Problem Set # 3

WCATSS 2014

1. Let G be a finite group, f the associated 2-dimensional finite gauge theory, and F the associated 3-dimensional finite gauge theory. These theories are defined on *unoriented* manifolds. (If we twist by a nonzero cohomology class, then orientations are required.)

   (a) Compute $f(M)$, where M is the Möbius band. Note $\partial M \simeq S^1$, which we view as incoming, so $f(M)$ is a linear functional on the vector space $f(S^1)$. Basis elements of $f(S^1)$ correspond to irreducible complex representations of G, so we get a number for each such representation. Interpret the result in terms of the representation theory of G. Try particular examples, such as $G = \mathbb{Z}/4\mathbb{Z}$, $G = Q$, where Q is the 8-element quaterion group.

   (b) The finite path integral can be interpreted as an inverse limit (or just 'limit' in modern usage). Use this to compute $f(pt)$, which is a category. The groupoid of G-bundles on pt is $*//G$, the groupoid whose single object $*$ has the group G of automorphisms. The finite path integral is the limit of the functor $*//G \to \mathrm{Cat}_{\mathbb{C}}$ which maps $*$ to the category $\mathrm{Vect}_{\mathbb{C}}$ with trivial G-action. Here $\mathrm{Cat}_{\mathbb{C}}$ is the *2-category* of linear categories. (Think informally about limits, if necessary.)

   (c) Several variations: (i) include a nonzero cohomology class, which can be represented by a central extension $\mathbb{T} \to \widetilde{G} \to G$; (ii) replace $\mathrm{Cat}_{\mathbb{C}}$ by the 2-category of complex algebras, bimodules, and intertwiners; (iii) compute $F(S^1)$ as a limit over the groupoid $G//G$, where G acts on G by conjugation; (iv) compute $F(pt)$ as the limit of a functor into the 3-category of tensor categories.

2. In this problem you will use the cobordism hypothesis to compute tqfts in various cases.

   (a) What are the 1-dimensional unoriented tqfts in the category of vector spaces? in super vector spaces? (Note the symmetry in the category of super vector spaces, which is an isomorphism $V \otimes W \to W \otimes V$ for each pair of super vector spaces V, W, uses the Koszul sign rule.)

   (b) Consider the super analog of the bicategory of algebras, bimodules, and maps. What are the fully-dualizable objects here? Is the Clifford algebra fully-dualizable? If so compute the Serre automorphism.

3. (a) Use the notation $C(L)$ for the (graded) chain complex whose (bigraded) homology groups are the Khovanov homology groups of the oriented link $L \subset \mathbb{R}^3$. Use the notation $L^\vee \subset \mathbb{R}^3$ for the link which is the composite embedding $L \hookrightarrow \mathbb{R}^3 \to \mathbb{R}^3$ where the latter isomorphism is given by reflection about one of the coordinate planes. Show that $C(L^\vee) \cong C(L)^\vee := \mathrm{Hom}_{\mathbb{Z}}(C(L), \mathbb{Z})$ is the linear dual chain complex. Also, show that for disjoint links L, L' we have $C(L \sqcup L') \cong C(L) \otimes C(L')$.

   (b) Write down a chain complex that computes the Khovanov homology of a trefoil (with a choice of orientation). Compute its homology.