# Search Games and Optimal Kakeya Sets

Yuval Peres [1]

Based on joint work with

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[1] Microsoft Research

# Kakeya sets – History

A subset $S \subseteq \mathbb{R}^2$ is called a **Kakeya** set if it contains a unit segment in every direction.

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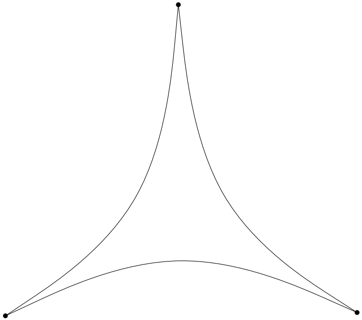
Kakeya's question (1917): Is the three-pointed deltoid shape a Kakeya set of minimal area?



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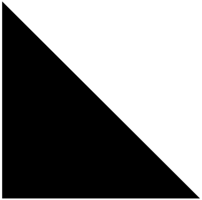

$n = 1$

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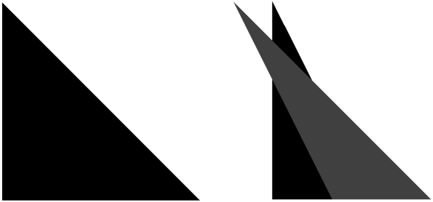

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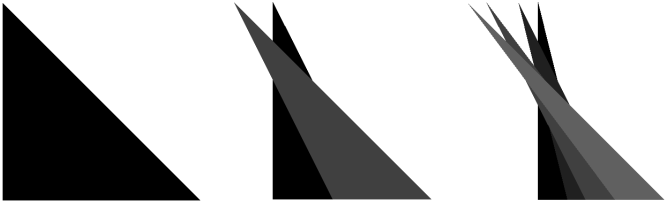

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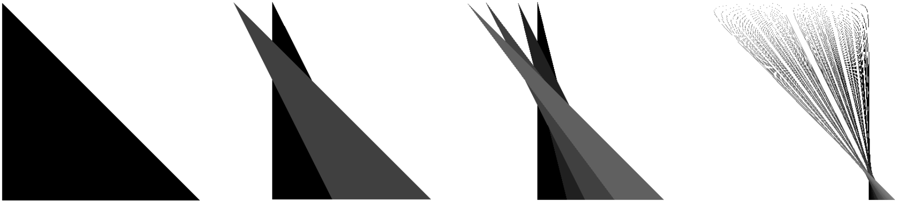

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(Figures due to Terry Tao)

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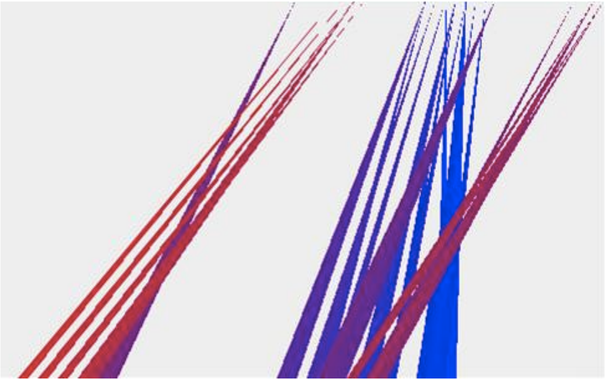

A. S. Besicovitch.
On Kakeya's problem and a similar one.
*Math. Z.*, 27(1):312–320, 1928.

Roy O. Davies.
Some remarks on the Kakeya problem.
*Proc. Cambridge Philos. Soc.*, 69:417–421, 1971.

Micah Adler, Harald Räcke, Naveen Sivadasan, Christian Sohler, and Berthold Vöcking.
Randomized pursuit-evasion in graphs.
*Combin. Probab. Comput.*, 12(3):225–244, 2003.

Yakov Babichenko, Yuval Peres, Ron Peretz, Perla Sousi, and Peter Winkler.
Hunter, Cauchy Rabbit and Optimal Kakeya Sets.
*Transactions AMS*, to appear; arXiv:1207.6389

# Definition of the game $G_n$

**Two players**

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Hunter

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# Definition of the game $G_n$

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**Where?**

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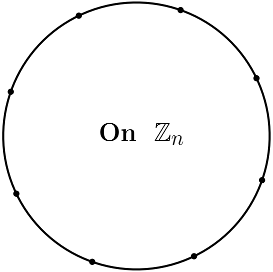

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At night – they cannot see each other....

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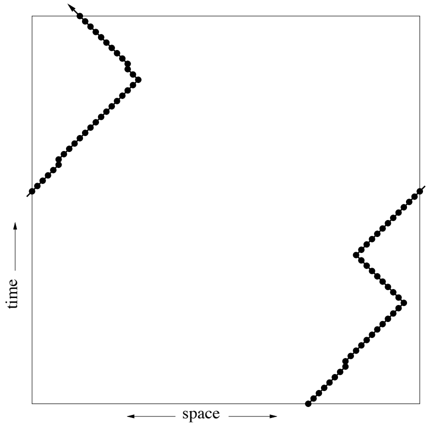
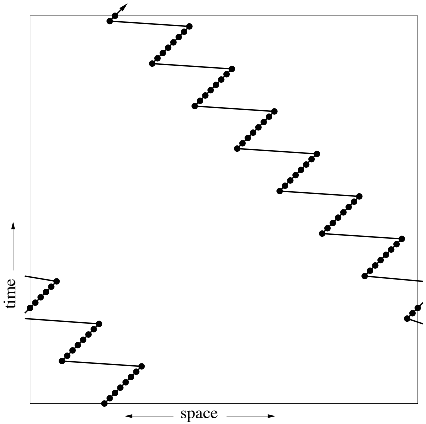
- **Zig-Zag hunter strategy:** He starts in a random direction, then switches direction with probability $1/n$ at each step.

  **Rabbit counter-strategy:** From a random starting node, the rabbit walks $\sqrt{n}$ steps to the right, then jumps $2\sqrt{n}$ to the left, and repeats. The probability of capture in $n$ steps is $\asymp n^{-1/2}$, so mean capture time is $n^{3/2}$.

# Zig-Zag hunter strategy



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*Use second moment method – calculate first and second moments of $K_n$.*

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Extend the strategies *until time* $2n$ and *define* $K_{2n}$ analogously.

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With the hunter's strategy above

**Recall** $K_n = \sum_{i=0}^{n-1} \mathbf{1}(H_i = R_i)$

$$\mathbb{P}(K_n > 0) \gtrsim \frac{1}{\log n}.$$

*This gave expected capture time at most* $n \log n$.

What about the **rabbit**? Can he **escape** for time of order $n \log n$?

Looking for a **rabbit** strategy with

$$\mathbb{P}(K_n > 0) \lesssim \frac{1}{\log n}.$$

Extend the strategies *until time* $2n$ and *define* $K_{2n}$ analogously. Obviously

$$\mathbb{P}(K_n > 0) \leq \frac{\mathbb{E}[K_{2n}]}{\mathbb{E}[K_{2n} \mid K_n > 0]}$$

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This means that given the **rabbit and hunter** collided, we want them to collide "a lot". The hunter can only move to neighbours or stay put.

So the **rabbit** should also choose a distribution for the jumps that favors short distances, yet grows linearly in time. This suggests a ***Cauchy random walk***.

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*This is what we want-* **But** in the discrete setting...

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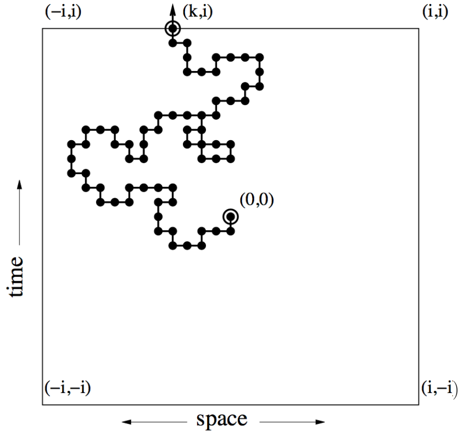

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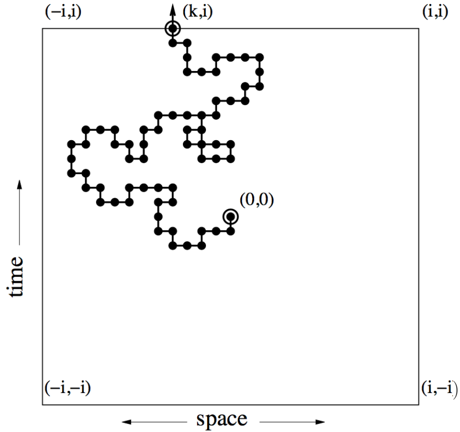

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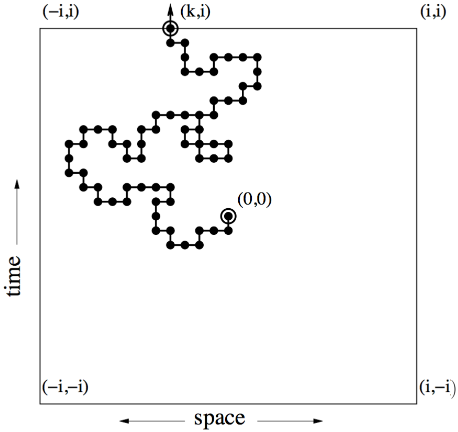

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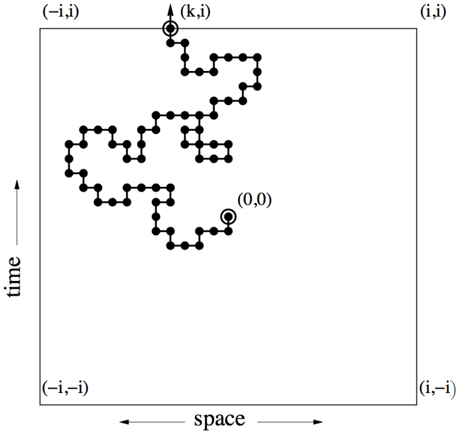
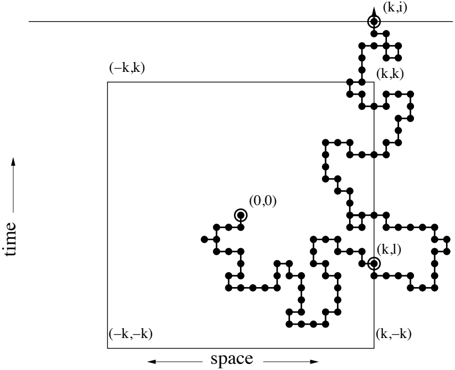
- Thus the hitting probability at $(0, i)$ is at least $1/(8i + 4)$.




(k,i)
(−k,k)
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(0,0)
(k,l)
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time
space

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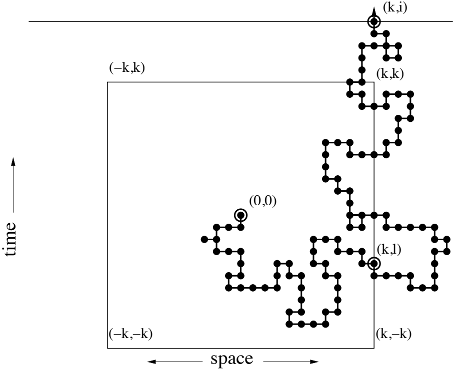

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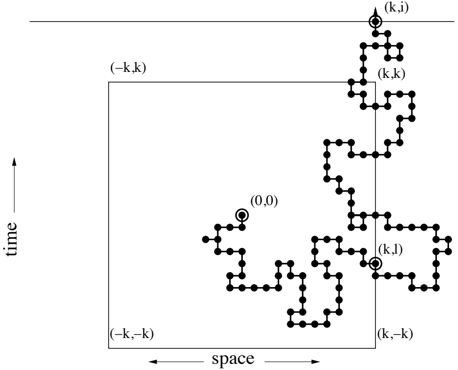
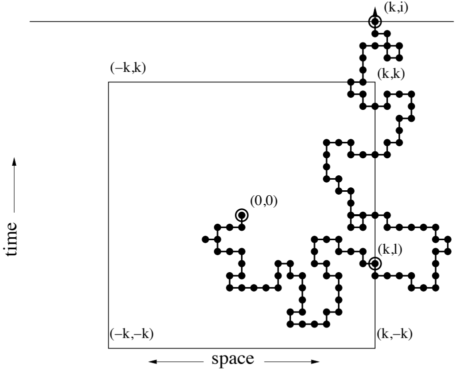


- Suppose $0 < k < i$.
- With probability $1/4$, SRW exits the square $[-k, k]^2$ via the right side.
- Repeating the previous argument, the hitting probability at $(k, i)$ is at least $c/i$.



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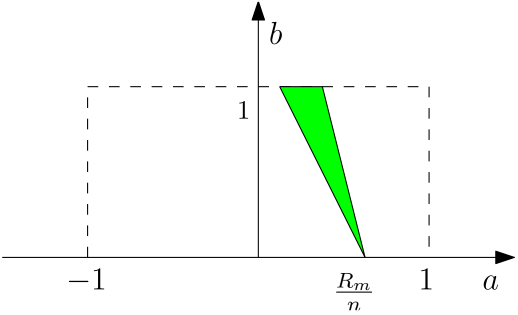

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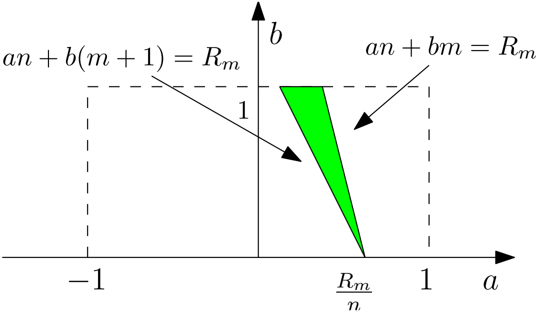

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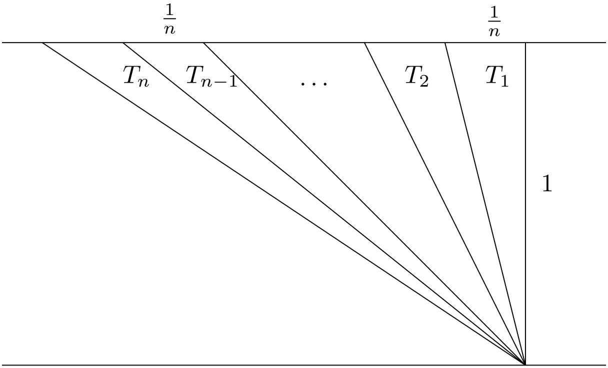
Hence the probability of collision in $[0, n)$ is half the area of the union of all such triangles, which are translates of



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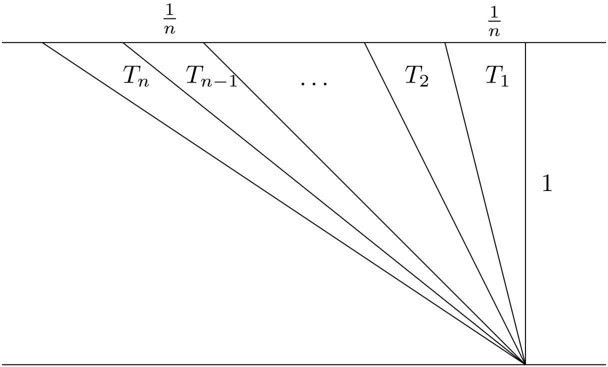


In these triangles we can find a unit segment in all directions that have an angle in $[0, \pi/4]$

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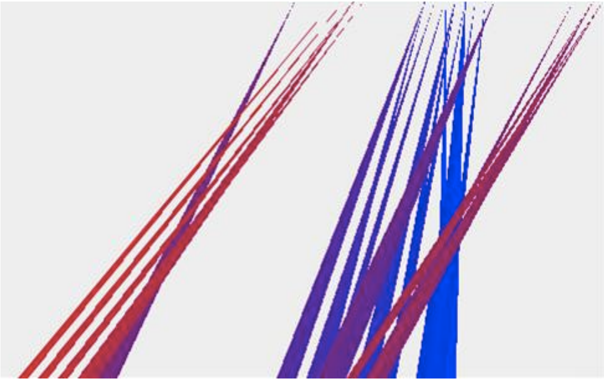

Simulation generated with $n = 32$

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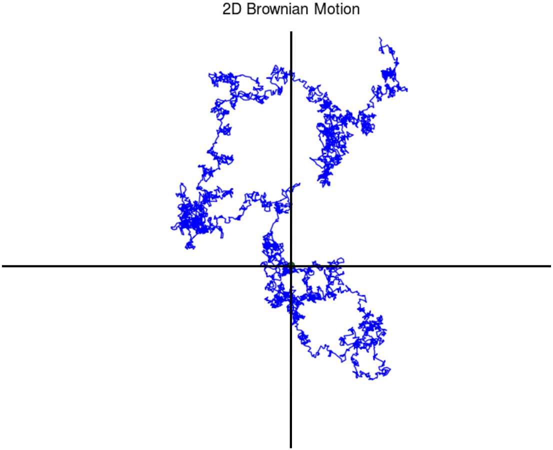

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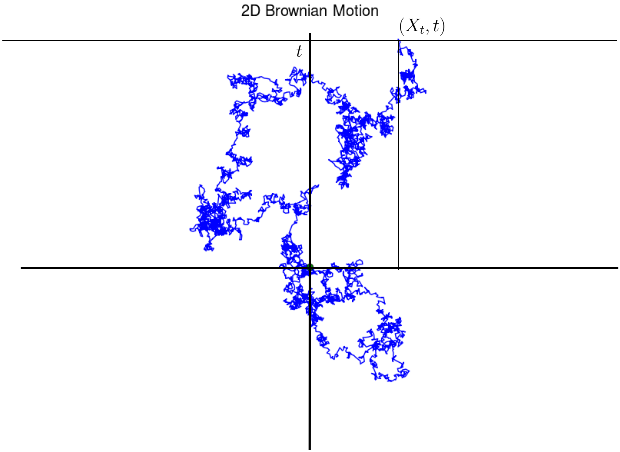


$X_{t+s} - X_t$ has the same law as $tX_1$ and $X_1$ has the Cauchy distribution (*density given by* $(\pi(1+x^2))^{-1}$).

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Motivated by the Cauchy strategy, let's see a **continuum** analog of the probabilistic Kakeya construction of the hunter and rabbit.

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$\text{Leb}(\Lambda) = 0$ and **most importantly** the $\varepsilon$-neighbourhood satisfies almost surely

$$\text{Leb}(\Lambda(\varepsilon)) \asymp \frac{1}{|\log \varepsilon|}$$

# Kakeya sets – Open problems

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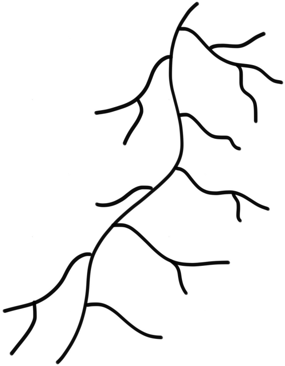

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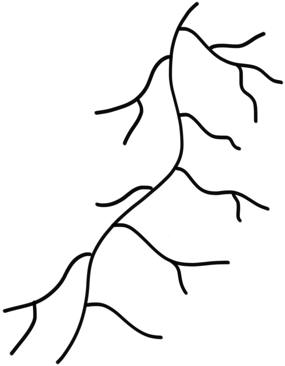

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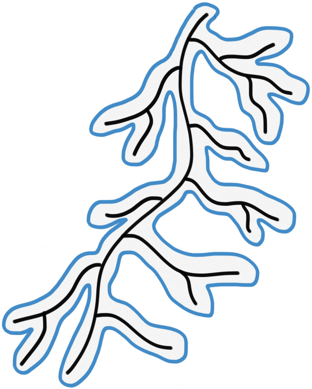

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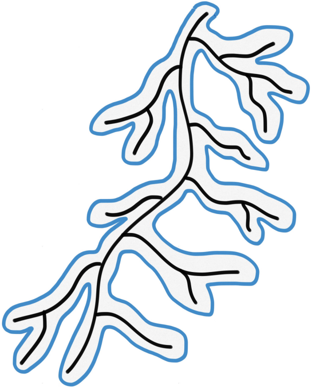

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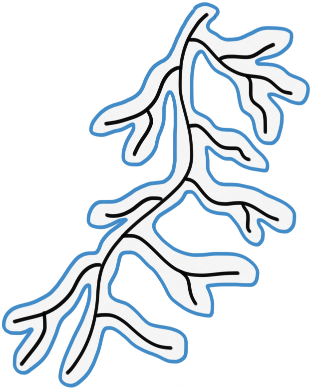
- The hunter can now employ his previous strategy on this path. This will give $O(n \log n)$ capture time.



**On any graph the hunter can catch the rabbit in time $O(n \log n)$.**

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**Open Question: If the hunter and rabbit both walk on the same graph, is the *expected capture time* $O(n)$?**