Problem Set # 4

WCATSS 2014

1. Let F be 3-dimensional quantum Chern-Simons theory for some fixed compact Lie group G and level. Let $V = F(S^1 \times S^1)$ be the complex vector space assigned to the standard 2-torus. It carries a representation of $SL_2(\mathbb{Z})$. All answers to this problem are in terms of V and this representation. (You'll also need to know that $F(S^2)$ is 1-dimensional.)
   (a) Compute $F(S^1 \times S^2)$.
   (b) Compute $F(S^3)$. (You may want to use the decomposition of S^3 as a union of 2 solid torus, a *Heegaard splitting* as explained in the last problem. Compare to a similar Heegaard splitting of $S^1 \times S^2$.)
   (c) Compute $F(L_{p,q})$, where $L_{p,q}$ is the lens space described by relatively prime positive integers p, q.

2. (a) Suppose that G is a compact Lie group. Construct a homeomorphism $LG \cong G \times \Omega G$, where ΩG is based at the identity element.
   (b) Prove that $H_{*+d}(LG; \mathbb{Q})$, an algebra with the string product, is isomorphic to $H_{*+d}(G; \mathbb{Q}) \otimes H_*(\Omega G; \mathbb{Q})$, where the former is given the intersection product and the latter the Pontryagin product coming from concatenation of loops.
   (c) Compute the string product on $H_{*+3}(LS^3; \mathbb{Q})$.

3. Suppose M is a compact oriented manifold of dimension d. Let μ denote the string product and Φ the string coproduct. The unit $\mathbf{1}$ for the loop product on $H_{*+d}(LM)$ is given by the fundamental class of M.
   (a) Prove that $\Phi(\mathbf{1}) = \chi(M) c_0 \otimes c_0$, where $c_0 \in H_0(LM)$ is the generator corresponding to the component of constant loops.
   (b) By decomposing an X-shaped cobordism with two incoming and two outgoing circles in different ways, use the TQFT structure to prove that (up to sign)

$$(\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Phi) = \Phi \circ \mu = (id \otimes \mu) \circ (\Phi \otimes id)$$

   (c) Use the previous two results to deduce that for any $a \in H_*(LM; \mathbb{Q})$ we have that $\Phi(a)$ is a multiple of $c_0 \otimes c_0$. Conclude that in particular the coproduct vanishes on elements that are not of homological degree d.
   (d) Use this to deduce that a genus one cobordism with one incoming and one outgoing circle induces a zero operation.

4. Suppose one is given a surface $\Sigma_{g,r}$ of genus g and with r ordered boundary components. Let $\Gamma_{g,r}$ denote its mapping class group, $\Gamma_{g,r} = \pi_0 \operatorname{Diff}(\Sigma_{g,r}, \partial\Sigma_{g,r})$. Glueing a genus one surfaces with two boundary components induces a homomorphism $t : \Gamma_{g,r} \to \Gamma_{g+1,r}$. Harer proved that the induced map $t_* : H_*(B\Gamma_{g,r}) \to H_*(B\Gamma_{g+1,r})$ is an isomorphism for $g \gg *$. Deduce that every class in the image of t_* induces a zero higher string operation. This is Tamanoi's vanishing theorem.

5. Suppose G is a compact connected Lie group.

   (a) Let $p : S^1 \sqcup S^1 \to S^1 \vee S^1$ be the pinch map, which identifies two points. Compute the homotopy fiber of the map $\operatorname{Map}(S^1 \vee S^1, BG) \to (LBG)^2$. (Hint: use that it fits into a homotopy pull back diagram with the diagonal map $BG \to (BG)^2$.)

   (b) Suppose that in a fibration $F \to E \to B$ the fiber has as top-dimensional non-zero rational homology group $H_d(F; \mathbb{Q}) \cong \mathbb{Q}$ and that the action of $\pi_1 B$ on this group is trivial. Use the Serre spectral sequence to construct a so-called transfer map $H_*(B) \to H_{*+d}(E)$.

   (c) Use this to define a string product on $H_{*-d}(LBG; \mathbb{Q})$.

6. Let Σ be an oriented closed surface of genus g. Let $(\alpha_1, \ldots, \alpha_g)$ a g-tuple of disjoint simple closed curves in Σ that generate the homology of Σ.

   (a) Consider the surface obtained by performing 1-surgery along each α_i. Show that this surface is a 2-sphere. Note that this surgery is witnessed as a 3-manifold which is compact with two boundary components: a 2-sphere and Σ. Show that this 3-manifold is independent of the ordering of the α-tuple.

   (b) Use the above point to construct a closed oriented connected 3-manifold from the data $(\Sigma; (\alpha_1, \ldots, \alpha_g), (\beta_1, \ldots, \beta_g))$ of a closed oriented surface, and two g-tuples of disjoint simple closed curves in Σ that generate the homology of Σ.

   (c) Show that every closed connected oriented 3-manifold is obtained via the construction in (b). Do this by explaining how a self-indexing Morse function on such a 3-manifold determines Heegaard data.

   (d) Give two Heegaard splittings of a closed connected orientable 3-manifold whose Heegaard data are not isomorphic.

   (e) Show that two closed connected oriented 3-manifolds are homeomorphic if they admit Heegaard splittings resulting in isomorphic Heegaard data.

   (f) Classify closed connected orientable 3-manifolds that admit Heegaard splittings along surfaces of genus less than two.