Subshifts of linear complexity and subgroups of finite index of free groups

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General idea

- Develop automata theory inside a restricted set of words (typically the factors of a shift)
- Find classes of shifts for which some problems are simpler (examples below).
- Find natural generalizations of classes like Sturmian shifts (like normal sets below).
- Understand the role played by free groups in symbolic systems (Sturmian or interval exchange shifts).

Complexity of shifts

The complexity of a shift space S is the sequence $(c_n)_{n\geq 0}$ where c_n is the number of factors (or blocks) of S of length n.
A (binary) Sturmian shift is a shift on a binary alphabet of complexity $n+1$..

Example

Set $A=\{a,b\}$. The Fibonacci shift is generated by the the fixpoint $x=f^\omega(a)$ of the morphism $f:A^*\rightarrow A^*$ defined by $f(a)=ab$ and $f(b)=a$.

$$x = abaababaabaababaababaabaababaabaab\cdots$$

Sturmian shifts

Generalization of Sturmian shifts to larger alphabets (called Arnoux-Rauzy words or episturmian words).
An irreducible shift S on the alphabet A is Sturmian if the set of its factors

- is closed under reversal
- contains, for each $n \geq 1$, exactly one word u of length n such that $ua \in F$ fore more than one letter a and for this word $uA \subset F$ (u is a right special word).

It is easy to see that the complexity of a Sturmian shift on an alphabet with $k+1$ letters, is $kn+1$.

Example

Set $A = \{a, b, c\}$. The morphism $f : A^* \to A^*$ defined by $f(a) = ab$, $f(b) = ac$ and $f(c) = a$ has the fixpoint

$$x = abacabaabacababacabaabacabacabaabacab\cdots$$

called the Tribonacci word. The corresponding shift is Sturmian.

Interval exchange transformations : generalization of rotations. Introduced by Oseledec (1966) following an earlier idea of Arnold (1963). These transformations form a generalization of rotations of the circle.

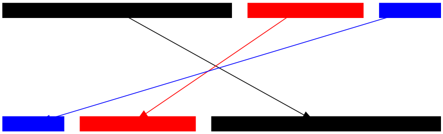


Fig.: A 3-interval exchange transformation

Interval exchange transformations preserve the Lebesgue measure. They are a.s. uniquely ergodic (Masur, Veech, 1982).

An interval exchange transformation is called minimal if the orbit of every point $z \in [0, 1[$ is dense.
It is regular (or idoc) if the orbits of the singular points are infinite and disjoint.

Theorem (Keane, 1975)

A regular interval exchange transformation is minimal.

Let T be an interval exchange transformation corresponding to a partition $(I_a)_{a \in A}$ of $[0,1[$. The natural coding of T w.r.t. to $z \in [0,1[$ is the infinite word x defined by

$$x_n = a \quad \text{if} \quad T^n(z) \in I_a.$$

If T is minimal, the subshift generated by x does not depend on z. It is called an interval exchange shift.

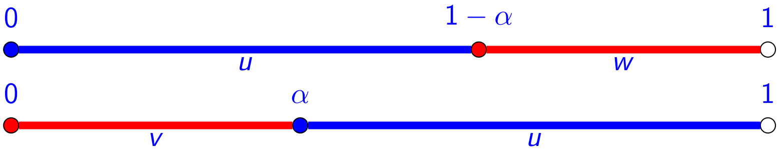


Fig.: A 2-interval exchange transformation (rotation of angle α).

The natural coding of T w.r.t. to α is the Fibonacci word.

The tree condition

Let S be a shift on the alphabet A. For a factor w of S, let

$$\begin{aligned} L(w) &= \{a \in A \mid aw \in F(S)\}, \\ R(w) &= \{a \in A \mid wa \in F(S)\}, \\ E(w) &= \{(a,b) \in A \times A \mid awb \in F(S)\} \end{aligned}$$

Let $G(w)$ be the graph on two disjoint copies of $L(w)$ and $R(w)$ with edges $E(w)$.
We say that S satisfies the tree condition if the graph $G(w)$ is a tree for any $w \in F(S)$.
If S staisfies the tree condition and $A \subset F(S)$, its complexity is $kn+1$
A Sturmian shift and a regular interval exchange shift satisfy the tree condition.

The graphs $G(\varepsilon), G(a)$ and $G(aba)$ for the Tribonacci shift are shown below.

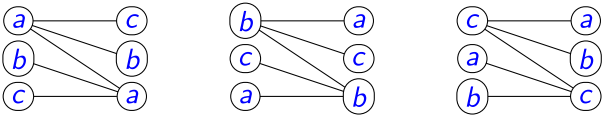


FIG.: The graphs $G(\varepsilon), G(a)$ and $G(aba)$ in the Tribonacci set.

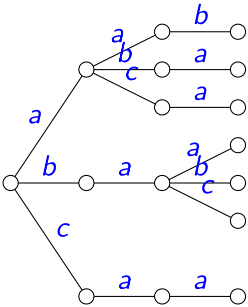


FIG.: The Tribonacci shift

Return words

Let S be a shift. For $w \in F(S)$, let

$$\Gamma_S(w) = \{z \in F(S) \mid wz \in F(S) \cap A^+ w\}$$

be the set of return words to w and

$$\mathcal{R}_S(w) = \Gamma_S(w) \setminus \Gamma_S(w) A^+$$

be the set of first return words to w.

Theorem (Justin, Vuillon, 2000)

If S is Sturmian on the alphabet A, for any word $w \in F(S)$, the set $R_S(w)$ is a basis of the free group on A.

For example, if F is the Fibonacci shift we have $R_S(aa) = \{baa, babaa\}$.

We say that an irreducible shift S on the alphabet A is normal if

(i) It satisfies the tree condition,

(ii) For any $w \in F(S)$, the set $R_{F(S)}(w)$ is a basis of the free group on A.

Thus Sturmian shifts are normal. Moreover

Theorem

Any regular interval exchange shift is normal.

The proof that it satisfies condition (ii) uses a generalization of Rauzy induction.

A rotation on three intervals (disposed vertically).

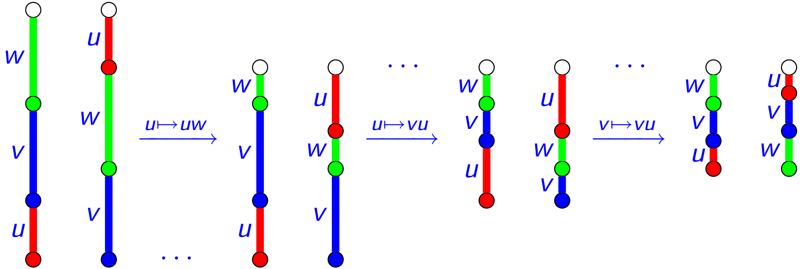


FIG.: Generalized Rauzy induction.

The sequence of 3 inductions (one right followed by two left ones) allows to reach the induced transformation on the interval coding points preceded by a w. The resuling first return set is $R_F(w) = \{vuw, vvuw, w\}$.

The finite index basis property

A shift S on the alphabet A has the finite index basis property if for any subgroup H of finite index of the free group on A, the indecomposable words of $H \cap F(S)$ form a basis of H.

Theorem

Normal sets have the finite index basis property.

For example, if H is the subgroup formed by the words of length multiple on n, then $H \cap F(S)$ is the set of words of length multiple of n. For an alphabet with $k+1$ letters, a basis of H has $kn+1$ elements.

Example

Let S be the Fibonacci shift. The set of words of length n is the basis of the subgroup of words of length multiple of n (in the free group on $\{a, b\}$).
For $n = 2$, we find $\{aa, ab, ba\}$ and $bb = ba(aa)^{-1}ab$.
For $n = 3$, we find $\{aab, aba, baa, bab\}$ and

$$
\begin{aligned}
abb &= aba(baa)^{-1}bab \\
bba &= bab(aab)^{-1}aba \\
bbb &= bba(aba)^{-1}abb
\end{aligned}
$$

The last formula uses the two first ones.

Decoding

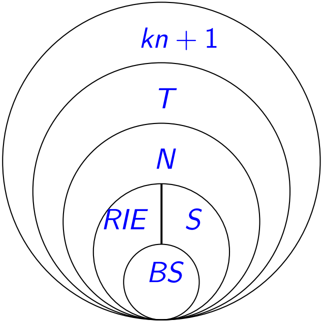
Let S be a shift on the alphabet A. Let $f : B^* \rightarrow A^*$ be an injective morphism such that $f(B)$ is the basis of a subgroup of finite index of the free group on A. Then $f^{-1}(S)$ is a shift called a finite index decoding of S.

The class of Sturmian shifts is not closed under finite index decoding but the class of regular interval exchange transformations is. More generally :

Theorem

The class of normal shifts is closed under finite index decoding.

The final picture



	RB	FIB	FID
S	yes	yes	no
RIE	yes	yes	yes
N	yes	yes	yes
T	yes	?	?
$kn+1$	no	no	no

FIG.: Binary Sturmian (BS), Regular interval exchange (RIE), Sturmian (S), Normal (N), Tree (T), of complexity $kn+1$ and the properties of Return words forming a basis (RB) finite index basis (FIB) and finite index decoding (FID).