

Problem Set # 5

WCATSS 2014

1. Compute the Verlinde ring of $G = SU_2$ in terms of twisted equivariant K -theory as follows.
 - (a) Show that there are two singleton conjugacy classes in G and every other conjugacy class is diffeomorphic to S^2 . Deduce a G -equivariant cell decomposition with two 0-cells with stabilizer G and a single 1-cell with stabilizer \mathbb{T} , the diagonal subgroup of G . (A k -cell with stabilizer $H \subset G$ is homeomorphic to $B^k \times G/H$, where B^k is the k -dimensional open ball.)
 - (b) Recall that the equivariant K -theory group $K_H(\text{pt})$ is isomorphic to the complex representation ring $R(H)$. Identify $R(SU_2)$, $R(\mathbb{T})$, and the restriction map $R(SU_2) \rightarrow R(\mathbb{T})$ in terms of polynomial and Laurent rings.
 - (c) In general an equivariant cell structure gives a spectral sequence for generalized cohomology, but because we have only 0- and 1-cells the spectral sequence collapses to a short exact sequence. Compute the untwisted equivariant K -theory group $K_G(G)$.
 - (d) A twisting is an equivariant line bundle over the 1-cell, so a character of \mathbb{T} . Let the twisting τ be the character $\lambda \mapsto \lambda^r$, $\lambda \in \mathbb{T}$, for some positive integer r . Compute $K_G^\tau(G)$. Express the result as a quotient of $R(SU_2)$.

2. Let $S^1 \xrightarrow{e} \mathbb{R}^3 \xleftarrow{e'} S^1$ be two disjoint knots. Let A be an E_3 -algebra in $(\text{Ch}_{\mathbb{Q}}, \otimes)$, the symmetric monoidal category of chain complexes over the rationals with tensor product. Use the defining properties of factorization homology to construct the sequence of maps

$$C_*(S^1 \times S^1; \mathbb{Q}) \otimes (A \otimes A) \rightarrow \int_{S^1 \sqcup S^1} A \rightarrow \int_{\mathbb{R}^3} A \simeq A .$$

Show that this composite map factors as the map $(A \otimes A)[2] \xrightarrow{\cdot \text{Lk}(e, e')} (A \otimes A)[2] \xrightarrow{[-, -]} A$, which is the linking number of e and e' times the Lie bracket operation on A .

3. Let M be a closed connected oriented 3-manifold equipped with a self-indexing Morse function $f: M \rightarrow \mathbb{R}$. As in problem 6 on Problem Set #4, such an f determines Heegaard data: $(\Sigma_g; \alpha, \beta)$.
 - (a) Explain that a point in the intersection $\mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset \text{Sym}^g(\Sigma_g)$ is the same data as a choice of gradient flow lines connecting all intermediate critical points of M .
 - (b) Removing a neighborhood of these flow lines, as well as a neighborhood of a flow line between the extremal critical points of f , results in a 3-manifold with boundary which is a disjoint union of 2-spheres, equipped with a non-vanishing vector field on it. Explain why the vector field on each boundary component has index zero, and so can be extended to a non-vanishing vector field on the entire 3-manifold.

- (c) Convince yourself that a holomorphic strip in $\text{Sym}^g(\Sigma)$ connecting two such intersection points exists only if these two (not-uniquely) associated non-vanishing vector fields are connected by a path among such, at least after removing some open balls from the 3-manifold.
- (d) Conclude that (each variant of the) Heegaard-Floer chain complex split as a direct sum indexed by equivalence classes of non-vanishing vector fields on the 3-manifold, where two are deemed equivalent if they are homotopic on the 3-manifold without some balls.
- (e) Consider the Lie group $\text{Spin}^c(n) := \text{Spin}(n) \times_{\mathbb{Z}/2} U(1)$ with the canonical central actions of $\mathbb{Z}/2$. The *moduli space of Spin^c -structures* on a rank n vector bundle $E \rightarrow B$ is the space of lifts of its classifying map $B \rightarrow EBO(n)$ to $B\text{Spin}^c(n)$, other moduli spaces are similar. Show that the moduli space of Spin^c -structures on $E \rightarrow B$ is the moduli space of pairs $(L \rightarrow B, B \xrightarrow{E \oplus L} B\text{Spin}(n+2))$ of complex line bundles on B together with Spin-structures on the direct sum. Show also that the moduli space of Spin^c -structures on an oriented rank n vector bundle $E \rightarrow B$ is the space of lifts of the composite map $w_2: B \rightarrow EBSO(n) \rightarrow B^2\mathbb{Z}/2$, the latter which classifies the universal cover of $SO(n)$ ($n > 2$), to $B^2\mathbb{Z}$ along the “modulo 2” map.
- (f) Now take $n = 3$. Show that $BSO(2) \rightarrow BSO(3) \rightarrow B^2\mathbb{Z}/2$ canonically factors through $B^2\mathbb{Z}$. Identify the fiber of the map $BSO(3) \rightarrow B^2\mathbb{Z}/2$ as $B^2\mathbb{Z}$ using the Hopf fibration. Use the previous two sentences to argue that there is a bijection between the set of path components of the moduli space of Spin^c -structures on a closed oriented 3-manifold, and the set of equivalence classes of non-vanishing vector fields on the 3-manifold, where two such are equivalent if they are homotopic on the 3-manifold minus some open balls.