Local Rules for Computable Planar Tilings

(Automata Theory and Symbolic Dynamics Workshop)

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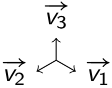
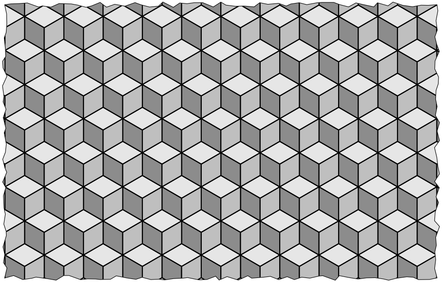
3 Juin 2013

Problematic

$n \to d$ tilings

Let $\vec{v_1}, \dots, \vec{v_n}$ be pairwise non-colinear vectors of $\mathbb{R}^d$ with $n > d > 0$.





- A *$n \to d$ tile* is a parallelotope generated by d of the $\vec{v_i}$'s, there are $\binom{n}{d}$ tiles.
- A *$n \to d$ tiling* is a face-to-face tiling of $\mathbb{R}^d$ by $n \to d$ tiles.
- The set $\mathcal{X}_{n \to d}$ of all tilings of $\mathbb{R}^d$ by $n \to d$-tiles is the *full $n \to d$ tiling space*.

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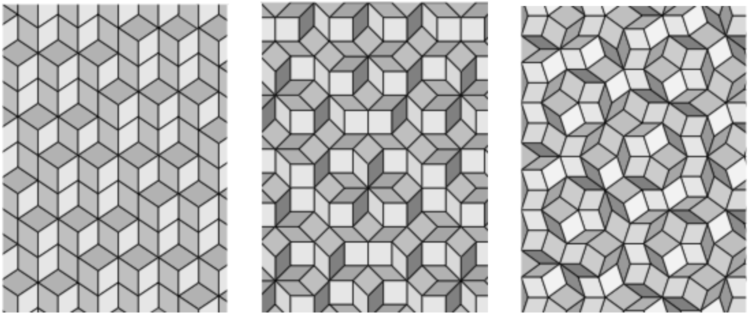

$3 \to 2$ $4 \to 2$ $5 \to 2$

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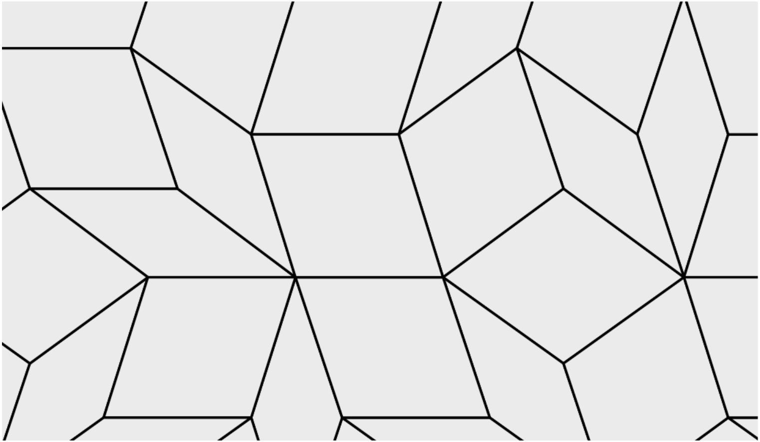
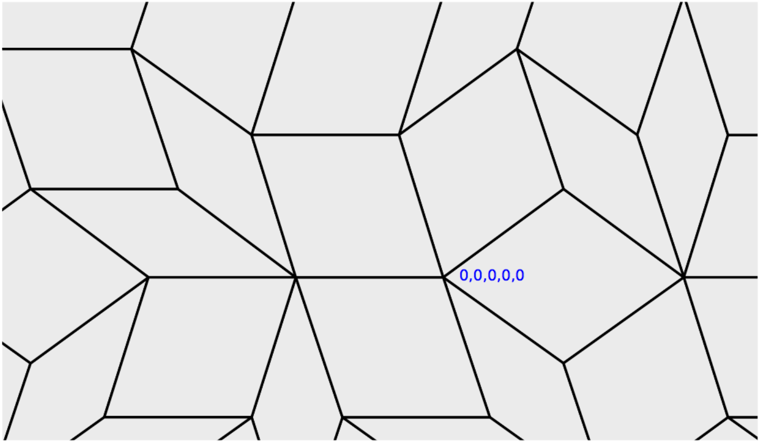


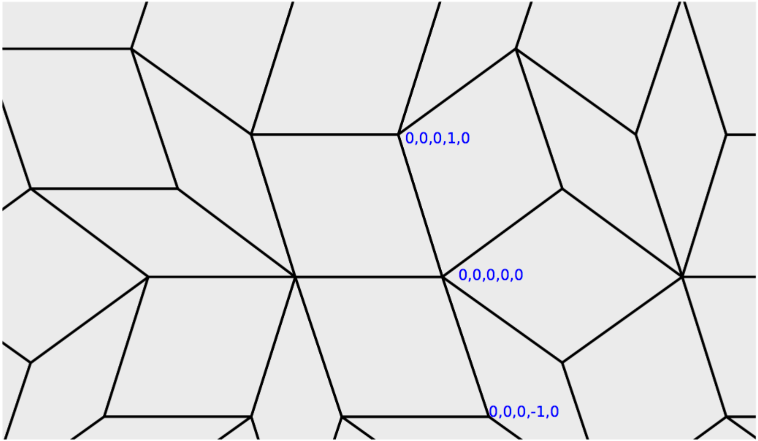
Consider a $n \to d$ tiling.

Lift



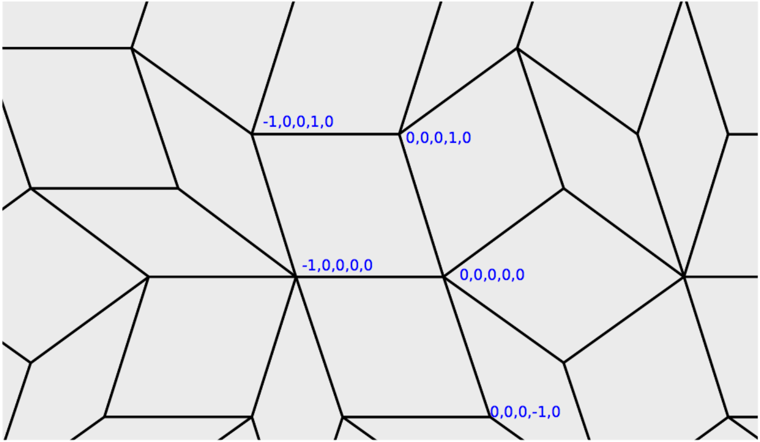
Map an arbitrary vertex onto an arbitrary vector of $\mathbb{Z}^n$.

Lift



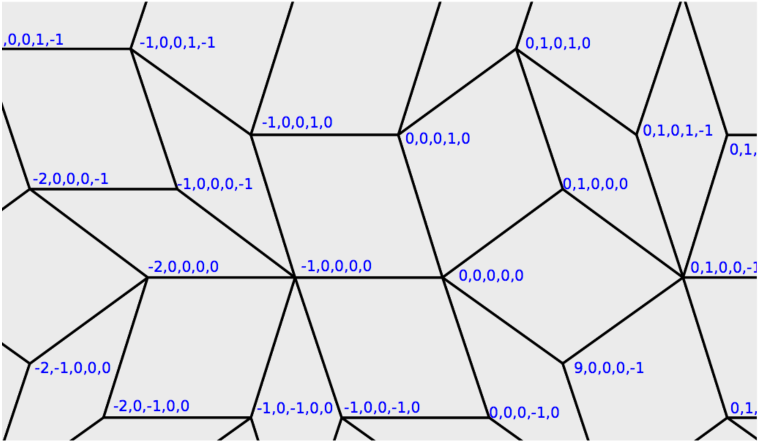
Modify the k^{th} entry when moving along the k^{th} direction.

Lift



$n \to d$ vertices are mapped onto vertices of $[0, 1]^n$.

Lift

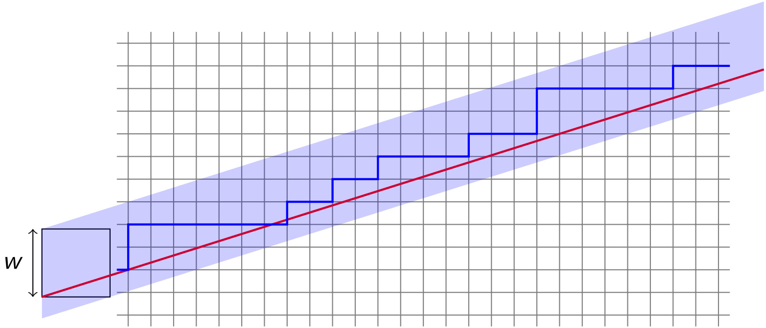


The whole tiling is mapped onto a stepped surface of $\mathbb{R}^n$: its *lift*.

Planar tilings

Definition

A $n \to d$ tilings set $\mathcal{T} \subset \mathcal{X}_{n\to d}$ is a *planar tiling space* if there are a d-dimensional vector subspace $V \subset \mathbb{R}^n$, the *slope* and a positive integer w, the *width*, such that all tiling $t \in \mathcal{T}$ can be lifted into the slice $V + [0, w)^n$.



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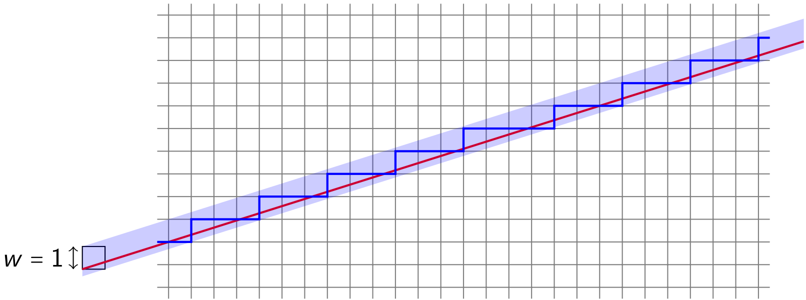

The $w = 1$ case corresponds to *strong planar tilings*.

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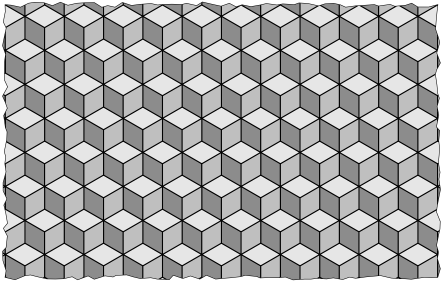

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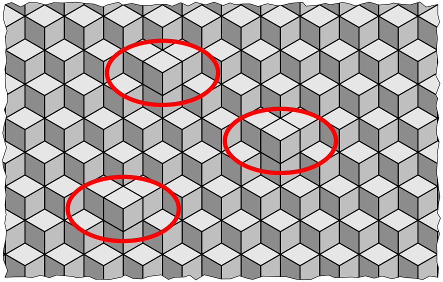

Here $w = 2$, if $w \geq 2$, this corresponds to *weak planar tilings*.

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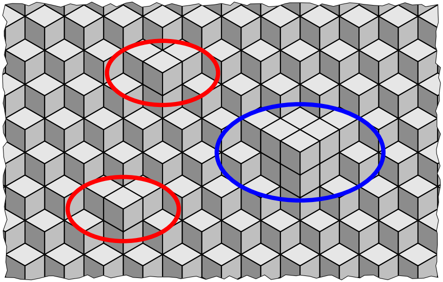


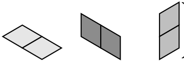
Here $w = 3$, if $w \geq 2$, this corresponds to *weak planar tilings*.

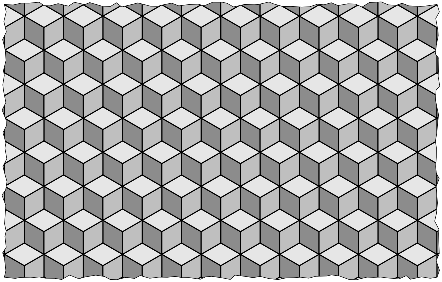
Local rules

- A *$n \to d$-pattern of size r* of a tiling $t \in \mathcal{X}_{n\to d}$ is a set of tiles lying inside a ball of radius $r > 0$. For $\mathcal{T} \subset \mathcal{X}_{n\to d}$ denote $\mathcal{P}_r(\mathcal{T})$ the set of $n \to d$-pattern of size r of each tiling of $\mathcal{T}$.
- The set of tilings of forbidden $n \to d$-patterns $\mathcal{F}$ is

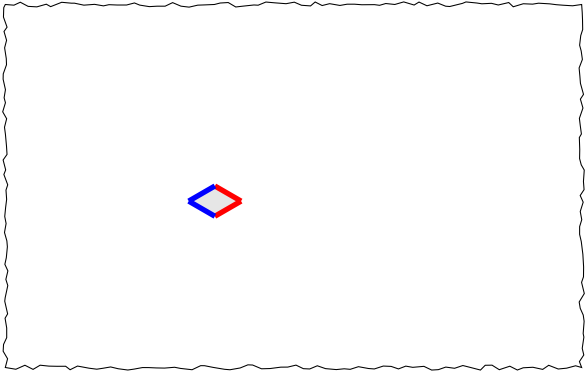
$$\mathcal{T}_{\mathcal{F}} = \{t \in \mathcal{X}_{n\to d} : \text{no patterns of } \mathcal{F} \text{ appears in } t\}$$

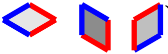
- $\mathcal{T}$ is a *set of tilings of finite type* if there exists $\mathcal{F}$ finite such that $\mathcal{T} = \mathcal{T}_{\mathcal{F}}$.

$\mathcal{F} = \Bigg\{$  $\Bigg\}$

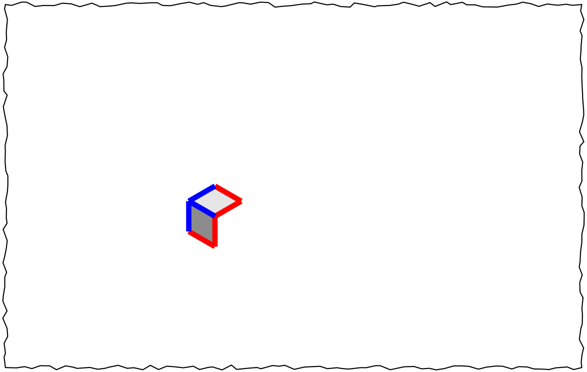


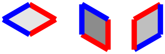
Colored local rules



Consider these decorated $3 \to 2$ tiles: $\left\{\right.$  $\left.\right\}$, which can match only if the corresponding edges have the same color.

Colored local rules



Consider these decorated 3 → 2 tiles: {  }, which can match only if the corresponding edges have the same color.

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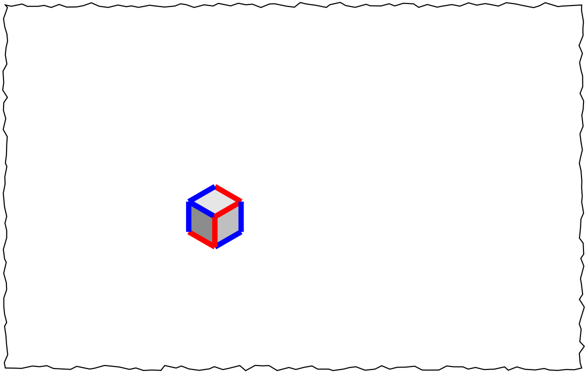
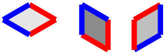


Consider these decorated $3 \to 2$ tiles: $\Big\{$  $\Big\}$, which can match only if the corresponding edges have the same color.

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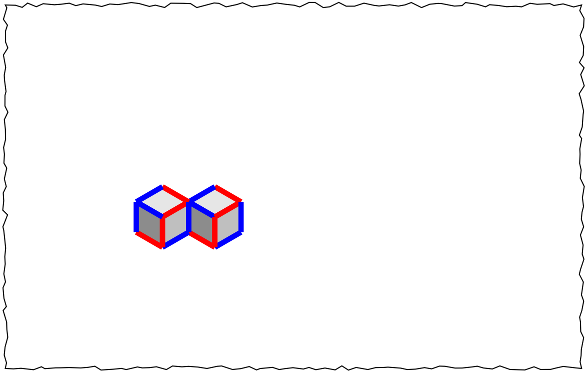
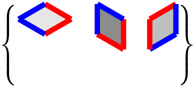


Consider these decorated $3 \to 2$ tiles: , which can match only if the corresponding edges have the same color.

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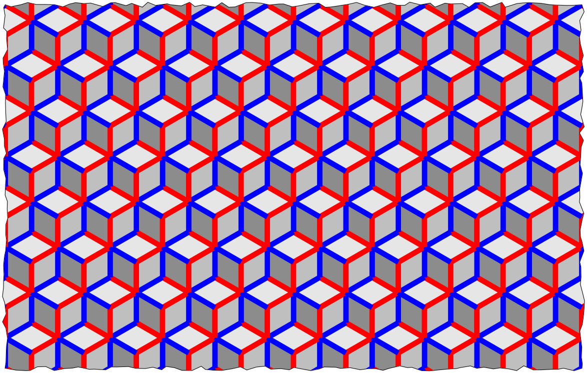
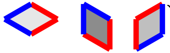
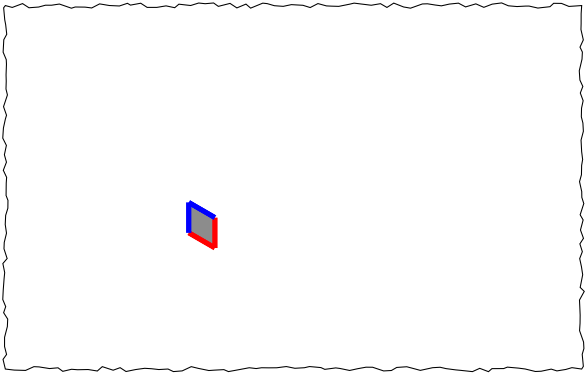
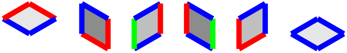


Consider these decorated $3 \to 2$ tiles: $\Big\{$  $\Big\}$, which can match only if the corresponding edges have the same color.

A set of tilings has *colored local rules* if it is possible to decorate tiles to obtain it.

Colored local rules



Consider these decorated $3 \to 2$ tiles: $\Big\{$  $\Big\}$, which can match only if the corresponding edges have the same color.

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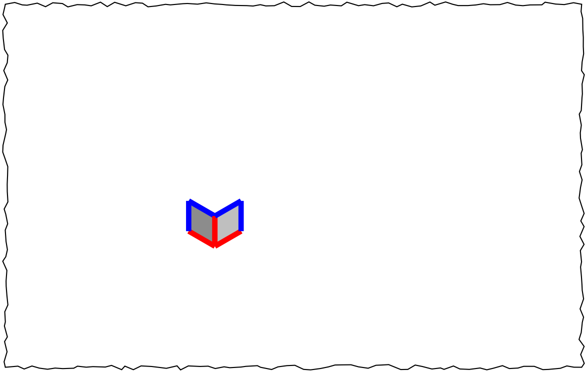
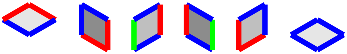


Consider these decorated $3 \to 2$ tiles: $\Big\{$  $\Big\}$, which can match only if the corresponding edges have the same color.

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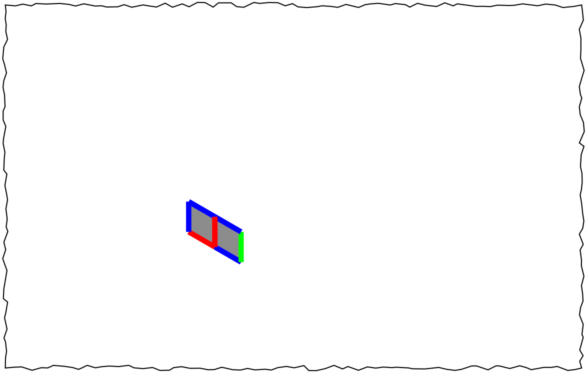
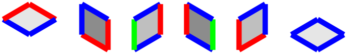


Consider these decorated 3 → 2 tiles: {}, which can match only if the corresponding edges have the same color.

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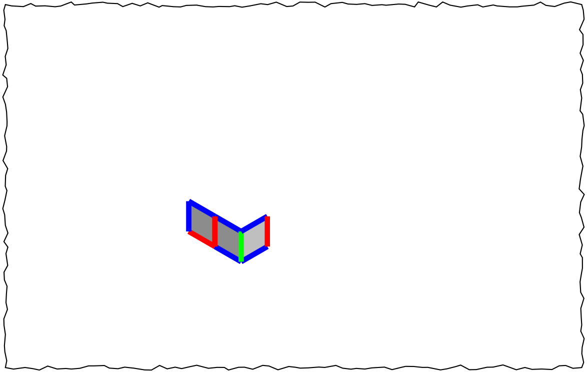
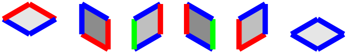


Consider these decorated $3 \to 2$ tiles: $\Big\{$  $\Big\}$, which can match only if the corresponding edges have the same color.

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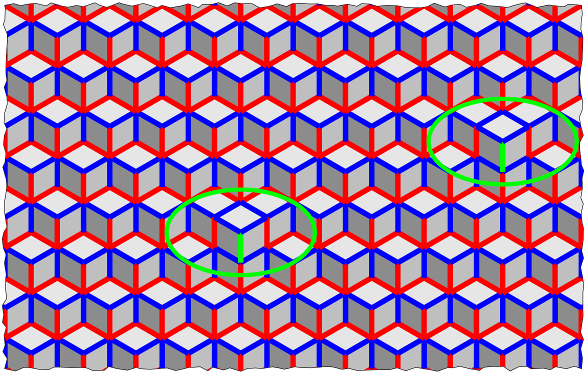
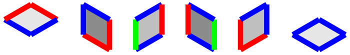


Consider these decorated $3 \to 2$ tiles: $\Bigg\{$  $\Bigg\}$, which can match only if the corresponding edges have the same color.
This allows only small fluctuations and obtain a weak planar set of tilings

Historic of the problem

Which vector space admits local rules or colored local rules?

n-fold tiling: plane tiling of slope $\mathbb{R}(u_1, \dots, u_n) + \mathbb{R}(v_1, \dots, v_n)$, where

$$u_k = \cos\left(\frac{2k\pi}{n}\right) \text{ and } v_k = \sin\left(\frac{2k\pi}{n}\right)$$

Slope of the Tiling	undecorated rules	decorated rules
$5, 10$-fold	stong	strong[1]
8-fold	none[2]	strong[3]
12-fold	none[3]	strong[4]
n-fold (with 4 not divide n)	weak[5]	strong?
quadratic slope in $\mathbb{R}^4$	weak[6]	strong[7]
non algebraic	none[8]	?

(1): *Penrose 1974* (2): *Burkov 1988* (3): *Le 1992* (4): *Socolar 1989*
(5): *Socolar 1990* (6): *Levitov 1988* (7): *Le & al. 1992* (8): *Le 1997*

Main results

- A vector $\overrightarrow{v} \in \mathbb{R}^n$ is *computable* if there exists a computable function $f : \mathbb{N} \longrightarrow \mathbb{Q}^n$ such that $\|\overrightarrow{v} - f(n)\|_\infty \leq 2^{-n}$ for all $n \in \mathbb{N}$.
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Theorem *(Fernique & S.)*

A d-dimensional vector space V admits $n \to d$ weak colored local rules (of width 3) for $n > d$ if and only if it is computable.

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non algebraic	none(8)	?
computable	weak	weak

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quadratic slope in $\mathbb{R}^4$	weak$^{(6)}$	strong$^{(7)}$
non algebraic	none$^{(8)}$	?
computable	weak (non natural)	weak

Notion of natural local rules

Local rules are said *natural* if there are verified by strong planar tiling.

Computability obstruction

Algorithm to obtain the slope

Input: Local rules of the planar tilings set $\mathcal{T} \subset \mathcal{X}_{n\to d}$, the width w and an integer m which corresponds to the precision.

Algorithm:

- $r_0 := 2wm$, $r := r_0$ and $d := 1$
- While $d \geq \frac{1}{2m}$ do
  - enumerate $\mathcal{P}_r(\mathcal{T})$, the set of all the diameter r patterns centered on 0 allowed by these local rules (this takes exponential but finite time in r)
  - enumerate $\mathbb{X}_{r_0}^r$, the set of d-dimensional vector spaces which admit a basis given by d vectors associated at a border vertex at distance r_0 of the origin in the lift of a pattern of P_r and compute

$$d = \max_{W_1, W_2 \in \mathbb{X}_r} \widetilde{d}(W_1, W_2)$$

  - r:=r+1
- Output: an element of $W \in \mathbb{X}_r$

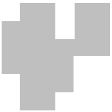
The algorithm halts: For sufficiently large r all vector spaces of $\mathbb{X}_r$ are near of V, if not by compacity one obtains one other slope for the $n \to d$ tiling.

The algorithm holds: There exists $W' \in \mathbb{X}_r$ such that $\widetilde{d}(W', V) \leq \frac{w}{r_0}$, thus

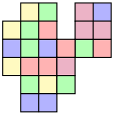
$$\widetilde{d}(W, V) \leq \widetilde{d}(W, W') + \widetilde{d}(W', V) \leq \frac{1}{2m} + \frac{w}{r_0} \leq \frac{1}{m}$$

An important tool:
Simulation of effective subshifts by SFT

Subshifts defined by forbidden patterns



$\mathbb{U} \subset \mathbb{Z}^d$ finite



finite pattern $u \in \mathcal{A}^{\mathbb{U}}$

Let $\mathcal{F}$ be a set of patterns, define the subshift where the forbidden patterns are $\mathcal{F}$:

$$\mathcal{T}_{\mathcal{A},\mathcal{F}} = \{x \in \mathcal{A}^{\mathbb{Z}^d} \ : \ \text{no pattern of } \mathcal{F} \text{ appear in } x\} \subseteq \mathcal{A}^{\mathbb{Z}^d}$$

Some class of subshifts:

$\mathcal{T}$ *fullshift* $\iff$ $\mathcal{F} = \emptyset$ and $\mathcal{T} = \mathcal{T}_{\mathcal{A},\mathcal{F}} = \mathcal{A}^{\mathbb{Z}^d}$,

$\mathcal{T}$ *subshift of finite type* $\iff$ there exists $\mathcal{F}$ finite set of patterns such that $\mathcal{T} = \mathcal{T}_{\mathcal{A},\mathcal{F}}$

$\mathcal{T}$ *effective subshift* $\iff$ there exists $\mathcal{F}$ recursive enumerable set of patterns such that $\mathcal{T} = \mathcal{T}_{\mathcal{A},\mathcal{F}}$.

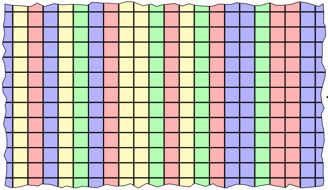
Realisation of effective subshift by sofic

Theorem *(Hochman-09, Durand-Romashchenko-Shen-2010, Aubrun-Sablik-2010)*

If $\mathcal{T} \subset \mathcal{A}^{\mathbb{Z}}$ is an effective subshift, there is a subshift of finite type $\mathcal{T}_{\texttt{Final}} \subset \mathcal{B}^{\mathbb{Z}^2}$ and a factor map $\pi : \mathcal{B} \to \mathcal{A}$ such that

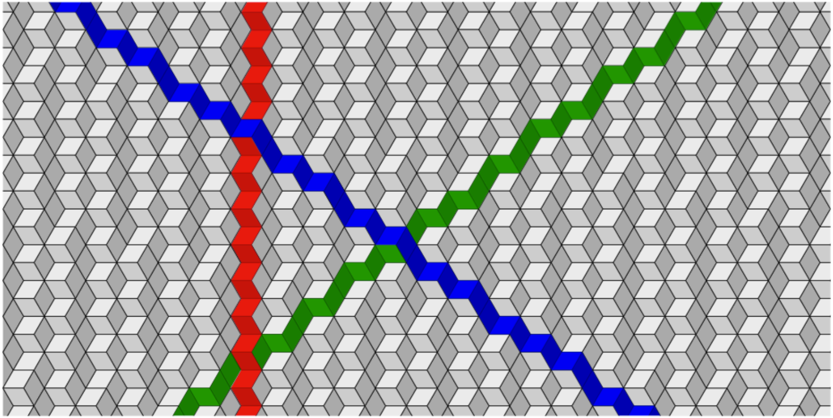
$$\pi(\mathcal{T}_{\texttt{Final}}) = \{x \in \mathcal{A}^{\mathbb{Z}^2} : \exists y \in \mathcal{T}, \forall i \in \mathbb{Z},\ x_{i+\mathbb{Z}} = y\}.$$

More precisely $x \in \mathcal{T}$ iff a "superposition" of x in one direction is in $\pi(\mathcal{T}_{\texttt{Final}})$.



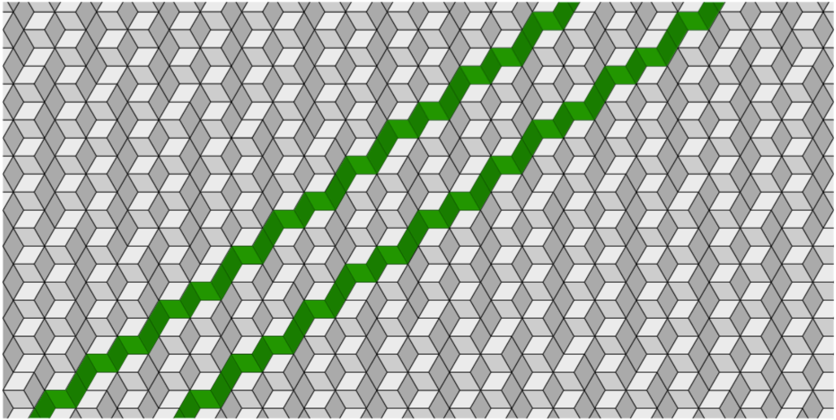
Realization of computable $3 \to 2$ planar tilings with colored local rules

Stripes of 3 → 2 strong planar tilling



For 3 → 2 strong planar tilling, intertwined stripes encoding Sturmian words.

Stripes of 3 → 2 strong planar tilling



Parallel stripes encode Sturmian words with the same slope.

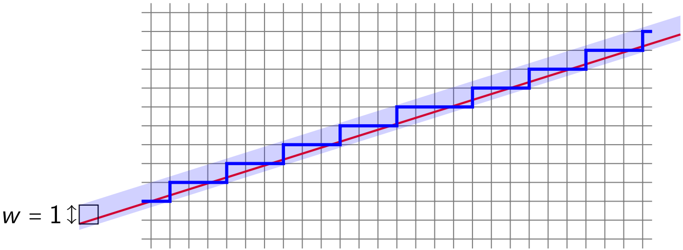
Quasi-Sturmian words

▶Define the *Sturmian word* $s_{\rho,\alpha} \in \{0,1\}^{\mathbb{Z}}$ of slope $\alpha \in [0,1]$ and intercept ρ by

$$s_{\rho,\alpha}(n) = 0 \;\Leftrightarrow\; (\rho + n\alpha) \mod 1 \in [0, 1-\alpha).$$

▶For $x, y \in \{0,1\}^{\mathbb{Z}}$ define $d(x,y) := \sup_{p\leq q} \left||x_p x_{p+1} \ldots x_q|_0 - |y_p y_{p+1} \ldots y_q|_0\right|$.

Fact: *Sturmian words with equal slopes are at distance at most one.*



$s_{\rho,\alpha}$: ...001000100010001000010001000100010...

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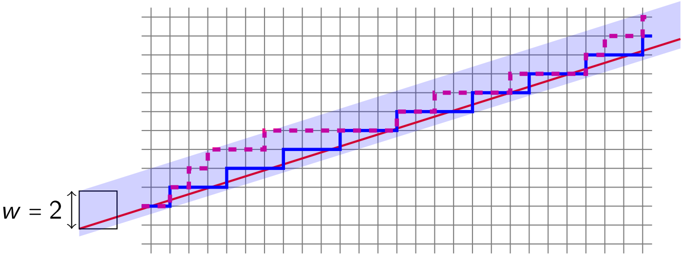

$s_{\rho,\alpha}$: ...001000100010001000010001000100010...
Quasi-Sturmian: ...001010100010000000010010000001001001010...

Quasi-Sturmian words

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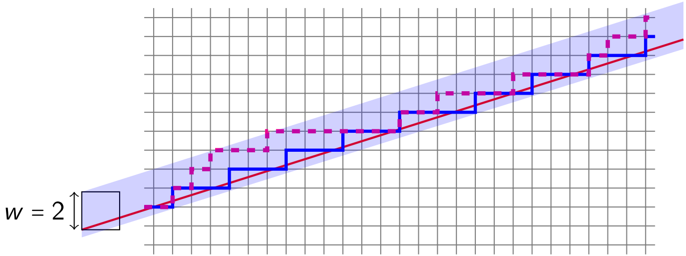

$s_{\rho,\alpha}$: ...0010001000100010001000010001000100010...

Quasi-Sturmian: ...0010101000100000001001000010001001010...

Changes: $\Uparrow$ $\Uparrow$ $\Uparrow$ $\Uparrow$ $\Uparrow\Uparrow$ $\Uparrow\Uparrow$ $\Uparrow$ ...

Quasi-Sturmian words

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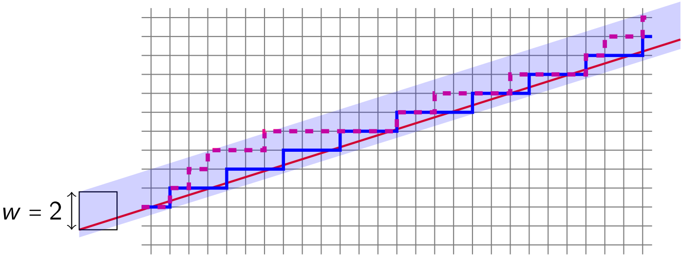
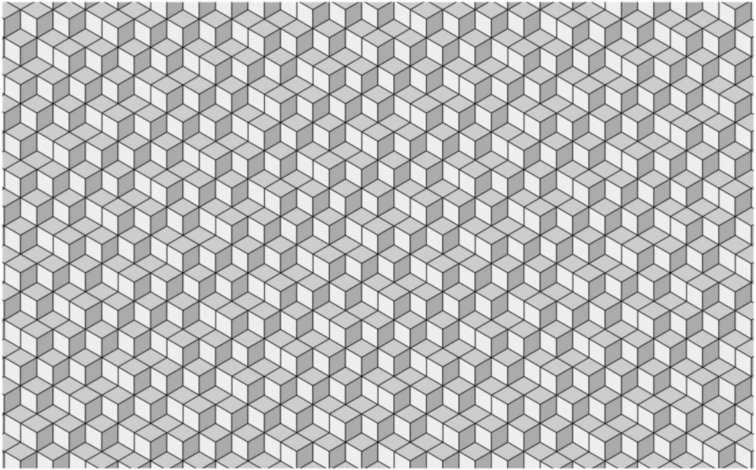


$s_{\rho,\alpha}$: …0010001000100010001000010001000010…

Quasi-Sturmian: …0010101000100000001001000010001001010…

Coding: …00000111111111110000000110001000100111…

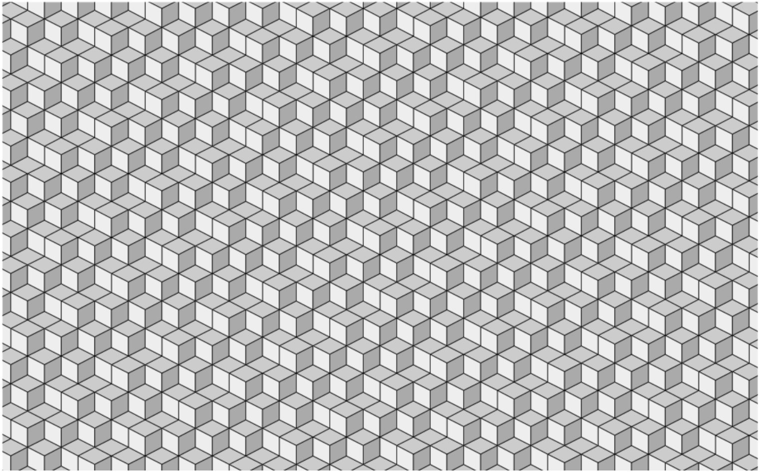
From strong planar 3 → 2 tilings to quasi-Sturmian subshifts



$$\mathcal{T} \subset \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \forall m \exists \rho \; x_{(.,m)} = s_{\rho,\alpha} \right\} \subset \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \forall m \; d(x_{(.,m)}, s_{\rho,\alpha}) \leq 1 \right\}$$

Sturmian subshift — Quasi-sturmian subshift

From strong planar 3 → 2 tilings to quasi-Sturmian subshifts



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Sturmian subshift | Quasi-sturmian subshift

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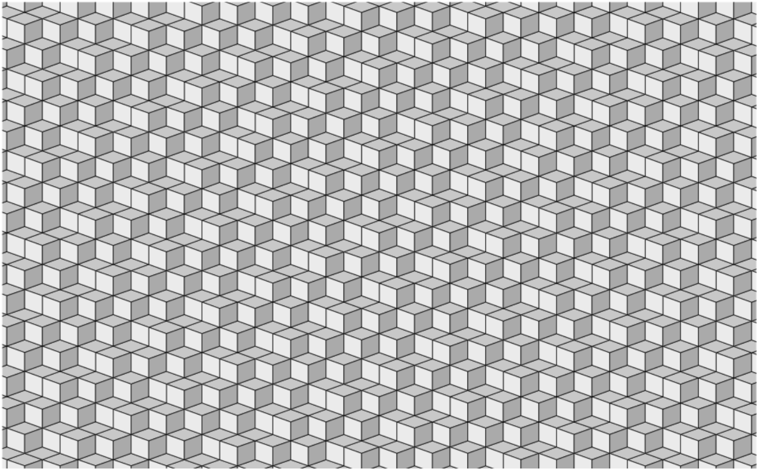

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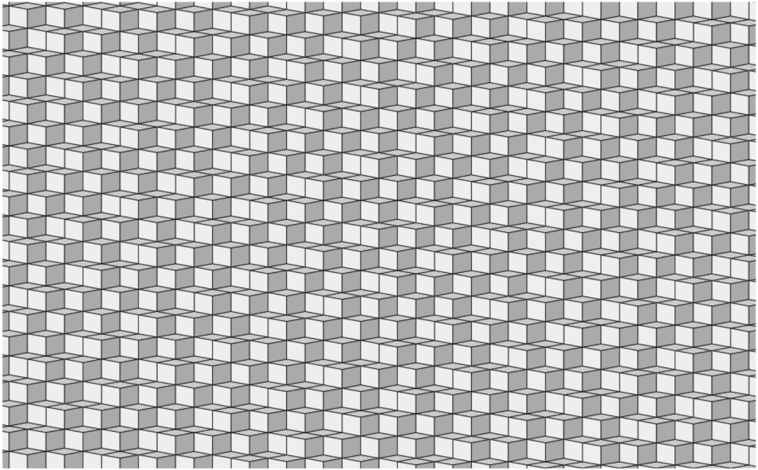

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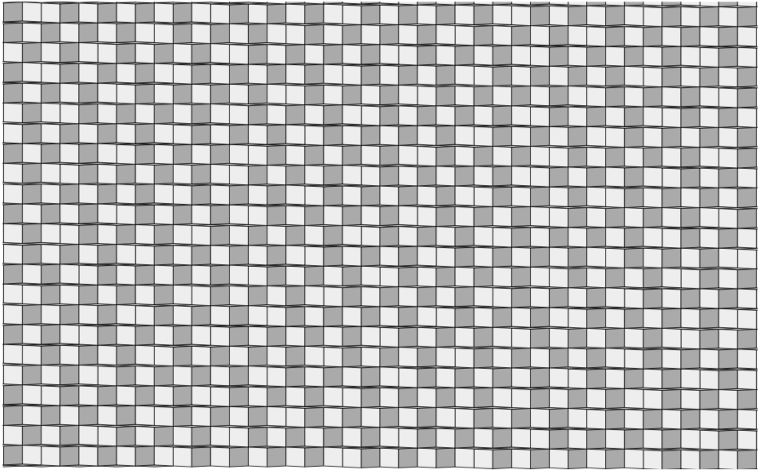

$$\mathcal{T} \quad \subset \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \forall m \exists \rho \; x_{(.,m)} = s_{\rho,\alpha} \right\} \subset \quad \left\{ x \in \{0,1\}^{\mathbb{Z}^2} : \forall m \; d(x_{(.,m)}, s_{\rho,\alpha}) \leq 1 \right\}$$

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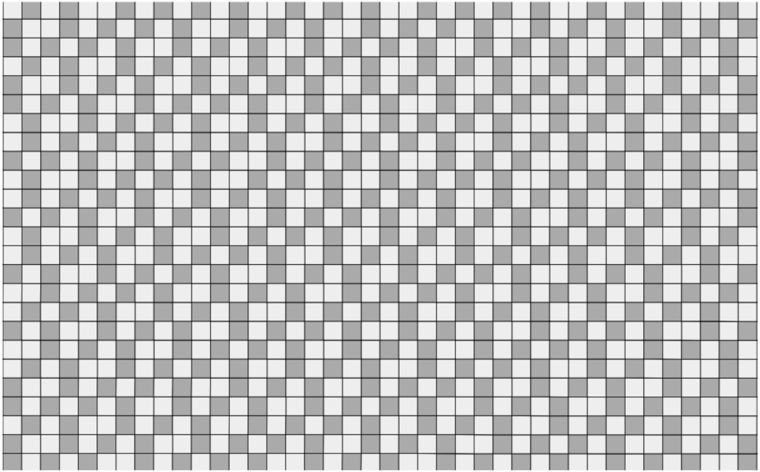

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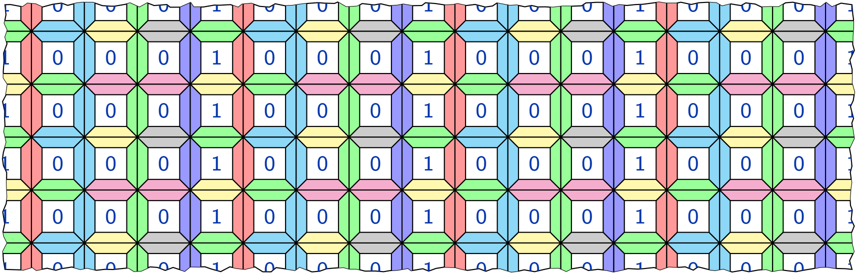
Sturmian subshift

Quasi-sturmian subshift

Independent quasi-Sturmian subshifts of slope α is sofic

If α is computable then $\left\{s_{\alpha,\rho} \in \{0,1\}^{\mathbb{Z}} : \rho \in \mathbb{R}\right\}$ is an effective subshift. So there exists an SFT $\mathcal{T}_{\{0,1\}\times\mathcal{B},\mathcal{F}} \subset (\{0,1\} \times \mathcal{B})^{\mathbb{Z}^2}$ such that:

$$\pi_1(\mathcal{T}_{\{0,1\}\times\mathcal{B},\mathcal{F}}) = \left\{x \in \{0,1\}^{\mathbb{Z}^2},\ \exists\rho \in \mathbb{R},\ \forall m \in \mathbb{Z},\ w_{\cdot,m} = s_{\alpha,\rho}\right\}.$$

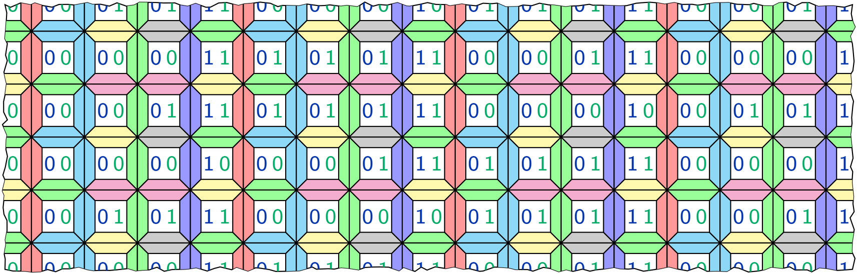


Each line is the same sturmian word $s_{\alpha,\rho}$

Independent quasi-Sturmian subshifts of slope α is sofic

Consider the SFT $\widetilde{Z}_\alpha \subset \left(\{0,1\} \times \mathcal{B} \times \{0,1\}\right)^{\mathbb{Z}^2}$ such that:

$$x \in \widetilde{Z}_\alpha \iff \begin{cases} \pi_{12}(x) \in \mathcal{T}_{\mathcal{B},\mathcal{F}}, \\ \pi_3(x_{m,n}) = 0 \text{ and } \pi_3(x_{m,n+1}) = 1 \;\Rightarrow\; \pi_1(x_{m,n}) = 0, \\ \pi_3(x_{m,n}) = 1 \text{ and } \pi_3(x_{m,n+1}) = 0 \;\Rightarrow\; \pi_1(x_{m,n}) = 1. \end{cases}$$

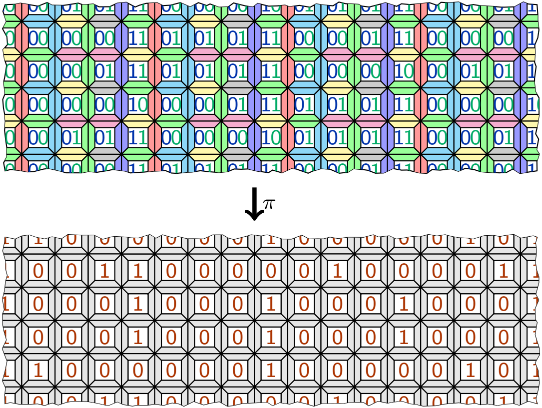


On each line we add an independent valid coding.

Independent quasi-Sturmian subshifts of slope α is sofic

Define $\pi(x)_{m,n} = \begin{cases} \pi_1(x_{m,n}) & \text{if } \pi_3(x_{m,n}) = \pi_3(x_{m,n+1}), \\ 1 - \pi_1(x_{m,n}) & \text{otherwise.} \end{cases}$

$$\pi(\widetilde{Z}_\alpha) = Z_\alpha = \left\{ x \in \{0,1\}^{\mathbb{Z}^2},\ \forall m \in \mathbb{Z},\ d(x_{(\cdot,m)}, s_{\alpha,0}) \leq 1 \right\}$$

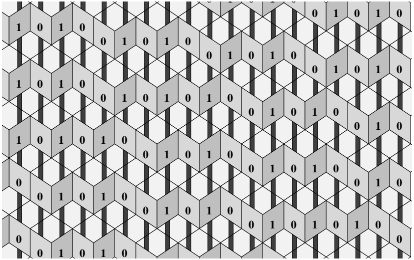


After the factor π, each line is an independent quasi-sturmian of slope α.

Transformation of tiles of $\widetilde{Z}_\alpha$ in $3 \to 2$-tiles

Each tiles of $\widetilde{Z}_\alpha$ can be viewed as a wang tile.
We construct a set $\tau_\alpha^{\vec{v_3}}$ of $3 \to 2$ colored tiles in the following way:

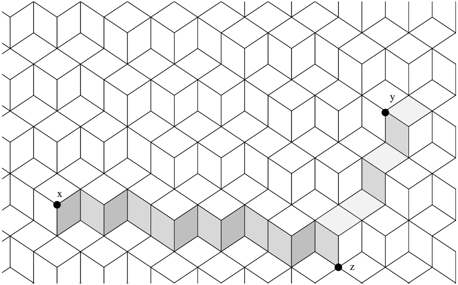
0 1 0 1 0
1 0 1 0 0 1 0 1 0 0 1 0 1 0 0 1 0 1 0
1 0 1 0 0 1 0 1 0 1 0 0 1 0 1 0 0 1 0
1 0 1 0 1 0 0 1 0 1 0 0 1 0 1 0 0 1 0
0 0 1 0 1 0 0 1 0 1 0 0 1 0 1 0 1 0 0
0 0 1 0 1 0



Call *$\vec{v}_i$-ribbon* of a $3 \to 2$ tiling a maximal sequence of tiles, with two consecutive tiles being adjacent along an edge $\vec{v}_i$.
Then, $\tau_\alpha^{\vec{v_3}}$ exactly forms the $3 \to 2$ tilings whose $\vec{v}_3$-ribbons has slope α.

Width of the planar tiling built width local rules

In the same way we construct the set of tiles $\tau_{\beta}^{\overrightarrow{v_2}}$ and $\tau_{\alpha/\beta}^{\overrightarrow{v_1}}$ and we consider colored $3 \to 2$ tilings formed with $\tau_{1,\alpha,\beta} = \tau_{\alpha/\beta}^{\overrightarrow{v_1}} \times \tau_{\beta}^{\overrightarrow{v_2}} \times \tau_{\alpha}^{\overrightarrow{v_3}}$.



These tilings are all planar tilings of slope orthogonal to $(1, \alpha, \beta)$. Moreover, the width of such a tiling is at most 3, since any two of its vertices can be connected by a path made of two ribbons.

Theorem *(Fernique & S.)*

A d-dimensional vector space V admits $n \to d$ weak colored local rules (of width 3) for $n > d$ if and only if it is computable.

How delete the colors?

Given a slope, it is possible to substitute each tile of a strong planar tiling by a "meta" tile arbitrary large. Thus decorations can be encoded by “fluctuations” at the cost of an increase of 1 in the width.

Theorem *(Fernique & S.)*

A d-dimensional vector space V admits $n \to d$ weak local rules (of width 4) for $n > d$ if and only if it is computable.

Perspectives

Decorated local rules

The computable slopes have natural decorated rules (of with 3) but it is possible to have strong decorated local rules (i.e., width 1)?

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Decorations can be encoded by "fluctuations" at the cost of an increase of 1 in the thickness, but the rules are no more natural.

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Natural undecorated local rules

Only algebraic slopes can have natural undecorated rules *(Le '95)*. Even fewer slopes can have strong undecorated rules *(Levitov '88)*. There is yet no complete characterization of these slopes.