Continuous-time Models in Corporate Finance

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OBJECTIVES

- Diffusion models in Finance
  - 1) Option Pricing (1973 →)
  - 2) Corporate Finance (more recent) [Impact of Financial Decisions on Firms' value]

This course:

a) synthesis of 2

b) bridge gap btw 2 and 1

Questions to students:

who knows 1? stochastic calculus? (Tutorials) 2?

LECTURE 1: INTRODUCTION

1°) Elements of Corporate Finance

- Balance sheet of a firm = instantaneous picture of what the firm owns (assets) and what the firm owes (liabilities)

Throughout the lectures = simple case: 4 items maximum of in balance sheet

ASSETS	LIABILITIES	
productive Assets A_t	Equity E_t	Net worth
liquid reserves M_t	Debt D_t	

- Income statement during period $t \to t+dt$

Net operating income dx_t	Profit/Losses → dM_t → dividends dL_t (if ≤ 0: equity issuance)
Financial income $rM_t dt$ (constant interest rate)	Interest paid $c_t dt$

For simplicity: debt structure decided once and for all at $t=0$

For the moment no taxes

2°) Pricing Risky Debt (Merton 1973)

This is an application of option pricing methods

Consider the simple case where $M_t \equiv 0$,

and debt consists of a single promised repayment D at $t = T$ (0-coupon). Assume

[denoted V in M,L]

(1) $dx_t = \beta A_t dt$; (2) $dA_t = \mu A_t dt + \sigma A_t dW_t$

$\beta = 0$ in Merton

NB: / shareholders are protected by LL [not adjusted]

/ Assumption: perfect secondary markets $\Rightarrow$ $\mu = r - \beta$

$\mathcal{E}(dA_t + dx_t) = rA_t dt$

if $A_T > D$: they repay and get $A_T - D$

if $A_T < D$: they default and debtholders get A_T

market value of debt at $t = 0$:

(3) $$D_0 = \mathcal{E}_0\left[e^{-rT}\operatorname{Min}(D, A_T)\right]$$

$$= e^{-rT}D - \mathcal{E}_0\left[e^{-rT}\operatorname{Max}(0, D - A_T)\right]$$

(indeed $\operatorname{Min}(D, A_T) = D - \operatorname{Max}(0, D - A_T)$)

$\Rightarrow$ BS formula for a put option

$$D_0 = e^{-rT}D\,N(x_1) + A_0 e^{-\beta T}N(x_2)$$

$x_1 = \frac{1}{\sigma\sqrt{T}}\ln\frac{De^{-rT}}{A_0 e^{-\beta T}} - \frac{1}{2}\sigma\sqrt{T}$; $x_2 = -x_1 - \sigma\sqrt{T}$

NB: by definition $D_0 = e^{-RT}D$

R: yield to maturity

"market" value of debt

(3) $\Rightarrow \frac{D_0}{De^{-rT}} = e^{-\underbrace{(R-r)}_{\text{default spread}}T} = N(x_1) + \frac{A_0}{De^{-rT}}N(x_2)$

De^{-rT}: "nominal" debt

$\frac{A_0}{De^{-rT}}$: "nominal" asset to debt ratio

$\Rightarrow$ comparative static analysis

3°) The Modigliani-Miller paradox

come back to the general case:
(no restrictions on x_t, A_t)

$$(4) \quad \underbrace{dx_t + rM_t dt - c_t dt}_{\text{net profit/losses}} = dM_t + dL_t$$

at $t=0$ firm raises D_0 in debt
and invests $I + M_0$

shareholder value $SV_0 = E_0 - (I + M_0 - D_0)$

Debt repays coupons $c_t dt$ until T
and at T final repayment $\min(A_T + M_T, D)$

$$D_0 = \mathcal{E}_0\left[\int_0^T e^{-rt} c_t dt + e^{-rT} \min(A_T + M_T, D)\right]$$

$$E_0 = \mathcal{E}_0\left[\int_0^T e^{-rt} dL_t + e^{-rT} \max(A_T + M_T - D, 0)\right]$$

$$\Rightarrow SV_0 = \mathcal{E}_0\left[\int_0^T e^{-rt}(c_t dt + dL_t) + e^{-rT}(A_T + M_T)\right] - I - M_0$$

$$(4) \Rightarrow c_t dt + dL_t = dx_t + rM_t dt - dM_t$$

$$\int_0^T e^{-rt}(rM_t dt - dM_t) = -\int_0^T d\left[e^{-rt} M_t\right] = M_0 - e^{-rT} M_T$$

$$\Rightarrow SV_0 = \mathcal{E}_0\left[\int_0^T e^{-rt} dx_t + e^{-rT}(-M_T + A_T + M_T)\right] + M_0 - I - M_0$$

$$\Rightarrow SV_0 = \mathcal{E}_0\left[\int_0^T e^{-rt} dx_t + e^{-rT} A_T\right] - I = A_0 - I$$

(MM) Irrelevance of financial policy: c_t, D (debt); dL_t (equity)

NB: MM also implies that risk management activities (reducing σ) are at best useless (if costless) or reduce shareholder value (if costly)

LECTURE 2 : The Trade-off Theory (Leland 1994)

$\neq$ notation

2 elements are missing in MM:

- taxes $\underset{\substack{\uparrow \\ \text{tax rate}}}{\theta}\,[\,dx_t - \underset{\substack{\uparrow \\ \text{constant coupon}}}{c}\,dt\,] \rightarrow$ $\boxed{\text{increase leverage to minimize taxes}}$

- liquidation costs : shareholders inject capital until $\tau = \inf\{t,\ A_t = A_B\}$ $\hookrightarrow$ bankruptcy

Then debtholders get $(1-\alpha)A_B$

$\hookrightarrow$ liquidation cost (including taxes)

$\rightarrow$ decrease leverage

optimal leverage trades-off $\left\langle \begin{array}{l} \text{tax benefits of debt} \\ \text{expected liquidation costs} \end{array}\right.$

1°) The model

shareholders are not cash constrained

$\Rightarrow M_t$ irrelevant (convention $M_t \equiv 0$)

$$dx_t = \underset{\hookrightarrow [NB: =0 \text{ in } L]}{\beta A_t\, dt} \quad : \quad dA_t = \underset{\hookrightarrow (r-\beta)}{\mu} A_t\, dt + \sigma dW_t$$

Assume constant coupon $c \Rightarrow$ stationary model

$$D_t = D(A_t) \qquad E_t = E(A_t)$$

$$\begin{cases} rD = \mu A D'(A) + \dfrac{\sigma^2 A^2}{2} D''(A) + c & A \geq A_B \quad (1) \\ D(A_B) = (1-\alpha)A_B & D(A) \sim \dfrac{c}{r} \quad A \to +\infty \end{cases}$$

$$D(A) = \mathcal{E}_A\left[\int_0^\tau e^{-rt} c\, dt + e^{-r\tau} A_B (1-\alpha)\right]$$

$$D(A) = \frac{c}{r} - E(e^{-r\tau})\left[\frac{c}{r} - A_B(1-\alpha)\right] \qquad (2)$$

$$E(A) = \mathcal{E}_A\left[\int_0^\tau e^{-rt}(\beta A_t - c)(1-\theta)\, dt\right]$$

$$\begin{cases} E(A_B) = 0 & E(A) \sim (1-\theta)\left(A - \dfrac{c}{r}\right),\ A \to +\infty \\ rE(A) = \mu A E'(A) + \dfrac{\sigma^2 A^2}{2} E''(A) + (1-\theta)[\beta A - c]_{A \geq A_B} & (3) \end{cases}$$

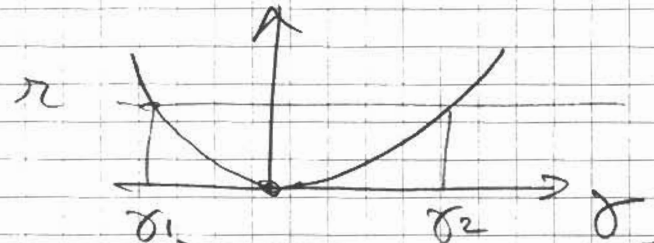
2°) Asset Pricing

NB: General solutions of linear equation

$$rF(A) = \mu A F'(A) + \frac{\sigma^2 A^2}{2} F''(A)$$

are $F(A) = k_1 A^{-\gamma_1} + k_2 A^{\gamma_2}$, where $-\gamma_1 < 0 < \gamma_2$

solutions of $r = \mu \gamma + \frac{\sigma^2}{2}\gamma(\gamma - 1)$



condition at infinity $\Rightarrow k_2 = 0$

(both for D and E)

boundary condition

$$\boxed{D(A) = \frac{c}{r} + k_1 A^{-\gamma_1} = \frac{c}{r} + \left[\frac{c}{r} - A_B(1-\alpha)\right]\left(\frac{A}{A_B}\right)^{-\gamma_1}} \quad (4)$$

risk-premium

NB: (2) $\Rightarrow$ $E_A(e^{-r\tau}) = \left(\frac{A}{A_B}\right)^{-\gamma_1}$

(3) $\Rightarrow$ $E(A) = k'_0 A + k'_1 A^{-\gamma_1} - (1-\theta)\frac{c}{r}$

$$r k'_0 A = \mu A k'_0 + \beta(1-\theta)A \Rightarrow k'_0 = \frac{\beta(1-\theta)}{r-\mu} = 1-\theta$$

$$E(A_B) = 0 \Rightarrow A_B + k'_1 A_B^{-\gamma_1} - \frac{c}{r} = 0$$

$$k'_1 = \left(\frac{c}{r} - A_B\right) A_B^{\gamma_1}$$

$$\boxed{E(A) = (1-\theta)\left[A - \frac{c}{r} + \underbrace{\left(\frac{c}{r} - A_B\right)\left(\frac{A}{A_B}\right)^{-\gamma_1}}_{\text{LL option}}\right]} \quad (5)$$

3°) Financial Decisions

Timing: 1) firm chooses c

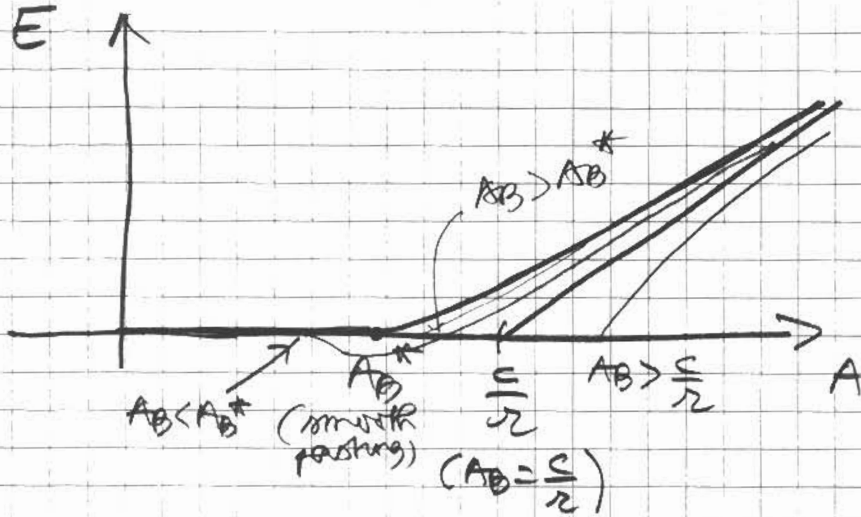
2) market lends D_0

3) shareholders choose A_B

• Backward induction:

Step 3: $A_B \rightarrow \max \left(\frac{c}{r} - A_B\right) A_B^{\gamma_1}$ (NB: $\gamma_1 > 0$)

$$\boxed{A_B^* = \left(\frac{\gamma_1}{1+\gamma_1}\right)\frac{c}{r}} < \frac{c}{r}$$



Equity prices for ≠ values of A_B

$$E'(A_B) = 1 + \gamma_1 \left(A_B - \frac{c}{r}\right) A^{-\gamma_1 - 1} A_B^{\gamma_1}$$

$$E'(A_B) \geq 0 \iff A_B(1+\gamma_1) \geq \gamma_1 \frac{c}{r} \iff A_B \geq A_B^*$$

optimal A_B $\Rightarrow$ option like equity price (~ call)

PB 1 with Trade-off Theory

- Default is always strategic (exercise of LL option)
- Equity prices are convex functions of A

$\Rightarrow$ Shareholders never want to manage risks (quite the opposite)

STEP 2

$$D_0 = \frac{c}{r} - \left[\frac{c}{r} - A_B(1-\alpha)\right]\left(\frac{A}{A_B}\right)^{-\gamma_1}$$

$$SV_0 = E_0 - (I - D_0) = E_0 + D_0 - I$$

$$= (1-\theta)\left[A_0 - \frac{c}{r} + \left(\frac{c}{r} - A_B\right)\left(\frac{A}{A_B}\right)^{-\gamma_1}\right] + \frac{c}{r} - \left[\frac{c}{r} - A_B(1-\alpha)\right]\left(\frac{A}{A_B}\right)^{-\gamma_1} - I$$

$$SV_0 = \underbrace{(1-\theta)A_0 - I}_{\text{After tax NPV}} + \theta\frac{c}{r} + \left(\frac{A_0}{A_B}\right)^{-\gamma_1}\left[(1-\theta)\left(\frac{c}{r} - A_B\right) - \frac{c}{r} + A_B(1-\alpha)\right]$$

$$SV_0 = (1-\theta)A_0 - I + \underbrace{\theta\frac{c}{r}\left[1 - \left(\frac{A_0}{A_B}\right)^{-\gamma_1}\right]}_{\text{expected tax rebate}} - \underbrace{(\alpha-\theta)A_0^{-\gamma_1} A_B^{1+\gamma_1}}_{\text{expected net liquidation cost}}$$

Then $\frac{c}{r}$ (or $A_B = \frac{\gamma_1}{1+\gamma_1}\frac{c}{r}$) is chosen so as to maximize SV_0:

$$SV_0 = (1-\theta)A_0 - I + \theta\left(1+\frac{1}{\gamma_1}\right)\left[A_B - A_B^{1+\gamma_1}A_0^{-\gamma_1}\right] - (\alpha-\theta)A_B^{1+\gamma_1}A_0^{-\gamma_1}$$

$$SV_0 = (1-\theta)A_0 - I + \frac{1+\gamma_1}{\gamma_1}\left[\theta A_B - \left(\frac{\theta+\alpha\gamma_1}{1+\gamma_1}\right)A_B^{1+\gamma_1}A_0^{-\gamma_1}\right]$$

max when $\quad \theta = (\theta+\alpha\gamma_1)\left(\frac{A_B}{A_0}\right)^{\gamma_1}$

$$\Rightarrow \quad \boxed{A_B^* = A_0\left(\frac{\theta}{\theta+\alpha\gamma_1}\right)^{1/\gamma_1}}$$

$$SV_0^* = (1-\theta)A_0 - I + \frac{1+\gamma_1}{\gamma_1}A_B^*\left[\theta - \underbrace{\frac{\theta+\alpha\gamma_1}{1+\gamma_1}\left(\frac{A_B^*}{A_0}\right)^{-\gamma_1}}_{=\theta}\right]$$

$$= (1-\theta)A_0 - I + \theta A_B^*$$

$$= A_0 - I - \theta A_0\left[1 - \left(\frac{\theta}{\theta+\alpha\gamma_1}\right)^{1/\gamma_1}\right].$$

NB: leverage proportional to A_0
increases with θ
decreases with α

calibration: $\gamma_1 \sim 1$, $\theta \sim \frac{1}{3}$ $\quad \alpha = \frac{1}{2}$

"nominal" leverage $\quad \frac{c}{rA_0} = \frac{1+\gamma_1}{\gamma_1}\frac{A_B}{A_0} = \frac{1+\gamma_1}{\gamma_1}\left(\frac{\theta}{\theta+\alpha\gamma_1}\right)^{1/\gamma_1}$

$$\frac{c}{rA_0} \sim \frac{1+1}{1}\left(\frac{1/3}{1/3+1/2}\right) = \frac{2}{1+3/2} = \frac{4}{5}$$

$\boxed{\text{PB 2 of trade-off theory: predicts excessive leverage}}$

NB: If $\theta = 0$ (no taxes) $\quad A_B^* = 0$ (no leverage). This is because shareholders are assumed to have no cash constraint