Convex Optimization for Finding Influential Nodes in Social Networks

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This talk represents work of myself, L. Elkin, T. K. Pong

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Social networks

- Define social networks to be directed graphs, e.g., Twitter posters and followers.
- The network can be either deterministic (arcs pass every message) or probabilistic (arcs flip coins to determine whether to pass messages).
- Maximizing influence means finding the set of nodes (with a constraint or cost on the size of the set) that reaches the largest number of nodes through the network. Many variations of the problem.

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- Influential nodes can be used to spread an important announcement.
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Influence in numerical analysis

Gene Golub, center, with some of his former students at his 60th Birthday celebration in Minneapolis, 1992. Others are, from left, Oliver Ernst, Alan George, Franklin Luk, Jim Varah, Eric Grosse, Petter Bjørstad, Margaret Wright, Gene Golub, Dan Boley, Dianne O'Leary, Steve Vavasis, Tony Chan, Bill Coughran, Michael Heath, Michael Overton, and Nick Trefethen.

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Unweighted deterministic case

- Input: Digraph $G = (V, \mathcal{A})$ and integer k.
- Output: The subset $V^* \subset V$, $|V^*| = k$ that maximizes the number of vertices V' such that there exists a directed path from a node in V^* to a node in V'
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IP formulation–setup

- Let A be the $|V| \times |V|$ matrix of 0's and 1's such that $A(i,j) = 1$ iff there is a directed path from i to j.
- Let $\mathbf{x}$ be the $0-1$ decision vector that indicates membership in the influential subset, and let $\mathbf{t}$ be the $0-1$ vector that indicates whether a node is reached by the influential subset.
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Convex formulation

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- Since 2005, many papers have appeared in the optimization literature showing that convex relaxation can solve an NP-hard problem assuming the problem data satisfies certain assumptions.
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Rationale for this style of analysis

- Many NP-hard data-mining problems are routinely solved in practice using heuristics.
- Users are happy with the results. How is this possible?
- Hypothesis: heuristic algorithms succeed because real data has underlying structure that algorithms can recover.
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Two-layer specialization

- Assume that graph is bipartite, i.e., $V = S \cup R$, and all arcs are directed from S, the *senders*, to R, the *receivers*.
- This is WLOG: Replace original graph by two copies of V; arcs denoted 'reaches' relation.
- Problem is NP-hard since it is equivalent to the set-cover problem: Given a universe $U = \{1, \ldots, n\}$ of objects and a collection $\mathcal{T}$ of subsets of U, find the subcollection $\mathcal{T}^* \subset \mathcal{T}$ minimizing $|\mathcal{T}^*|$ such that $\bigcup_{T \in \mathcal{T}^*} T = U$.

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Generative model: labeling the nodes

- Assume the sender and receiver nodes are partitioned into k *interest groups*: $S = S_1 \cup \cdots \cup S_k$; $R = R_1 \cup \cdots \cup R_k$.
- Assume S_l has exactly one distinguished *influencer*, $l = 1, \ldots, k$.
- Remaining S_l-nodes are called *subordinates.*
- Sought-after IP solution: $x_i = 1$ if i is an influencer; $x_i = 0$ if i is a subordinate.

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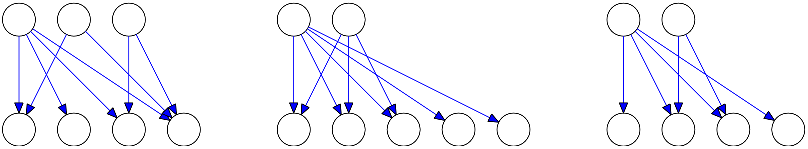

Theorem for simple deterministic case

- **Theorem.** For this model, sought-after IP solution is the unique solution of the LP relaxation.
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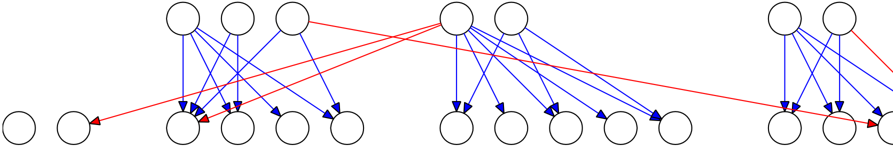

Theorem for noise

- **Theorem.** Convex relaxation exactly recovers IP solution with probability exponentially close to 1 assuming certain bounds on the noise.
- Noise arcs and subordinate-receiver arcs inserted at random according to rules to ensure that that each receiver has degree approximately $\sigma \cdot \min_l |S_l|$ where σ a fixed scalar in $(0, 1)$.
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- Each arc is labeled with a probability.
- Each sender i passes its message to a receiver j with specified probability p_{ij} for all $(i, j) \in \mathcal{A}$. Called the *cascade* model by Kempe, Kleinberg and Tardos (2003).
- Problem: given a digraph labeled with probabilities and an integer k, select the k nodes that reach the greatest expected number of followers (expectation over random choices of successful message transmission).
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Stochastic integer program

- Let $\mathcal{Y}$ denote the finite distribution of possible networks resulting from coin-flips.
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Explicit calculation of expected value

- In the two-layer case, the expected value can be computed in closed form.
- $E[\mathbf{e}^T\mathbf{t}] = \sum_j E[t_j]$.
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Rewritten stochastic IP

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Not SDP-expressible.

Convex solvers

- Interior-point methods for SDP discovered by Alizadeh; Nesterov & Nemirovsky, early 1990s
- Primal-dual path-following proposed by Alizadeh-Haeberly-Overton; H..K..M; and Nesterov-Todd late 1990s.
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Simplifying assumption

Assume all $p_{ij} = p$. Rewritten:

$$\begin{array}{ll} \min & \sum_{j \in R} (1-p)^{A(:,j)^T \mathbf{x}} \\ \text{s.t.} & \mathbf{e}^T \mathbf{x} = k, \\ & \mathbf{x} \in [0,1]^{|S|}. \end{array}$$

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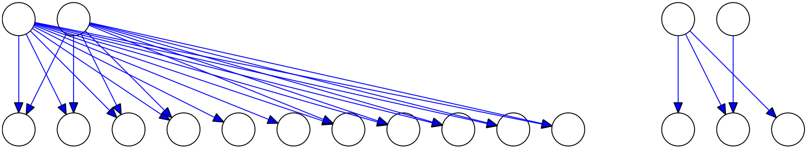

Say $p = 0.5$.

Recovery of influencers

- **Theorem.** Suppose that $\frac{\min_l |S_l|}{\max_l |S_l|} \geq \sqrt{1-p}$ and $\mathbf{x}$ is the solution to the convex relaxation. Then $\mathrm{round}(\mathbf{x})$ contains 1 exactly in the positions of the influencers.
- Notation: $\mathrm{round}(\cdot)$ means round each entry to the nearest integer (0 or 1).
- Proof is an elementary application of KKT conditions but not a clean duality argument.

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- **Theorem.** Suppose that $\frac{\min_l |S_l|}{\max_l |S_l|} \geq \sqrt{1-p}$ and $\mathbf{x}$ is the solution to the convex relaxation. Then $\mathrm{round}(\mathbf{x})$ contains 1 exactly in the positions of the influencers.
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Including noise

- Provided that the number of noise arcs coming into R_l is strongly dominated by the number of nodes of R_l that follow only the influencer, $l = 1, \ldots, k$, a similar result holds with a weaker constant.

Application of convex relaxation to general instances

- Obviously, cannot guarantee exact solution in general case.
- However, if convex relaxation succeeds, can obtain a certificate that optimal integer solution was found.

Future directions

- Multilayer stochastic networks. Kempe et al. have a guaranteed approximation algorithm. Issue: how to evaluate objective function? Sampling?
- More realistic generative model, e.g., forest fire model of Leskovec, Kleinberg, Faloutsos.