc*	*dccc*	*dddc*
ab	*abb*	*aabb*	*abbb*	*dd*	*ddc*	*dcdc*	*dddd*

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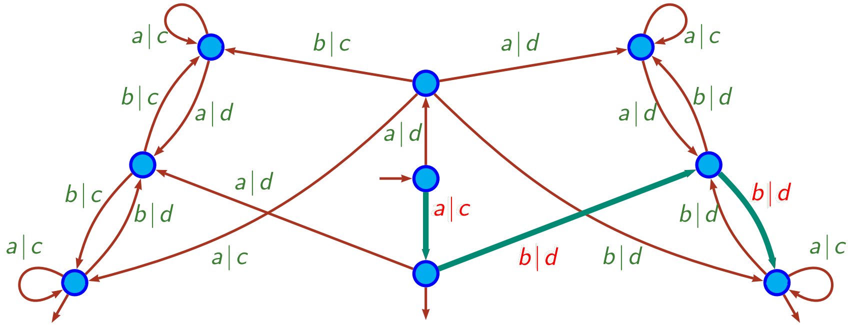

a	aaa	$aaaa$	$abaa$	c	cdc	$cdcc$	$dcdd$
	aab	$aaab$	$abab$		cdd	$cddc$	$ddcc$
aa	aba	$aaba$	$abba$	dc	dcc	$dccc$	$dddc$
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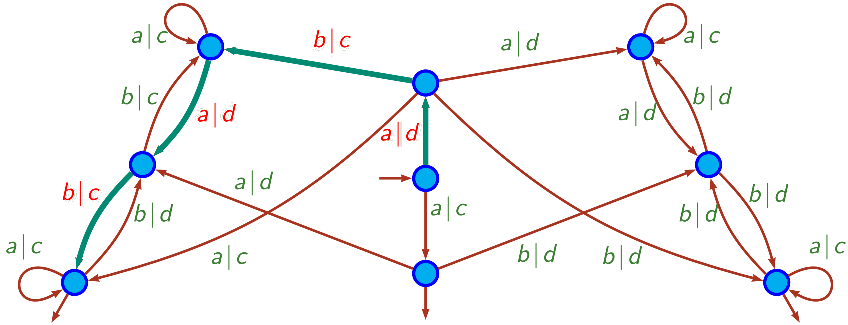
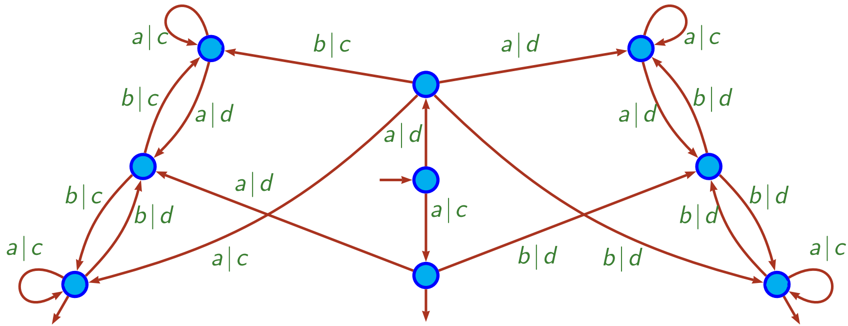
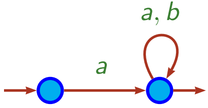


a	*aaa*	*aaaa*	*abaa*	*c*	*cdc*	*cdcc*	*dcdd*
	aab	*aaab*	*abab*		*cdd*	*cddc*	*ddcc*
aa	*aba*	*aaba*	*abba*	*dc*	*dcc*	*dccc*	*dddc*
ab	*abb*	*aabb*	*abbb*	*dd*	*ddc*	*dcdc*	*dddd*

The RBT on this example: construction of the transducer

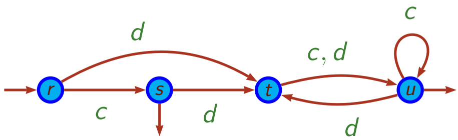


from the automata



$\mathcal{A}$

and

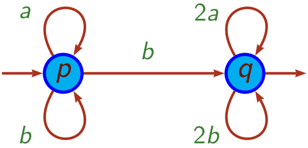


$\mathcal{B}$

Proof of the Rational Bijection Theorem

1. The model of weighted automaton:
   Bridge between growth function and finite automata
2. Decidability of equivalence of generating series
   Taken for granted
3. The conjugacy theorem
4. Definition of morphisms and the FET for weighted automata
5. The harvest

Step 1: the model of weighted automaton



$$\xrightarrow{1} p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \xrightarrow{1}$$

$$\xrightarrow{1} p \xrightarrow{b} q \xrightarrow{2a} q \xrightarrow{2b} q \xrightarrow{1}$$

$$b\,a\,b \quad \longmapsto \quad 5 \qquad\qquad \forall w \in A^* \qquad w \quad \longmapsto \quad \langle w \rangle_2$$

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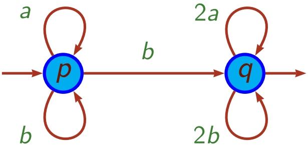

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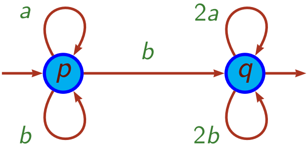
$$s\colon A^* \longrightarrow \mathbb{N} \qquad s \;:\; w \quad \longmapsto \quad <s, w> \qquad\qquad s \in \mathbb{N}^{A^*}$$

$$s = b + a\,b + 2\,b\,a + 3\,b\,b + a\,a\,b$$
$$+ 2\,a\,b\,a + 3\,a\,b\,b + 4\,b\,a\,a + 5\,b\,a\,b + \ldots$$

Step 1: the model of weighted automaton



$$\xrightarrow{1} p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \xrightarrow{1}$$

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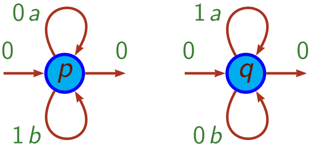
$$+ 2\,a\,b\,a + 3\,a\,b\,b + 4\,b\,a\,a + 5\,b\,a\,b + \ldots$$

Step 1: the model of weighted automaton(cont.)

$\mathcal{M} = \langle\, \mathbb{N} \cup \{+\infty\}, \min, +\,\rangle$



$$\xrightarrow{0} p \xrightarrow{1b} p \xrightarrow{0a} p \xrightarrow{1b} p \xrightarrow{0}$$

$$\xrightarrow{0} q \xrightarrow{0b} q \xrightarrow{1a} q \xrightarrow{0b} q \xrightarrow{0}$$

$b\,a\,b \quad \longmapsto \quad 1 \qquad\qquad \forall w \in A^* \qquad w \quad \longmapsto \quad \min\{|w|_a, |w|_b\}$

$s\colon A^* \longrightarrow \mathcal{M} \qquad s \,:\, w \quad \longmapsto \quad <s, w> \qquad\qquad s \in \mathcal{M}\langle\langle A^* \rangle\rangle$

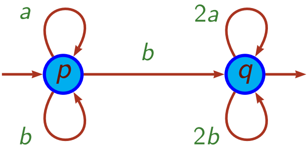
$$s = 0 1_{A^*} \oplus 0a \oplus 0b \oplus 0aa \oplus 1ab \oplus 1ba \oplus 0bb$$
$$\oplus\ 0aaa \oplus 1aab \oplus 1aba \oplus 1abb \oplus \ \ldots$$

Step 1: the model of weighted automaton

Series play the role of languages

$\mathbb{K}\langle\langle A^* \rangle\rangle$ **plays the role of** $\mathfrak{P}(A^*)$

Step 1: the model of weighted automaton



Automata are matrices

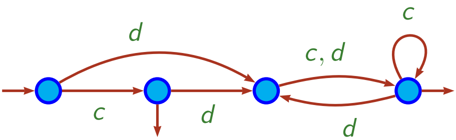
$$\mathcal{A} = \langle\, I, E, T \,\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} a+b & b \\ 0 & 2\,a+2\,b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\left|\mathcal{A}\right| = I\, E^{*}\, T$$

Step 2: the generating series

A language $K = (c + d\,c + d\,d)^* \setminus \{c\,c\,(c+d)^* \cup 1_{B^*}\}$ that is,

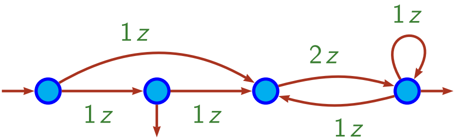
an unambiguous automaton:



$\mathcal{B}$

is transformed into an automaton over $\{z\}^*$ with weight in $\mathbb{N}$

$\mathcal{B}'$



which realises the generating series $\mathsf{G}_K(z) = \sum_{n \in \mathbb{N}} \mathsf{g}_K(n)\, z^n$.

Step 2: the generating series

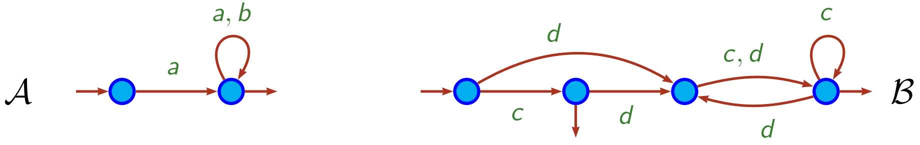
Growth functions

are realised

by weighted automata.

Step 2: the generating series

(i) Two unambiguous finite automata $\mathcal{A}$ and $\mathcal{B}$,

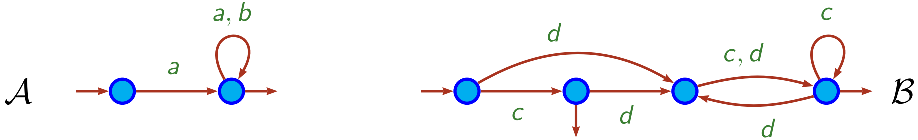


Step 2: the generating series

(i) Two unambiguous finite automata $\mathcal{A}$ and $\mathcal{B}$,

(ii) transformed into $\mathcal{A}'$ and $\mathcal{B}'$, over $\{z\}^*$ with multiplicity in $\mathbb{N}$, which realise the *generating functions* $\mathsf{G}_L(z)$ and $\mathsf{G}_K(z)$:

$$\mathsf{G}_L(z) = \sum_{n \in \mathbb{N}} \mathsf{g}_L(n)\, z^n \qquad \text{and} \qquad \mathsf{G}_K(z) = \sum_{n \in \mathbb{N}} \mathsf{g}_K(n)\, z^n \ ,$$



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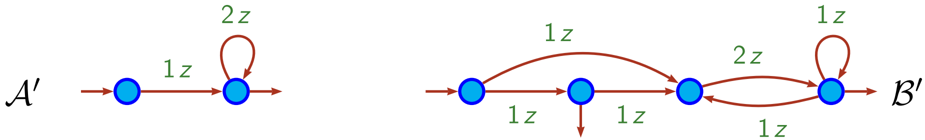

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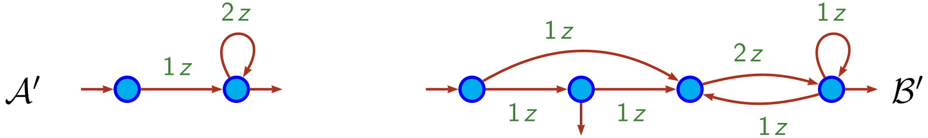
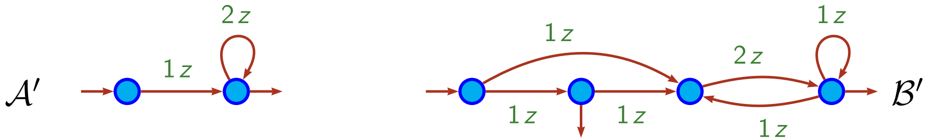
(iii) and whose equivalence is decidable (Chomsky–Miller 1958).



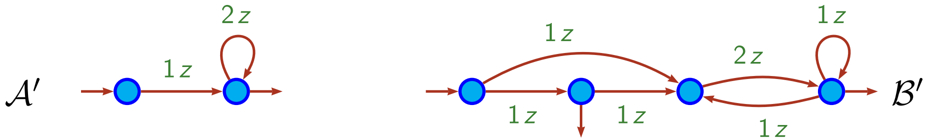
Step 3: The conjugacy theorem



Step 3: The conjugacy theorem

Theorem (BLS)

Two $\mathbb{N}$-automata are equivalent if and only if they are conjugate to a same third $\mathbb{N}$-automaton.



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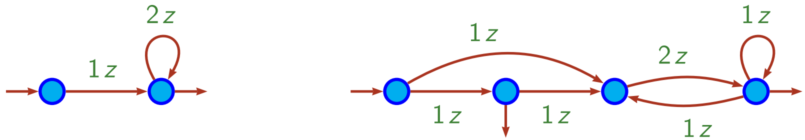
Automata are matrices

$\mathcal{A}'$



$\mathcal{B}'$

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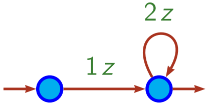
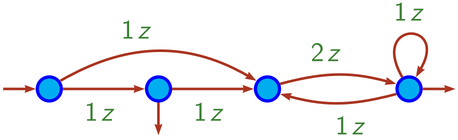




$\mathcal{B}'$

Step 3: The conjugacy theorem

Theorem (BLS)

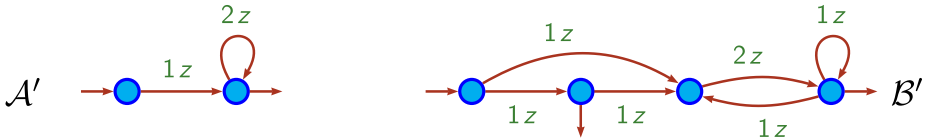
Two $\mathbb{N}$-automata are equivalent if and only if they are conjugate to a same third $\mathbb{N}$-automaton.

Definition

Let $\mathcal{A} = \langle\, I, E, T \,\rangle$ and $\mathcal{B} = \langle\, J, F, U \,\rangle$ be two $\mathbb{K}$-automata.

$\mathcal{A}$ is conjugate to $\mathcal{B}$ if

$$\exists X \quad \mathbb{K}\text{-matrix} \qquad I\,X = J, \qquad E\,X = X\,F, \quad \text{and} \quad T = X\,U$$



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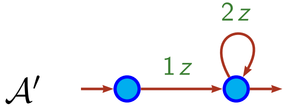
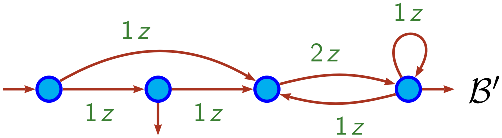
$$\exists X \quad \mathbb{K}\text{-matrix} \qquad I\,X = J, \qquad E\,X = X\,F, \quad \text{and} \quad T = X\,U$$

This is denoted as $\mathcal{A} \overset{X}{\Longrightarrow} \mathcal{B}$.



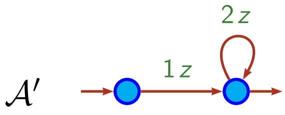


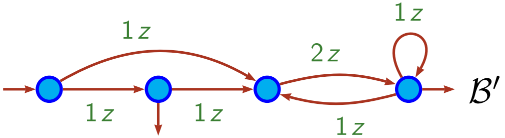
Step 3: The conjugacy theorem

Theorem (BLS)

Two $\mathbb{N}$-automata are equivalent if and only if they are conjugate to a same third $\mathbb{N}$-automaton.

- Conjugacy is a *preorder* (transitive and reflexive, but not symmetric).





Step 3: The conjugacy theorem

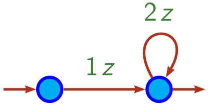
Theorem (BLS)

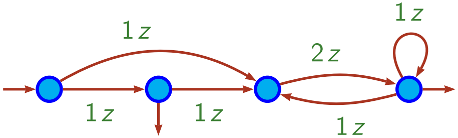
Two $\mathbb{N}$-automata are equivalent if and only if they are conjugate to a same third $\mathbb{N}$-automaton.

- Conjugacy is a *preorder* (transitive and reflexive, but not symmetric).
- $\mathcal{A} \stackrel{X}{\Longrightarrow} \mathcal{B}$ implies that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent*.

$$I\,E\,E\,T = I\,E\,E\,X\,U = I\,E\,X\,F\,U = I\,X\,F\,F\,U = J\,F\,F\,U$$

$\mathcal{A}'$





$\mathcal{B}'$

Step 3: The conjugacy theorem

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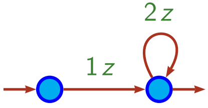
- Conjugacy is a *preorder* (transitive and reflexive, but not symmetric).

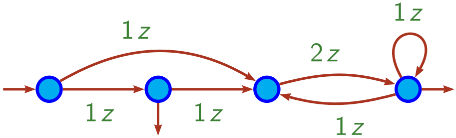
- $\mathcal{A} \overset{X}{\Longrightarrow} \mathcal{B}$ implies that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent*.

$$I\,E\,E\,T = I\,E\,E\,X\,U = I\,E\,X\,F\,U = I\,X\,F\,F\,U = J\,F\,F\,U$$

and then $\quad I\,E^*\,T = J\,F^*\,U$

$\mathcal{A}'$





$\mathcal{B}'$

Step 3: The conjugacy theorem

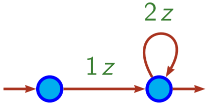
Theorem (BLS)

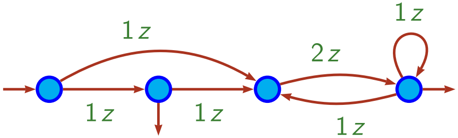
Two $\mathbb{N}$-automata $\mathcal{A}$ and $\mathcal{B}$ are equivalent if and only if there exists an $\mathbb{N}$-automaton $\mathcal{C}$ (and $\mathbb{N}$-matrices X and Y) such that

$$\mathcal{A} \overset{X}{\Longleftarrow} \mathcal{C} \overset{Y}{\Longrightarrow} \mathcal{B}$$

Moreover, $\mathcal{C}$ is effectively computable from $\mathcal{A}$ and $\mathcal{B}$.

$\mathcal{A}'$





$\mathcal{B}'$

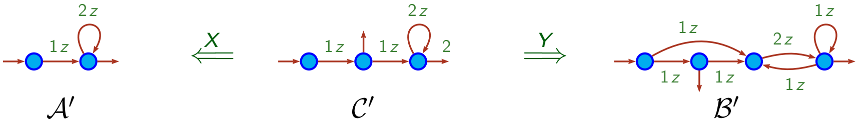
Step 3: The conjugacy theorem

Theorem (BLS)

Two $\mathbb{N}$-automata $\mathcal{A}$ and $\mathcal{B}$ are equivalent if and only if there exists an $\mathbb{N}$-automaton $\mathcal{C}$ (and $\mathbb{N}$-matrices X and Y) such that

$$\mathcal{A} \overset{X}{\Longleftarrow} \mathcal{C} \overset{Y}{\Longrightarrow} \mathcal{B}$$

Moreover, $\mathcal{C}$ is effectively computable from $\mathcal{A}$ and $\mathcal{B}$.



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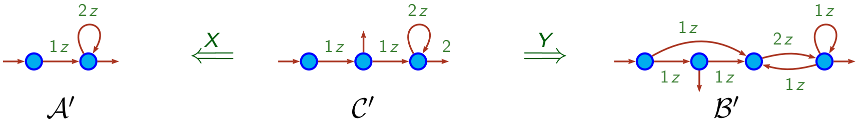
Moreover, $\mathcal{C}$ is effectively computable from $\mathcal{A}$ and $\mathcal{B}$.

$$\text{with} \qquad X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$



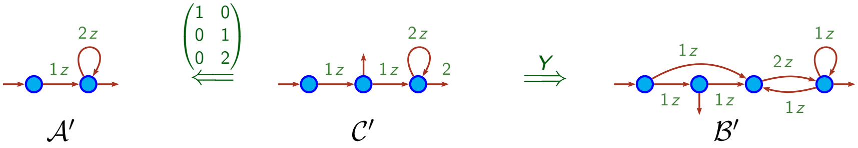
Step 3: The conjugacy theorem

$$\mathcal{C}' = \left\langle \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z & 0 \\ 0 & 0 & z \\ 0 & 0 & 2z \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle \qquad \mathcal{A}' = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & z \\ 0 & 2z \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

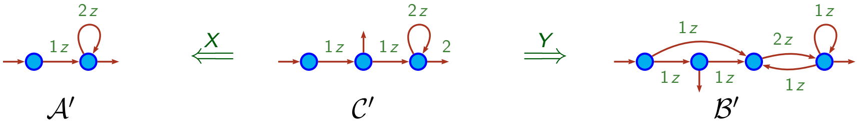
$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & z & 0 \\ 0 & 0 & z \\ 0 & 0 & 2z \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & z \\ 0 & 2z \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



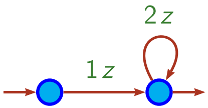
Step 4: Morphisms and the Decomposition theorem



Step 4:

1. Morphisms

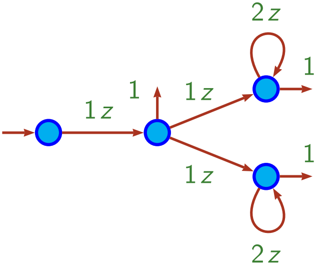
$\mathcal{A}'$



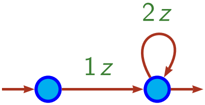
Step 4:

1. Morphisms

$\mathcal{D}'$



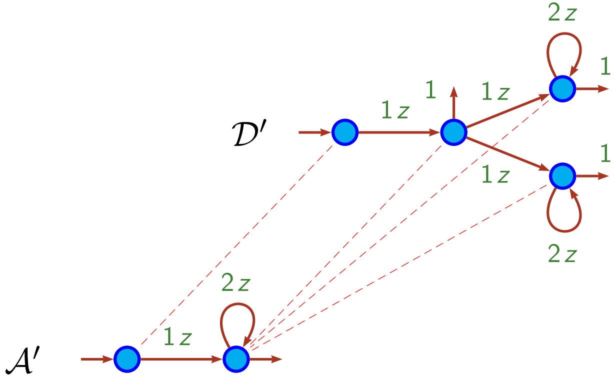
$\mathcal{A}'$



Step 4: 1. Morphisms

A map $\varphi\colon \mathcal{D}' \to \mathcal{A}'$ defines a matrix H_φ :

$$H_\varphi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$



Step 4: 1. Morphisms

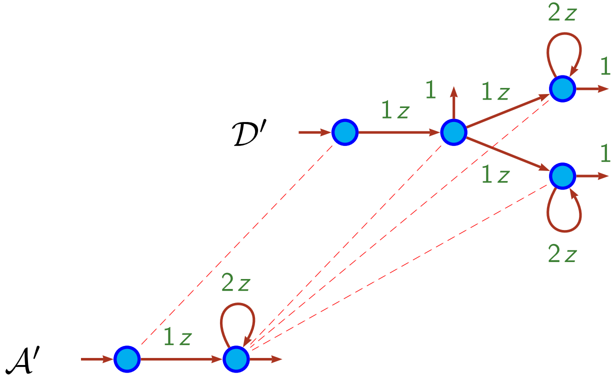
Definition

Let $\mathcal{D}' = \langle\, I, E, T \,\rangle$ and $\mathcal{A}' = \langle\, J, F, U \,\rangle$ be two $\mathbb{K}$-automata.

$\varphi\colon \mathcal{D}' \to \mathcal{A}'$ is an Out-morphism or $\mathcal{A}'$ is a quotient of $\mathcal{D}'$

if $\mathcal{D}'$ is conjugate to $\mathcal{A}'$ by H_φ : $\mathcal{D}' \overset{H_\varphi}{\Longrightarrow} \mathcal{A}'$

$$I\, H_\varphi = J, \qquad E\, H_\varphi = H_\varphi\, F, \quad \text{and} \quad T = H_\varphi\, U \ .$$



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Let $\mathcal{D}' = \langle\, I, E, T \,\rangle$ and $\mathcal{A}' = \langle\, J, F, U \,\rangle$ be two $\mathbb{K}$-automata.

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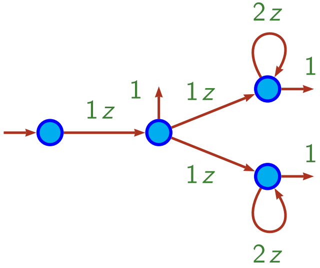
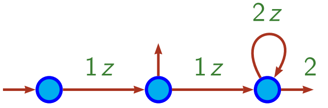
$\mathcal{D}'$





$\mathcal{C}'$

1. Morphisms

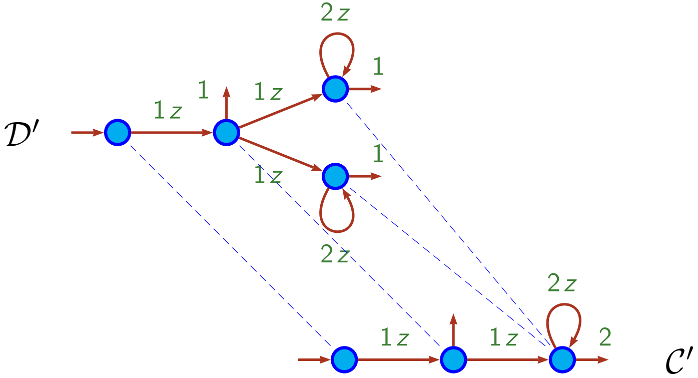
Definition

Let $\mathcal{D}' = \langle\, I, E, T\,\rangle$ and $\mathcal{C}' = \langle\, J, F, U\,\rangle$ be two $\mathbb{K}$-automata.

$\varphi\colon \mathcal{D}' \to \mathcal{C}'$ is an In-morphism or $\mathcal{C}'$ is a co-quotient of $\mathcal{D}'$

if $\mathcal{C}'$ is conjugate to $\mathcal{D}'$ by ${}^{\mathrm{t}}H_\varphi$: $\mathcal{C}' \xrightarrow{{}^{\mathrm{t}}H_\varphi} \mathcal{D}'$

$$I\,H_\varphi = J, \qquad E\,H_\varphi = H_\varphi\,F, \quad \text{and} \quad T = H_\varphi\,U \;.$$



$\mathcal{A}'$

$\mathcal{C}'$

Step 4:

2. The Decomposition theorem

Theorem (BLS)

Let $\mathcal{C}'$ and $\mathcal{A}'$ be two $\mathbb{N}$-automata, $\mathcal{C}'$ conjugate to $\mathcal{A}'$.
Then, there exists an $\mathbb{N}$-automaton $\mathcal{D}'$
such that $\mathcal{A}'$ is a quotient of $\mathcal{D}'$
and $\mathcal{C}'$ is an co-quotient of $\mathcal{D}'$.
Moreover, $\mathcal{D}'$ is effectively computable from $\mathcal{C}'$ and $\mathcal{A}'$.

Theorem (BLS)

Let $\mathcal{C}'$ *and* $\mathcal{A}'$ *be two* $\mathbb{N}$*-automata,* $\mathcal{C}'$ *conjugate to* $\mathcal{A}'$ *.*
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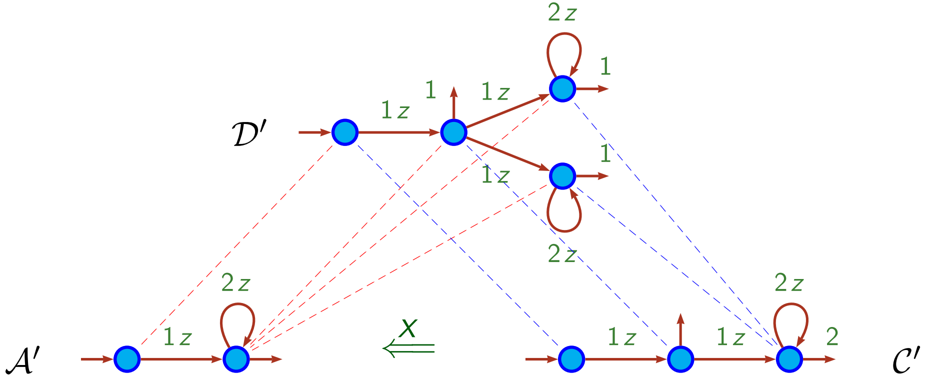
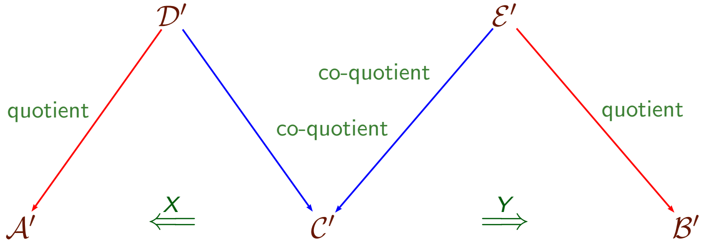
and $\mathcal{C}'$ *is an co-quotient of* $\mathcal{D}'$ *.*
Moreover, $\mathcal{D}'$ *is effectively computable from* $\mathcal{C}'$ *and* $\mathcal{A}'$ *.*

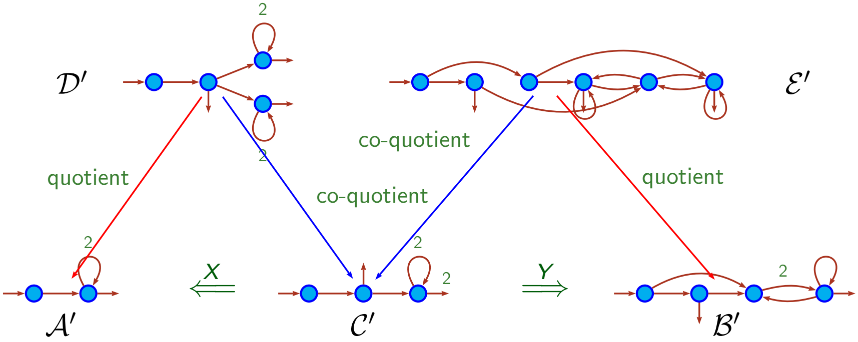


Step 4: 3. Conjugacy and Decomposition theorems together

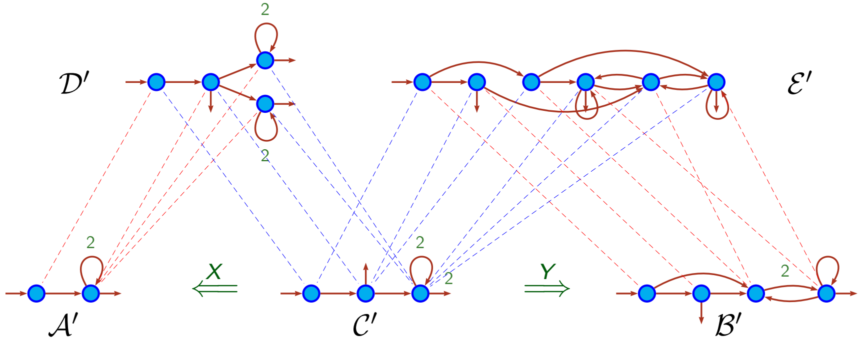
A structural interpretation of equivalence



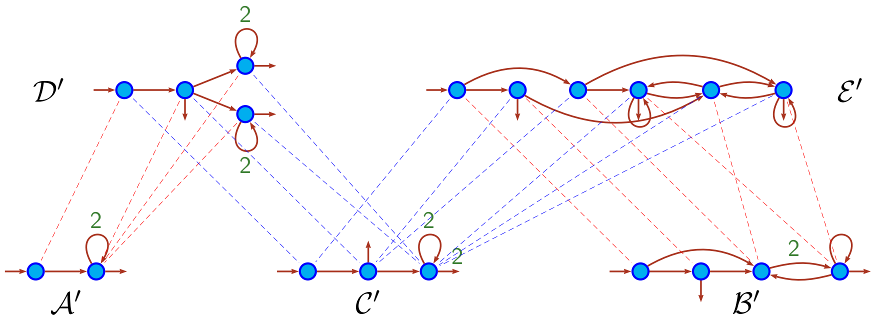
A structural interpretation of equivalence



A structural interpretation of equivalence



Step 5: 1. A technical proposition



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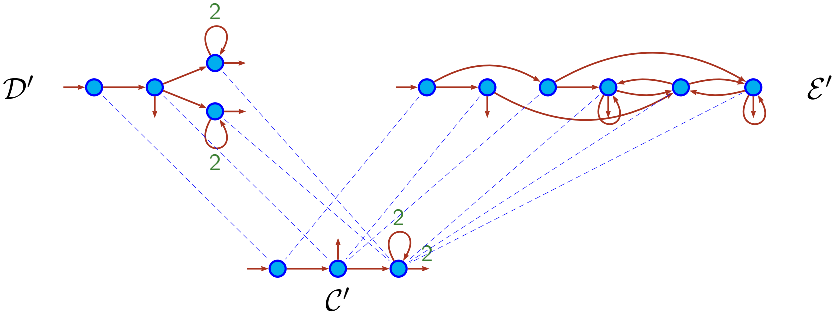

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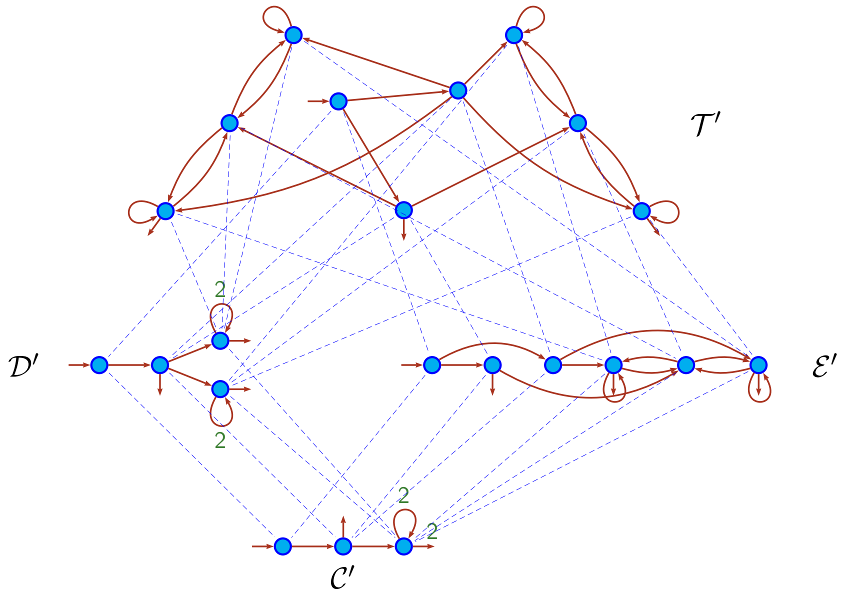
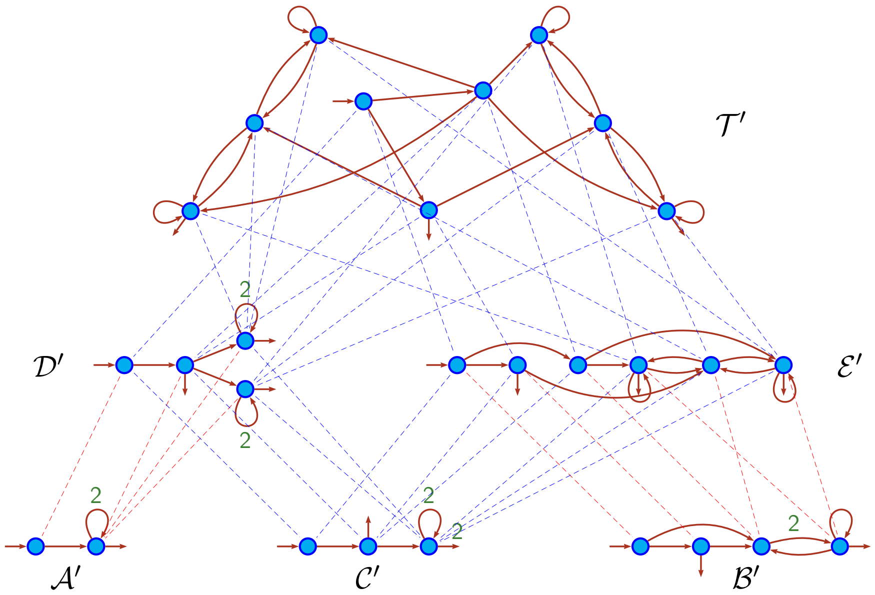


Step 5: 2. The harvest



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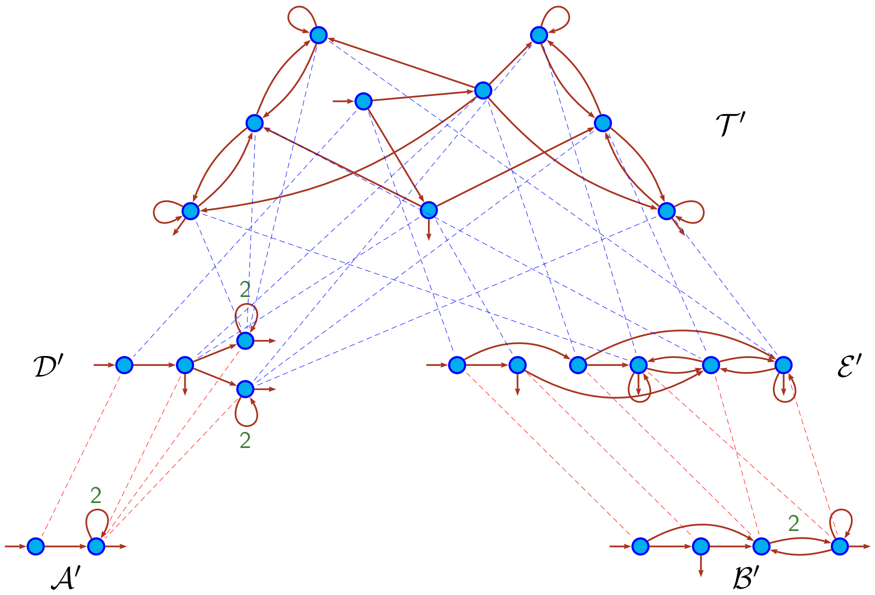

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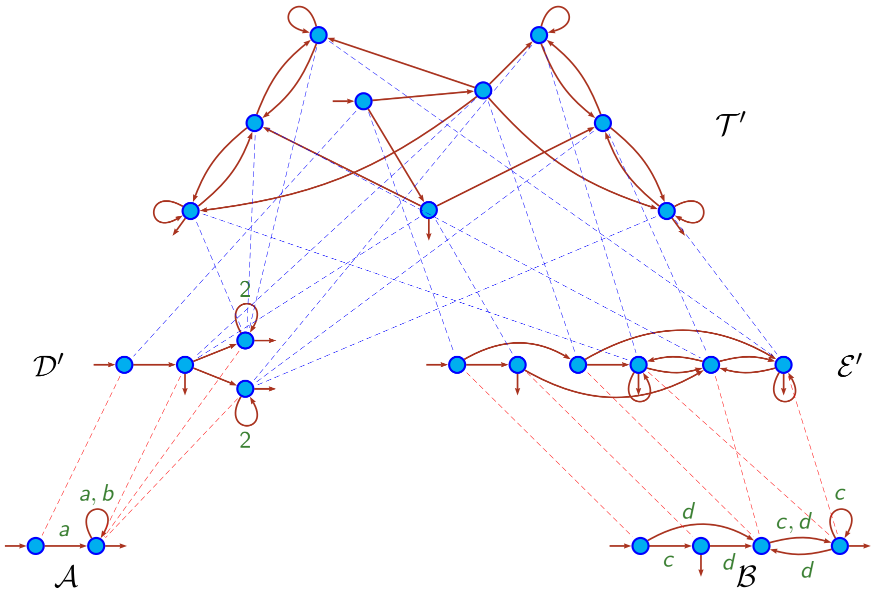

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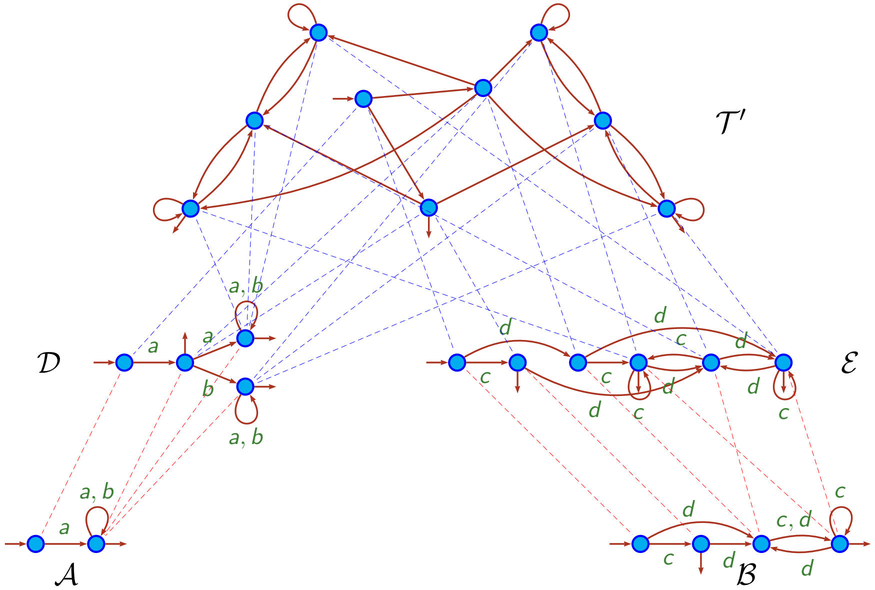

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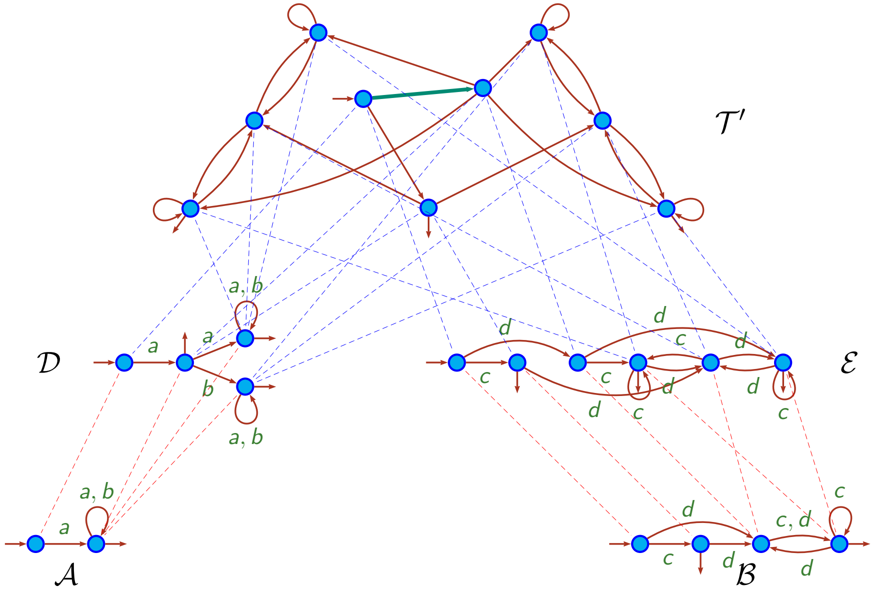

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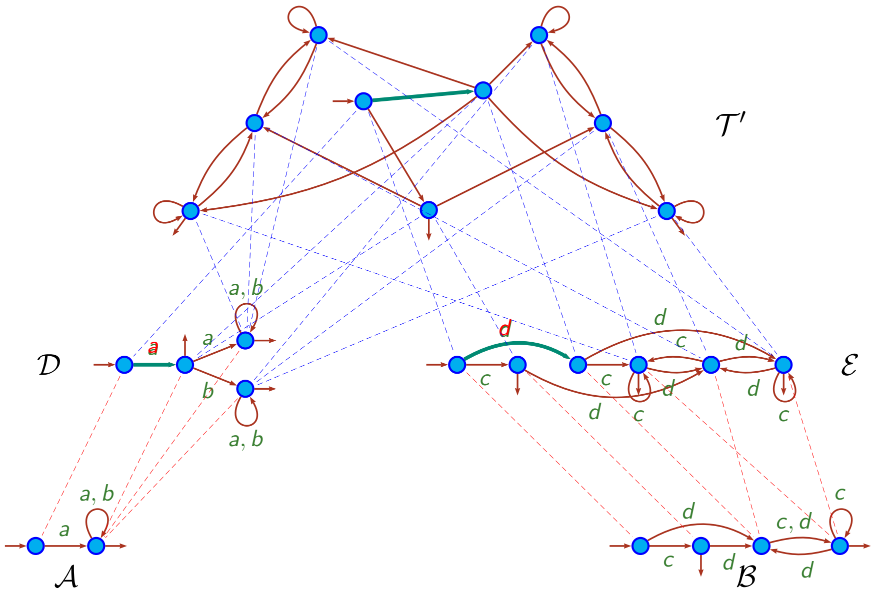

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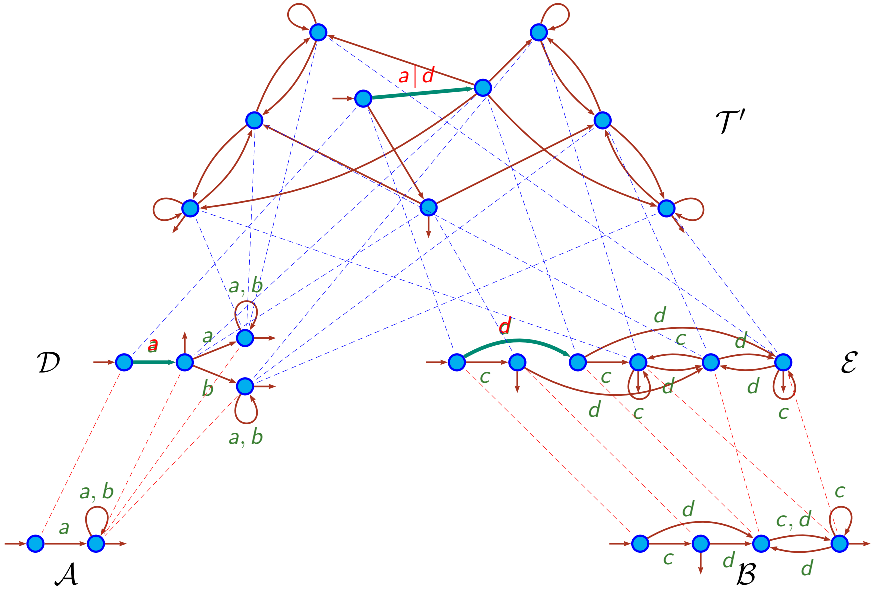

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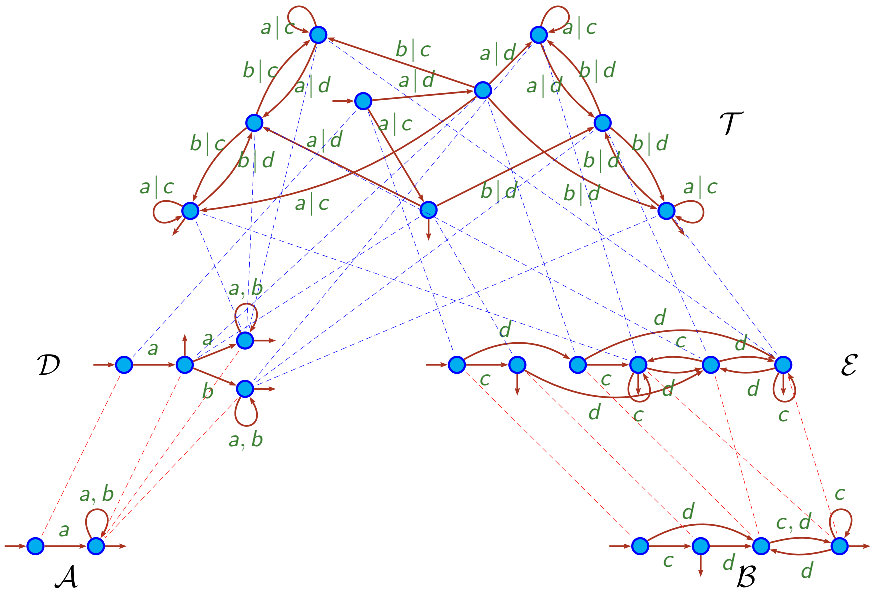


Part II

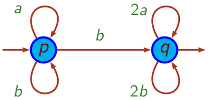
The foundations

1. Representation
   The representability theorem
2. Reduction
   Decidability of equivalence
3. Joint reduction
   The conjugacy theorem
4. Morphisms
   The decomposition theorem

Chapter I

Representation

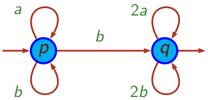
Automata are matrices



$$\mathcal{C}_1 = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} a+b & b \\ 0 & 2\,a+2\,b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \quad .$$

$$\mathcal{A} = \langle\, I, E, T \,\rangle \qquad\qquad |\mathcal{A}| = \sum_{n\in\mathbb{N}} I \cdot E^n \cdot T = I \cdot E^* \cdot T$$

Automata over free monoids are representations



$$\mu_1(a) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} , \quad \mu_1(b) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} , \quad I_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} , \quad T_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathcal{A} = \langle \, I, \mu, T \, \rangle \qquad w \mapsto I \cdot \mu(w) \cdot T \qquad |\mathcal{A}| = \sum_{w \in A^*} (I \cdot \mu(w) \cdot T) \, w$$

The control morphism

$$\mathcal{A} = \langle\, I, \mu, T \,\rangle$$

Reachability set

Reachability space

$$\mathbf{R}_{\mathcal{A}} = \{ I \cdot \mu(w) \,|\, w \in A^* \}$$

$$\mathbf{R}_{\mathcal{A}} \subseteq \mathbb{K}^Q \qquad \langle \mathbf{R}_{\mathcal{A}} \rangle$$

$$\Psi_{\mathcal{A}} \colon \mathbb{K}\langle A^* \rangle \longrightarrow \mathbb{K}^Q \qquad \forall w \in A^* \quad \Psi_{\mathcal{A}}(w) = I \cdot \mu(w)$$

$$\mathbf{R}_{\mathcal{A}} = \Psi_{\mathcal{A}}(A^*) \qquad \operatorname{Im} \Psi_{\mathcal{A}} = \Psi_{\mathcal{A}}(\mathbb{K}\langle A^* \rangle) = \langle \mathbf{R}_{\mathcal{A}} \rangle$$

$$\mathbb{K}\langle A^* \rangle \xrightarrow{\;\Psi_{\mathcal{A}}\;} \mathbb{K}^Q \qquad\qquad u \xrightarrow{\;\Psi_{\mathcal{A}}\;} x$$

The control morphism

The control morphism

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Reachability set

Reachability space

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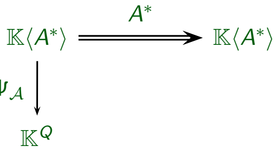

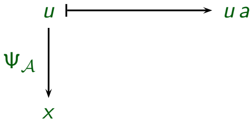

The control morphism

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Reachability space

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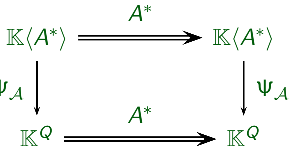

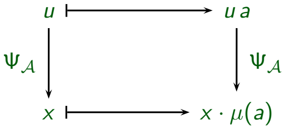

The control morphism is a morphism of actions

The observation morphism

Quotient of series

$$s \in \mathbb{K}\langle\langle A^* \rangle\rangle \qquad v \in A^* \quad v^{-1}s = \sum_{w \in A^*} \langle s, v\,w \rangle\, w$$

$v^{-1} \colon \mathbb{K}\langle\langle A^* \rangle\rangle \longrightarrow \mathbb{K}\langle\langle A^* \rangle\rangle$ endomorphism of $\mathbb{K}$-modules

$$\mathbb{K}\langle\langle A^* \rangle\rangle \overset{A^*}{\Longrightarrow} \mathbb{K}\langle\langle A^* \rangle\rangle \qquad s \longmapsto v^{-1}s$$

Quotient is a (right) action of A^* on $\mathbb{K}\langle\langle A^* \rangle\rangle$

The observation morphism

$$\mathcal{A} = \langle\, I, \mu, T \,\rangle$$

$$\Phi_{\mathcal{A}} \colon \mathbb{K}^{Q} \longrightarrow \mathbb{K}\langle\langle A^* \rangle\rangle \qquad \Phi_{\mathcal{A}}(x) = \left| \langle\, x, \mu, T \,\rangle \right| = \sum_{w \in A^*} (x \cdot \mu(w) \cdot T)\, w$$

$$s = \left| \langle\, I, \mu, T \,\rangle \right| = \Phi_{\mathcal{A}}(I)$$

$$\begin{array}{c} \mathbb{K}^{Q} \\ \Phi_{\mathcal{A}} \Big\downarrow \\ \mathbb{K}\langle\langle A^* \rangle\rangle \end{array} \qquad\qquad \begin{array}{c} x \\ \Phi_{\mathcal{A}} \Big\downarrow \\ t \end{array}$$

The observation morphism

$$\mathcal{A} = \langle\, I, \mu, T \,\rangle$$

$$\Phi_{\mathcal{A}} \colon \mathbb{K}^{Q} \longrightarrow \mathbb{K}\langle\!\langle A^* \rangle\!\rangle \qquad \Phi_{\mathcal{A}}(x) = \big|\langle\, x, \mu, T \,\rangle\big| = \sum_{w \in A^*} (x \cdot \mu(w) \cdot T)\, w$$

$$s = \big|\langle\, I, \mu, T \,\rangle\big| = \Phi_{\mathcal{A}}(I) \qquad w^{-1}s = \big|(\, I \cdot \mu(w), \mu, T \,)\big|$$

$$w^{-1}\Phi_{\mathcal{A}}(x) = \Phi_{\mathcal{A}}(x \cdot \mu(w))$$

$$\begin{array}{c} \mathbb{K}^{Q} \\ \Phi_{\mathcal{A}} \big\downarrow \\ \mathbb{K}\langle\!\langle A^* \rangle\!\rangle \end{array} \qquad\qquad \begin{array}{c} x \\ \Phi_{\mathcal{A}} \big\downarrow \\ t \end{array}$$

The observation morphism

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$$\Phi_{\mathcal{A}} \colon \mathbb{K}^{Q} \longrightarrow \mathbb{K}\langle\langle A^* \rangle\rangle \qquad \Phi_{\mathcal{A}}(x) = \big|\langle\, x, \mu, T \,\rangle\big| = \sum_{w \in A^*} (x \cdot \mu(w) \cdot T)\, w$$

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$$w^{-1}\Phi_{\mathcal{A}}(x) = \Phi_{\mathcal{A}}(x \cdot \mu(w))$$

$$\begin{array}{ccc} \mathbb{K}^{Q} & \overset{A^*}{\Longrightarrow} & \mathbb{K}^{Q} \\ \Phi_{\mathcal{A}} \downarrow & & \downarrow \Phi_{\mathcal{A}} \\ \mathbb{K}\langle\langle A^* \rangle\rangle & \overset{A^*}{\Longrightarrow} & \mathbb{K}\langle\langle A^* \rangle\rangle \end{array} \qquad \begin{array}{ccc} x & \longmapsto & x \cdot \mu(a) \\ \Phi_{\mathcal{A}} \downarrow & & \downarrow \Phi_{\mathcal{A}} \\ t & \longmapsto & a^{-1}t \end{array}$$

The observation morphism is a morphism of actions

The observation morphism

$$\mathcal{A} = \langle\, I, \mu, T \,\rangle$$

$$\Phi_{\mathcal{A}} \colon \mathbb{K}^{Q} \longrightarrow \mathbb{K}\langle\langle A^* \rangle\rangle \qquad\qquad \Phi_{\mathcal{A}}(x) = \big|\langle\, x, \mu, T \,\rangle\big| = \sum_{w \in A^*} (x \cdot \mu(w) \cdot T)\, w$$

$$s = \big|\langle\, I, \mu, T \,\rangle\big| = \Phi_{\mathcal{A}}(I) \qquad\qquad w^{-1}s = \big|(\, I \cdot \mu(w), \mu, T \,)\big|$$

$$w^{-1}\Phi_{\mathcal{A}}(x) = \Phi_{\mathcal{A}}(x \cdot \mu(w))$$

$$\begin{array}{ccc}
\mathbb{K}\langle A^* \rangle & \xrightarrow{\;A^*\;} & \mathbb{K}\langle A^* \rangle \\
\downarrow{\scriptstyle \Psi_{\mathcal{A}}} & & \downarrow{\scriptstyle \Psi_{\mathcal{A}}} \\
\mathbb{K}^{Q} & \xrightarrow{\;A^*\;} & \mathbb{K}^{Q} \\
\downarrow{\scriptstyle \Phi_{\mathcal{A}}} & & \downarrow{\scriptstyle \Phi_{\mathcal{A}}} \\
\mathbb{K}\langle\langle A^* \rangle\rangle & \xrightarrow{\;A^*\;} & \mathbb{K}\langle\langle A^* \rangle\rangle
\end{array}
\qquad\qquad
\begin{array}{ccc}
u & \longmapsto & w\,a \\
\downarrow{\scriptstyle \Psi_{\mathcal{A}}} & & \downarrow{\scriptstyle \Psi_{\mathcal{A}}} \\
x & \longmapsto & x \cdot \mu(a) \\
\downarrow{\scriptstyle \Phi_{\mathcal{A}}} & & \downarrow{\scriptstyle \Phi_{\mathcal{A}}} \\
t & \longmapsto & a^{-1}t
\end{array}$$

The observation morphism is a morphism of actions

The representability theorem

$U \subseteq \mathbb{K}\langle\langle A^* \rangle\rangle$ submodule $\qquad U$ stable (by quotient)

Theorem (Schützenberger 61, Fliess 71, Jacob 74)

$s \in \mathbb{K}\mathrm{Rec}\, A^* \iff \exists U$ *stable* *finitely generated* $s \in U$

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$$
\begin{array}{ccc}
\mathbb{K}\langle A^* \rangle & \overset{A^*}{\Longrightarrow} & \mathbb{K}\langle A^* \rangle \\
\Psi_{\mathcal{A}} \downarrow & & \downarrow \Psi_{\mathcal{A}} \\
\mathbb{K}^{Q} & \overset{A^*}{\Longrightarrow} & \mathbb{K}^{Q} \\
\Phi_{\mathcal{A}} \downarrow & & \downarrow \Phi_{\mathcal{A}} \\
\mathbb{K}\langle\langle A^* \rangle\rangle & \overset{A^*}{\Longrightarrow} & \mathbb{K}\langle\langle A^* \rangle\rangle
\end{array}
$$

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$s \in \mathbb{K}\mathrm{Rec}\, A^* \quad \Longrightarrow \quad \exists U$ *stable finitely generated* $s \in U$

$$\begin{array}{ccccc}
1_{A^*} \in & \mathbb{K}\langle A^* \rangle & \overset{A^*}{\Longrightarrow} & \mathbb{K}\langle A^* \rangle \\
& \downarrow \Psi_{\mathcal{A}} & & \downarrow \Psi_{\mathcal{A}} \\
I \in \mathrm{Im}\,\Psi_{\mathcal{A}} & \mathbb{K}^Q & \overset{A^*}{\Longrightarrow} & \mathbb{K}^Q \\
& \downarrow \Phi_{\mathcal{A}} & & \downarrow \Phi_{\mathcal{A}} \\
s \in \Phi_{\mathcal{A}}(\mathrm{Im}\,\Psi_{\mathcal{A}}) & \mathbb{K}\langle\langle A^* \rangle\rangle & \overset{A^*}{\Longrightarrow} & \mathbb{K}\langle\langle A^* \rangle\rangle
\end{array}$$

The representability theorem

$U \subseteq \mathbb{K}\langle\langle A^* \rangle\rangle$ submodule $\qquad$ U stable (by quotient)

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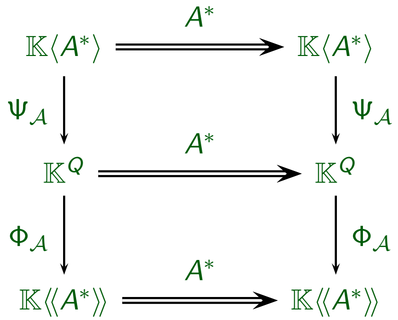

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$$\begin{array}{ccc}
\mathbb{K}\langle A^* \rangle & \overset{A^*}{\Longrightarrow} & \mathbb{K}\langle A^* \rangle \\
\downarrow \Psi_{\mathcal{A}} & & \downarrow \Psi_{\mathcal{A}} \\
\mathbb{K}^Q & \overset{A^*}{\Longrightarrow} & \mathbb{K}^Q \\
\downarrow \Phi_{\mathcal{A}} & & \downarrow \Phi_{\mathcal{A}} \\
\mathbb{K}\langle\langle A^* \rangle\rangle & \overset{A^*}{\Longrightarrow} & \mathbb{K}\langle\langle A^* \rangle\rangle
\end{array}$$

Chapter II

Reduction

The representability theorem for recognisable series

Proposition

$\mathcal{A} = \langle\, I, \mu, T \,\rangle$ *dimension* Q $\qquad$ $s = |\mathcal{A}|$

$\langle \mathbf{R}_{\mathcal{A}} \rangle$ *generated by* $G \subset \mathbb{K}^Q$

$\exists\ \mathcal{A}_G$ *of dimension* G $\qquad$ $s = |\mathcal{A}_G|$ $\qquad$ $\mathcal{A} \overset{M_G}{\Longleftarrow} \mathcal{A}_G$

The exploration procedure

$\mathbb{K}$-automaton $\mathcal{A} = \langle I, \mu, T \rangle$ Search for $P \subseteq A^*$

$$\begin{array}{ccc} & \mathbb{K}\langle A^* \rangle & P \\ & \big\downarrow \Psi_{\mathcal{A}} & \big\downarrow \Psi_{\mathcal{A}} \\ \operatorname{Im}\Psi_{\mathcal{A}} & \mathbb{K}^Q & \Psi_{\mathcal{A}}(P) \end{array}$$

$\Psi_{\mathcal{A}}(P)$ generating set of $\operatorname{Im}\Psi_{\mathcal{A}}$

The exploration procedure

$\mathbb{K}$-automaton $\mathcal{A} = \langle\, I, \mu, T \,\rangle$ Search for $P \subseteq A^*$

$$
\begin{array}{ccc}
\mathbb{K}\langle A^* \rangle & & P \\
\Psi_{\mathcal{A}} \downarrow & & \Psi_{\mathcal{A}} \downarrow \\
\mathbb{K}^Q & & \Psi_{\mathcal{A}}(P)
\end{array}
$$

$\operatorname{Im} \Psi_{\mathcal{A}}$

$\Psi_{\mathcal{A}}(P)$ generating set of $\operatorname{Im} \Psi_{\mathcal{A}}$

Halting criterium

The exploration procedure

$\mathbb{K}$-automaton $\mathcal{A} = \langle I, \mu, T \rangle$ Search for $P \subseteq A^*$

$$\begin{array}{ccc} & \mathbb{K}\langle A^* \rangle & P \\ & \Psi_{\mathcal{A}} \downarrow & \Psi_{\mathcal{A}} \downarrow \\ \operatorname{Im}\Psi_{\mathcal{A}} & \mathbb{K}^Q & \Psi_{\mathcal{A}}(P) \end{array}$$

generating set of $\operatorname{Im}\Psi_{\mathcal{A}}$

Halting criterium

- $\mathbb{B}$ finite finite $\operatorname{Im}\Psi_{\mathcal{A}}$

The exploration procedure

$\mathbb{K}$-automaton $\mathcal{A} = \langle\, I, \mu, T \,\rangle$ Search for $P \subseteq A^*$

$$\mathbb{K}\langle A^* \rangle \xrightarrow{\Psi_{\mathcal{A}}} \mathbb{K}^Q \qquad P \xrightarrow{\Psi_{\mathcal{A}}} \Psi_{\mathcal{A}}(P)$$

$\operatorname{Im}\Psi_{\mathcal{A}}$ $\mathbb{K}^Q$ $\Psi_{\mathcal{A}}(P)$ generating set of $\operatorname{Im}\Psi_{\mathcal{A}}$

Halting criterium

- $\mathbb{B}$ finite finite $\operatorname{Im}\Psi_{\mathcal{A}}$
- $\mathbb{F}$ field finite dimension

The exploration procedure

$\mathbb{K}$-automaton $\mathcal{A} = \langle\, I, \mu, T \,\rangle$ Search for $P \subseteq A^*$

$$\begin{array}{cccc} & \mathbb{K}\langle A^* \rangle & & P \\ & \Big\downarrow {\scriptstyle \Psi_{\mathcal{A}}} & & \Big\downarrow {\scriptstyle \Psi_{\mathcal{A}}} \\ \operatorname{Im} \Psi_{\mathcal{A}} & \mathbb{K}^Q & & \langle \Psi_{\mathcal{A}}(P) \rangle \end{array}$$

generating set of $\operatorname{Im} \Psi_{\mathcal{A}}$

Halting criterium

- $\mathbb{B}$ finite — finite $\operatorname{Im} \Psi_{\mathcal{A}}$
- $\mathbb{F}$ field — finite dimension
- $\mathbb{Z}$ ED — Noetherian

The exploration procedure

$\mathbb{K}$-automaton $\mathcal{A} = \langle\, I, \mu, T \,\rangle$ Search for $P \subseteq A^*$

$$\begin{array}{cccc} & \mathbb{K}\langle A^* \rangle & P & \\ & \Big\downarrow \Psi_{\mathcal{A}} & \Big\downarrow \Psi_{\mathcal{A}} & \\ \operatorname{Im}\Psi_{\mathcal{A}} & \mathbb{K}^Q & \langle \Psi_{\mathcal{A}}(P) \rangle & \text{generating set of } \operatorname{Im}\Psi_{\mathcal{A}} \end{array}$$

Halting criterium

- $\mathbb{B}$ finite — finite $\operatorname{Im}\Psi_{\mathcal{A}}$
- $\mathbb{F}$ field — finite dimension
- $\mathbb{Z}$ ED — Noetherian
- $\mathbb{N}$ — well partial ordered set

The exploration procedure

$\mathbb{K}$-automaton $\mathcal{A} = \langle\, I, \mu, T\,\rangle$ Search for $P \subseteq A^*$

$$\begin{array}{ccc} \mathbb{K}\langle A^*\rangle & & P \\ \Psi_{\mathcal{A}} \big\downarrow & & \Psi_{\mathcal{A}} \big\downarrow \\ \mathbb{K}^Q & & \langle \Psi_{\mathcal{A}}(P) \rangle \end{array}$$

$\operatorname{Im}\Psi_{\mathcal{A}}$ $\quad\langle \Psi_{\mathcal{A}}(P) \rangle$ generating set of $\operatorname{Im}\Psi_{\mathcal{A}}$

Result

$$\mathcal{A} \overset{M_P}{\Longleftarrow} \mathcal{C}$$

Chapter III

Joint reduction

The joint exploration

$\mathbb{K}$-automata $\mathcal{A} = \langle\, I, \mu, T \,\rangle$ and $\mathcal{B} = \langle\, J, \pi, U \,\rangle$ Search for

$P \subseteq A^*$

$$\begin{array}{ccc}
\mathbb{K}\langle A^*\rangle & P & \mathbb{K}\langle A^*\rangle \\
\Psi_{\mathcal{A}} \big\downarrow & \big\downarrow \Psi_{\mathcal{A}} \mid \Psi_{\mathcal{B}} & \big\downarrow \Psi_{\mathcal{B}} \\
\mathbb{K}^Q & \langle \Psi_{\mathcal{A}}(P) \rangle \mid \langle \Psi_{\mathcal{B}}(P) \rangle & \mathbb{K}\langle A^*\rangle
\end{array}$$

Result

$$\mathcal{A} \overset{M_P}{\Longleftarrow} \mathcal{C} \overset{N_P}{\Longrightarrow} \mathcal{B}$$

The conjugacy theorem

Theorem

Let $\mathbb{K}$ *be* $\mathbb{B}$, $\mathbb{N}$, $\mathbb{Z}$ *, or any (skew) fields.*

Two $\mathbb{K}$*-automata* $\mathcal{A}$ *and* $\mathcal{B}$ *are equivalent if, and only if, there exist a* $\mathbb{K}$*-automaton* $\mathcal{C}$ *(and* $\mathbb{K}$*-matrices* X *and* Y*) such that*

$$\mathcal{A} \overset{X}{\Longleftarrow} \mathcal{C} \overset{Y}{\Longrightarrow} \mathcal{B}$$

Moreover, $\mathcal{C}$ *is effectively computable from* $\mathcal{A}$ *and* $\mathcal{B}$.

Chapter IV

Morphisms

Morphisms of weighted automata

Definition

$\mathcal{A} = \langle\, I, E, T \,\rangle$ and $\mathcal{B} = \langle\, J, F, U \,\rangle$ $\mathbb{K}$-automata of dimension Q and R.

A map $\varphi\colon Q \to R$ defines an Out-morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$

if $\mathcal{A}$ is conjugate to $\mathcal{B}$ by the matrix H_φ : $\mathcal{A} \overset{H_\varphi}{\Longrightarrow} \mathcal{B}$

$$I\, H_\varphi = J, \qquad E\, H_\varphi = H_\varphi\, F, \qquad T = H_\varphi\, U$$

$\mathcal{B}$ is a quotient of $\mathcal{A}$

Morphisms of weighted automata

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Directed notion

Morphisms of weighted automata

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Directed notion Price to pay for the weight

Morphisms of weighted automata

Definition

$\mathcal{A} = \langle\, I, E, T \,\rangle$ and $\mathcal{B} = \langle\, J, F, U \,\rangle$ $\mathbb{K}$-automata of dimension Q and R.

A map $\varphi\colon Q \to R$ defines an In-morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$

if $\mathcal{A}$ is conjugate to $\mathcal{B}$ by the matrix H_φ : $\mathcal{A} \overset{H_\varphi}{\Longrightarrow} \mathcal{B}$

$$I\, H_\varphi = J, \qquad E\, H_\varphi = H_\varphi\, F, \qquad T = H_\varphi\, U$$

$\mathcal{B}$ is a quotient of $\mathcal{A}$

Directed notion

Price to pay for the weight

Morphisms of weighted automata

Definition

$\mathcal{A} = \langle\, I, E, T \,\rangle$ and $\mathcal{B} = \langle\, J, F, U \,\rangle$ $\mathbb{K}$-automata of dimension Q and R.

A map $\varphi \colon Q \to R$ defines an In-morphism $\varphi \colon \mathcal{A} \to \mathcal{B}$

if $\mathcal{B}$ is conjugate to $\mathcal{A}$ by the matrix ${}^{\mathsf{t}}H_\varphi$: $\mathcal{B} \xRightarrow{{}^{\mathsf{t}}H_\varphi} \mathcal{A}$

$$J \; {}^{\mathsf{t}}H_\varphi = I, \qquad F \; {}^{\mathsf{t}}H_\varphi = {}^{\mathsf{t}}H_\varphi \; E, \qquad U = {}^{\mathsf{t}}H_\varphi \; T$$

$\mathcal{B}$ is a co-quotient of $\mathcal{A}$

Directed notion

Price to pay for the weight

Morphisms of weighted automata

Definition

$\mathcal{A} = \langle\, I, E, T \,\rangle$ and $\mathcal{B} = \langle\, J, F, U \,\rangle$ $\mathbb{K}$-automata of dimension Q and R.

A map $\varphi \colon Q \to R$ defines an In-morphism $\varphi \colon \mathcal{A} \to \mathcal{B}$

if $\mathcal{B}$ is conjugate to $\mathcal{A}$ by the matrix ${}^{\mathrm{t}}H_{\varphi}$: $\mathcal{B} \overset{{}^{\mathrm{t}}H_{\varphi}}{\Longrightarrow} \mathcal{A}$

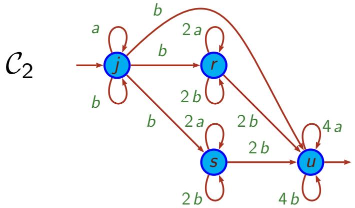
$$J\ {}^{\mathrm{t}}H_{\varphi} = I, \qquad F\ {}^{\mathrm{t}}H_{\varphi} = {}^{\mathrm{t}}H_{\varphi}\ E, \qquad U = {}^{\mathrm{t}}H_{\varphi}\ T$$

$\mathcal{B}$ is a co-quotient of $\mathcal{A}$

Proposition

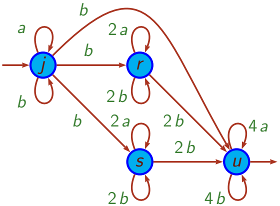
Every $\mathbb{K}$-automaton has a minimal (co-)quotient that is effectively computable (by the Moore algorithm).

Morphisms of weighted automata



Morphisms of weighted automata

$\mathcal{C}_2$

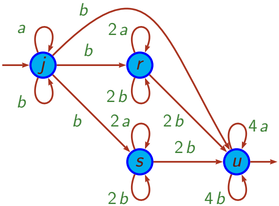


$\varphi_2 \colon \{j, r, s, u\} \to \{i, q, t\}$

$$H_{\varphi_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Morphisms of weighted automata

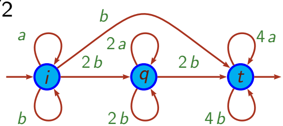
$\mathcal{C}_2$



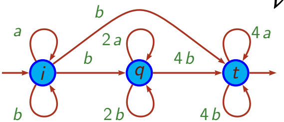
$\varphi_2 \colon \{j, r, s, u\} \to \{i, q, t\}$

$$H_{\varphi_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathcal{V}_2$



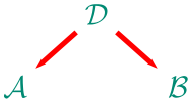
$\mathcal{V}_2'$

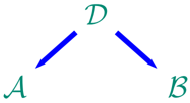


$\mathcal{C}_2 \xrightarrow{H_{\varphi_2}} \mathcal{V}_2$ $\qquad$ $\mathcal{V}_2' \xrightarrow{{}^{\mathrm{t}}H_{\varphi_2}} \mathcal{C}_2$

Morphisms of weighted automata

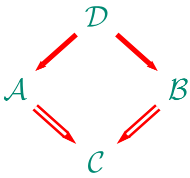
Minimal quotients and co-quotients

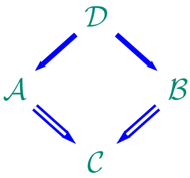




Morphisms of weighted automata

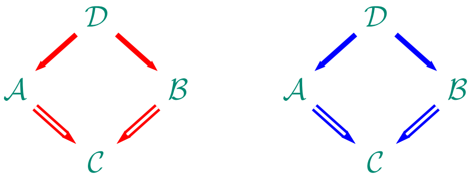
Minimal quotients and co-quotients





Morphisms of weighted automata

Minimal quotients and co-quotients

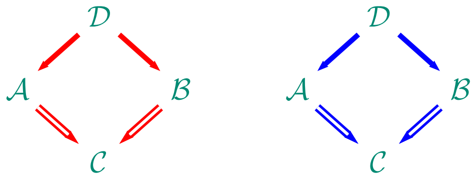


Equisubtractive commutative monoid, semiring

$$p + q = r + s \quad \Longrightarrow$$
$$\exists x, y, z, t \qquad p = x + y, q = z + t, r = x + z, s = y + t$$

Morphisms of weighted automata

Minimal quotients and co-quotients



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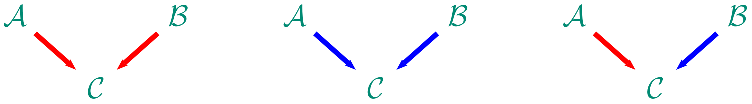
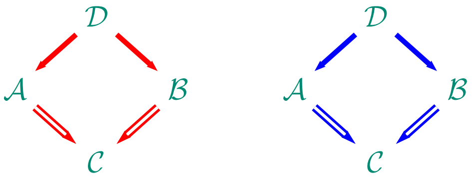
Filling diagrams backwards



Morphisms of weighted automata

Minimal quotients and co-quotients



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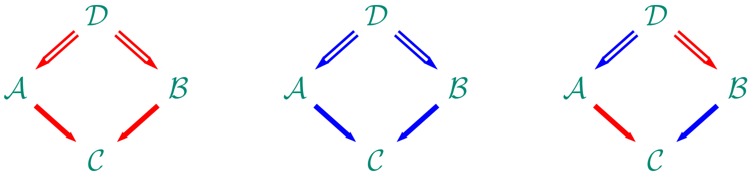
Filling diagrams backwards



The Decomposition theorem

Theorem

$\mathbb{K} = \mathbb{B}$ *or* $\mathbb{N}$, $\mathcal{A}$ *and* $\mathcal{B}$ *two trim* $\mathbb{K}$*-automata.*

$\mathcal{A} \overset{X}{\Longrightarrow} \mathcal{B} \iff$

$\exists\, \mathcal{C}$ $\mathcal{A}$ *co-quotient of* $\mathcal{C}$ *and* $\mathcal{B}$ *quotient of* $\mathcal{C}$.

The Decomposition theorem

Theorem

$\mathbb{K} = \mathbb{B}$ *or* $\mathbb{N}$, $\mathcal{A}$ *and* $\mathcal{B}$ *two trim* $\mathbb{K}$*-automata.*

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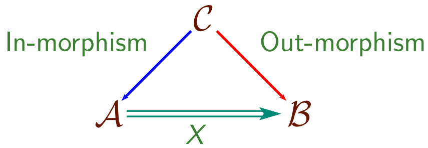


The Decomposition theorem

Theorem
$\mathbb{K} = \mathbb{Z}$ *or field* $\mathbb{F}$, $\mathcal{A}$ *and* $\mathcal{B}$ *two* $\mathbb{K}$*-automata.*
$\mathcal{A} \overset{X}{\Longrightarrow} \mathcal{B} \iff \exists\, \mathcal{C}$, $\mathcal{D}$ *, and a circulation matrix* D
$\mathcal{A}$ *co-quotient of* $\mathcal{C}$, $\mathcal{B}$ *quotient of* $\mathcal{D}$ *, and* $\mathcal{C} \overset{D}{\Longrightarrow} \mathcal{D}$

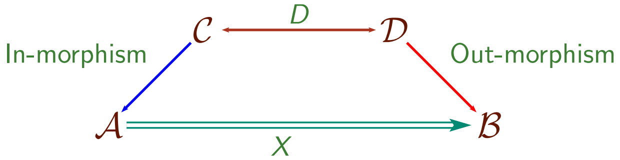
The Decomposition theorem

Theorem

$\mathbb{K} = \mathbb{Z}$ or field $\mathbb{F}$, $\quad \mathcal{A}$ and $\mathcal{B}$ two $\mathbb{K}$-automata.

$\mathcal{A} \stackrel{X}{\Longrightarrow} \mathcal{B} \iff \exists\, \mathcal{C}$, $\mathcal{D}$, and a circulation matrix D

$\mathcal{A}$ co-quotient of $\mathcal{C}$, $\mathcal{B}$ quotient of $\mathcal{D}$, and $\mathcal{C} \stackrel{D}{\Longrightarrow} \mathcal{D}$



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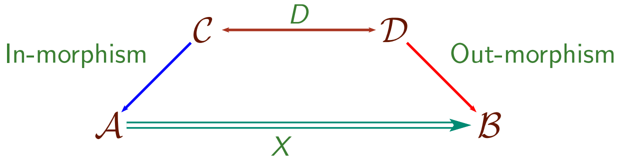
$\mathbb{K} = \mathbb{Z}$ *or field* $\mathbb{F}$, $\mathcal{A}$ *and* $\mathcal{B}$ *two* $\mathbb{K}$*-automata.*

$\mathcal{A} \stackrel{X}{\Longrightarrow} \mathcal{B} \iff \exists\, \mathcal{C}$, $\mathcal{D}$ *, and a circulation matrix* D

$\mathcal{A}$ *co-quotient of* $\mathcal{C}$, $\mathcal{B}$ *quotient of* $\mathcal{D}$ *, and* $\mathcal{C} \stackrel{D}{\Longrightarrow} \mathcal{D}$



circulation matrix = diagonal matrix of units

The Decomposition theorem

Theorem

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$\mathcal{A} \overset{X}{\Longrightarrow} \mathcal{B} \iff \exists\, \mathcal{C}$, $\mathcal{D}$, *and a circulation matrix* D

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The Decomposition theorem

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$\mathcal{A}$ *co-quotient of* $\mathcal{C}$, $\mathcal{B}$ *quotient of* $\mathcal{D}$, *and* $\mathcal{C} \overset{D}{\Longrightarrow} \mathcal{D}$

$\mathbb{K}$ has property (SU) = every element of $\mathbb{K}$ is a sum of units

The Decomposition theorem

Theorem
$\mathbb{K} = \mathbb{Z}$ or field $\mathbb{F}$, $\mathcal{A}$ and $\mathcal{B}$ two $\mathbb{K}$-automata.
$\mathcal{A} \stackrel{X}{\Longrightarrow} \mathcal{B} \iff \exists\, \mathcal{C}$, $\mathcal{D}$, and a circulation matrix D
$\mathcal{A}$ co-quotient of $\mathcal{C}$, $\mathcal{B}$ quotient of $\mathcal{D}$, and $\mathcal{C} \stackrel{D}{\Longrightarrow} \mathcal{D}$

$\mathbb{K}$ has property (SU) $=$ every element of $\mathbb{K}$ is a sum of units

$\mathbb{K}$ (SU) $\Longrightarrow$ $\forall\, X$ matrix $X = C\,D\,R$
C co-amalgamation D circulation R amalgamation

The Decomposition theorem

Theorem

$\mathbb{K} = \mathbb{Z}$ *or field* $\mathbb{F}$ *,* $\mathcal{A}$ *and* $\mathcal{B}$ *two* $\mathbb{K}$*-automata.*

$\mathcal{A} \stackrel{X}{\Longrightarrow} \mathcal{B} \iff \exists\, \mathcal{C}$ *,* $\mathcal{D}$ *, and a circulation matrix* D

$\mathcal{A}$ *co-quotient of* $\mathcal{C}$ *,* $\mathcal{B}$ *quotient of* $\mathcal{D}$ *, and* $\mathcal{C} \stackrel{D}{\Longrightarrow} \mathcal{D}$

$\mathbb{K}$ has property (SU) $=$ every element of $\mathbb{K}$ is a sum of units

$\mathbb{K}$ (SU) $\Longrightarrow \forall\, X$ matrix $X = C\,D\,R$

C co-amalgamation D circulation R amalgamation

$\mathbb{K}$ equisubtractive $\Longrightarrow$

given C co-amalgamation and R amalgamation matrices,

one can construct $\mathcal{C}$ and $\mathcal{D}$

The Decomposition theorem

Theorem

$\mathbb{K} = \mathbb{B}$ *or* $\mathbb{N}$, $\mathcal{A}$ *and* $\mathcal{B}$ *two trim* $\mathbb{K}$*-automata.*

$\mathcal{A} \overset{X}{\Longrightarrow} \mathcal{B} \iff$

$\exists\, \mathcal{C}$ $\mathcal{A}$ *co-quotient of* $\mathcal{C}$ *and* $\mathcal{B}$ *quotient of* $\mathcal{C}$.

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The Finite Equivalence Theorem

Theorem

Two irreducible sofic shifts are finitely equivalent if, and only if, they have the same entropy.

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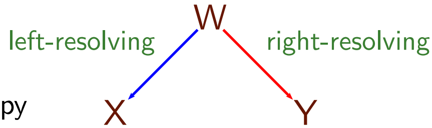
same entropy



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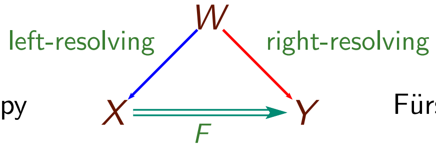
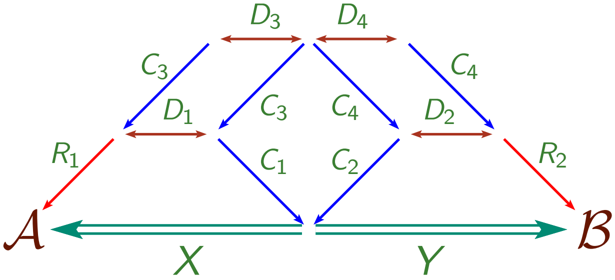


Fürstenberg lemma

The Conjugacy and Decomposition theorems together

A structural interpretation of equivalence



Part III

Questions

Richness of the model of weighted automata

- $\mathbb{B}$ 'classic' automata
- $\mathbb{N}$ 'usual' counting
- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ numerical multiplicity
- $\langle \mathbb{Z} \cup +\infty, \min, + \rangle$ Min-plus automata
- $\mathfrak{P}(B^*) = \mathbb{B}\langle\langle B^* \rangle\rangle$ transducers
- $\mathbb{N}\langle\langle B^* \rangle\rangle$ weighted transducers
- $\mathfrak{P}(F(B))$ pushdown automata

Equivalence of weighted automata

Equivalence of weighted automata with weights in

the Boolean semiring $\mathbb{B}$	decidable
a subsemiring of a field	decidable
$(\mathbb{Z}, \min, +)$	undecidable
$\mathrm{Rat}\, B^*$	undecidable
$\mathbb{N}\mathrm{Rat}\, B^*$	decidable

Equivalence of

transducers	undecidable
transducers with multiplicity in $\mathbb{N}$	decidable
functional transducers	decidable
polynomially ambiguous $(\mathbb{Z}, \min, +)$	decidable