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Differential equations for the approximation of the distance to the closest defective matrix.

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and

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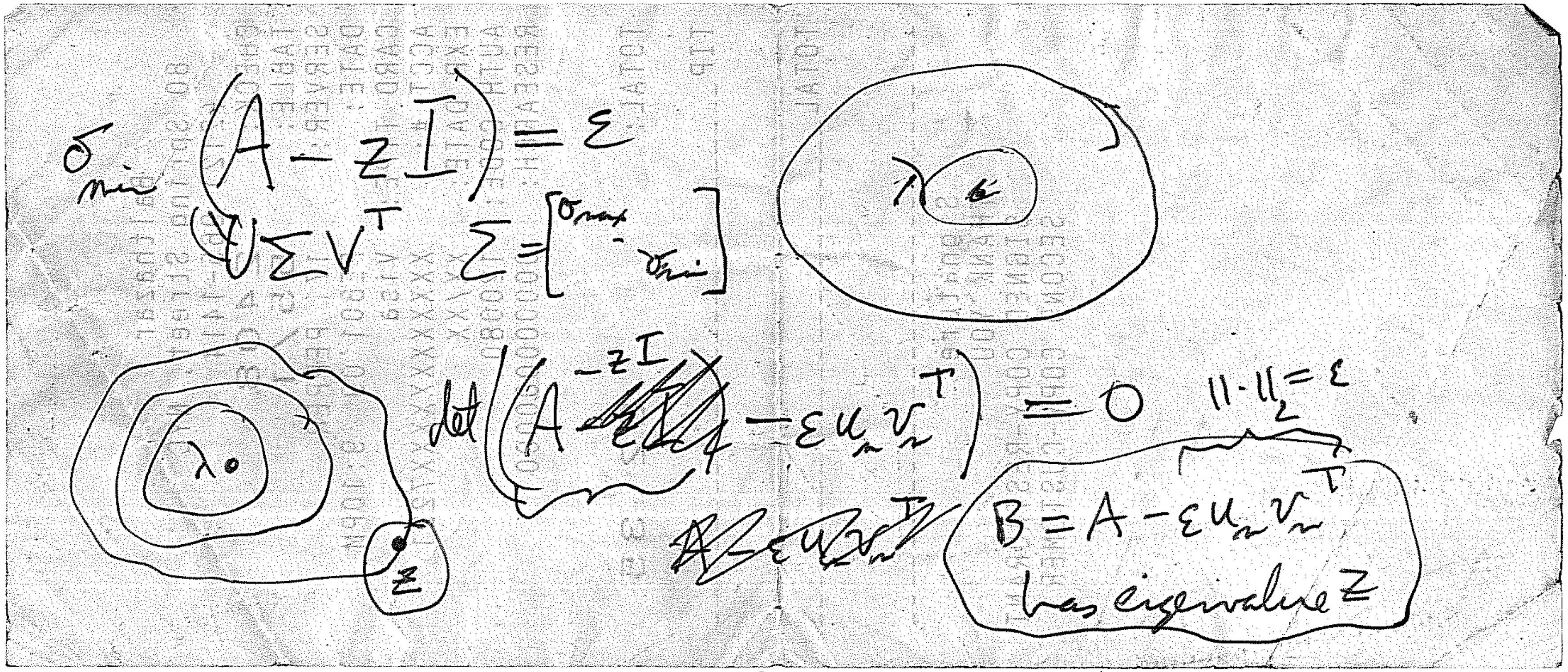
Dedicated to Michael Overton.

Preamble

In Winter 2009 I visited Michael; during a party at Courant, I asked Michael how to obtain extremal perturbations associated to a boundary point in the ε-pseudospectrum . . .



This is his answer on a receipt of Whole Foods.

Summary

- Problem and literature.
- Low-rank odes and extremal pseudo-eigenvalues.
- Theoretical properties and examples.
- Extension to structured problems.

Problem

Framework: Let $A \in \mathbb{K}^{n,n}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$) a matrix with all distinct eigenvalues. We denote by $\Lambda(A)$ the spectrum of A.

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The distance to defectivity is defined as

$$w_{\mathbb{K}}(A) \; = \; \inf\left\{ \|A - B\| : \, B \in \mathbb{K}^{n,n} \text{ is defective} \right\}$$

where, in this talk, $\| \cdot \|$ denotes here the Frobenius norm.
If $\mathbb{K} = \mathbb{C}$ the 2-norm is equivalent, that means $w_{\mathbb{K}}(A)$ is the same number; but this not true in general for $\mathbb{K} = \mathbb{R}$.

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Generically we expect that an extremizer $B_{\text{opt}} \in \mathbb{K}^{n,n}$ (if exists) has a coalescent defective pair of eigenvalues.

Some literature

First $w_{\mathbb{C}}(A)$ was introduced by **Demmel** (1983) in his very well-known PhD thesis under the name $diss(A, path)$, $path$ referring to the path traveled by the eigenvalues in the complex plane under a smoothly varying perturbation to A.

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The very interesting recent article by **Alam, Byers, Bora & Overton** (2011) shows that for $\mathbb{K} = \mathbb{C}$ the infimum is indeed a minimum. For approximating $w_{\mathbb{C}}(A)$, they also proposed an algorithm which is well-suited to problems of moderate size.

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Apparently the case $\mathbb{K} = \mathbb{R}$ is unexplored. Similarly there seem to be no methods to approximate any structured distance.

Methodology: two steps

(i) For a given ε we aim to approximate the quantity

$$r(\varepsilon) = \min\Big\{ y^* x : y \text{ and } x \text{ left/right eigenvectors to } \lambda \in \Lambda\left(A + \varepsilon E\right) \text{ for some } E : \ \|E\| \leq 1 \Big\},$$

with x and y normalized as: $\|x\| = \|y\| = 1, y^* x \geq 0$.

Connection: ε-pseudospectrum (**Trefethen & Embree** (2005))

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Meaning. If $\mathbb{K} = \mathbb{C}$ at a *locally minimal* solution two discs in ε-pseudospectrum have a contact point (**Alam & Bora** (2005))

Also interesting to consider $r(\varepsilon) = \delta$ for a small threshold δ.

Constructing a path for the eigenvalues

Part (i): we construct a smooth matrix valued function

$$A + \varepsilon E(t) \qquad \text{where} \quad \|E(t)\| = 1.$$

Normalization: any selected pair of left/right eigenvectors of $A + \varepsilon E(t)$ is such that $\|x(t)\| = \|y(t)\| = 1,\ y(t)^* x(t) > 0$.

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Desired properties

(a) the function $y(t)^* x(t)$ is decreasing:

(b) $\lim_{t \to \infty} E(t) = E_\infty$

(c) $y_\infty^* x_\infty$ *local* minimum of the function $y^* x(E) : \mathbb{K}^{n,n} \to \mathbb{R}^+$

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Idea: look for steepest descent direction $\dot{E}$ for $y(t)^* x(t)$, using

$$\frac{d}{dt}\left(y(t)^* x(t)\right) = \dot{y}(t)^* x(t) + y(t)^* \dot{x}(t).$$

Derivatives of eigenvectors

Proposition (**Meyer & Stewart** (1988))

Let the matrix $M(t)$ be smooth w.r.t. $t \in \mathbb{R}$, $\lambda(t)$ a simple eigenvalue with normalized left/right eigenvectors $y(t), x(t)$.

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Then the following hold:

$$\dot{x} = x^* G \dot{M} x x - G \dot{M} x$$

$$\dot{y}^* = y^* \dot{M} G y y^* - y^* \dot{M} G$$

where we omit the explicit dependence on t.

Steepest descent direction lemma

Let y and x left and right eigenvectors of $A + \varepsilon E$ associated to λ and G the group-inverse of $A + \varepsilon E - \lambda I$. Then set

$$S = yy^* G^* + G^* xx^* \ .$$

Let $\mathcal{B}$ the unit ball of the Frobenius norm.

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Then (1) for any smooth path $E(t) \in \mathcal{B}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(y^*x) \;\; = \;\; \varepsilon\, y^*x\,\mathrm{Re}\;\langle \dot{E}, S\rangle \ .$$

where $\langle A, B\rangle = \mathrm{trace}\,(A^*B)$ is the Frobenius inner product.

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where $\langle A, B \rangle = \mathrm{trace}\,(A^* B)$ is the Frobenius inner product.

Moreover (2) the steepest descent direction for $y^* x$ in the tangent hyperplane $T_E \mathcal{B}$ is given by

$$\dot{E} = D = -\mu \big(S - \mathrm{Re}\, \langle E, S \rangle E \big) \quad \text{with } \mu \text{ normalizing factor.}$$

Steepest descent ode

We consider the ODE

$$\dot{E} = -\big(S - \mathrm{Re}\,\langle E, S\rangle E\big), \qquad E(0) \in \mathcal{B}\ .$$

Let $c(t) = y(t)^* x(t)$, $y(t), x(t)$ being the normalized left/right eigenvectors associated to an eigenvalue $\lambda(t)$ of $A + \varepsilon E(t)$.

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Properties of ODE

(1) Norm conservation: $\|E(t)\| = 1$ for all t;

(2) Monotonicity: $c(t)$ decreasing along solutions of ODE;

(3) Stationary points: the matrix S does never vanish and the following statements are equivalent:

$$\dot{c} = 0 \quad \Longleftrightarrow \quad \dot{E} = 0 \quad \Longleftrightarrow \quad E \text{ is real multiple of } S.$$

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The associated λ represents an extremal ε-pseudo-eigenvalue.

Projection onto the tangent space of $\mathcal{M}_2$

Key property: stationary points have rank-2.

Consider a new ODE on the manifold $\mathcal{M}_2$ of rank-2 matrices by F-orthogonal projection $\mathbf{P}_E$ to tangent space $T_E\mathcal{M}_2$:

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Writing

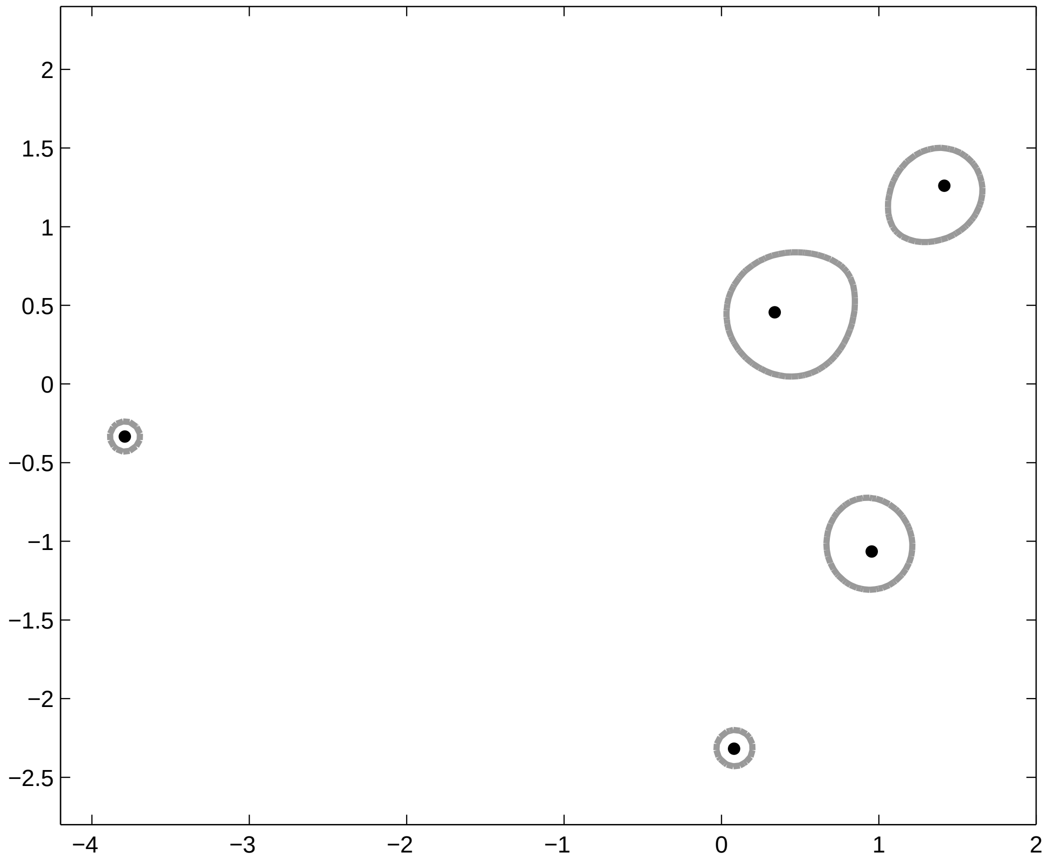
$$E = UTV^*$$

where $U, V \in \mathbb{C}^{n\times 2}$ have orthonormal columns and $T \in \mathbb{C}^{2\times 2}$ invertible, we are able to write a system of ODEs for U, V, T.

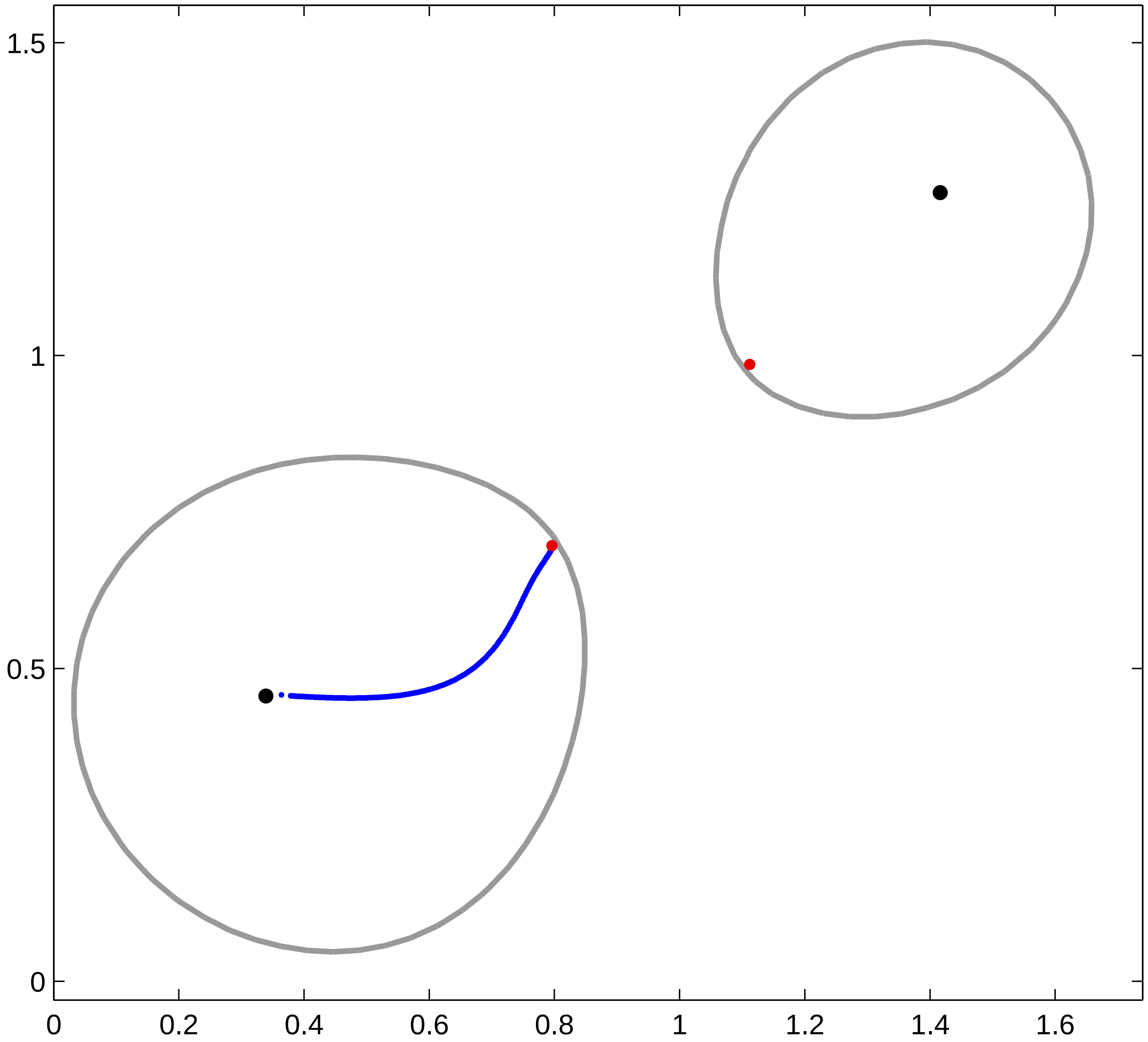
Example 1

$$A = \begin{pmatrix} 0 & 1+\mathbf{i} & 2+\mathbf{i} & 1+2\mathbf{i} & 1 \\ -1 & -1-\mathbf{i} & 1-\mathbf{i} & -\mathbf{i} & 0 \\ 1-\mathbf{i} & -1-2\mathbf{i} & 1+2\mathbf{i} & -2\mathbf{i} & 0 \\ 1-2\mathbf{i} & 1-\mathbf{i} & -1+2\mathbf{i} & -1-\mathbf{i} & 0 \\ 1 & -1-\mathbf{i} & 2\mathbf{i} & -1-\mathbf{i} & -2\mathbf{i} \end{pmatrix}$$

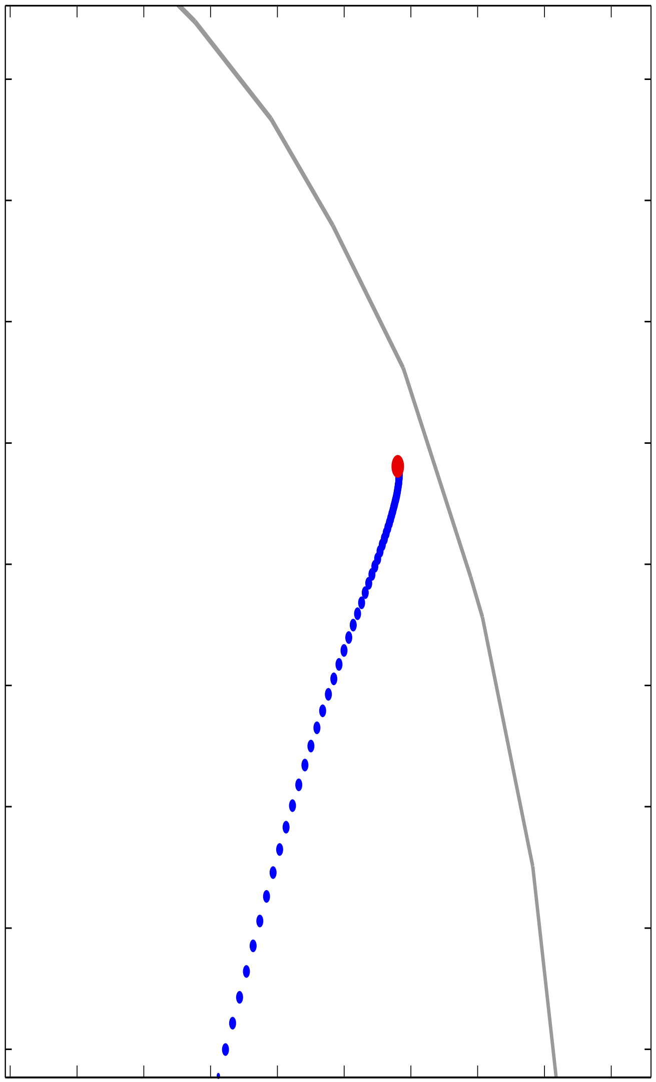
Pseudospectrum
($\varepsilon = 0.075$)



Trajectory of the ODE



Trajectory
in the ε-pseudospectrum



Zoom close
to boundary

Approximating the distance to defectivity

Part (ii). Let $\delta \geq 0$. In order to find an approximate solution of the minimization problem (slight generalization of $\delta = 0$)

$$\varepsilon^{\delta,*} \longrightarrow \min\{\varepsilon : \; r(\varepsilon) = \delta\}$$

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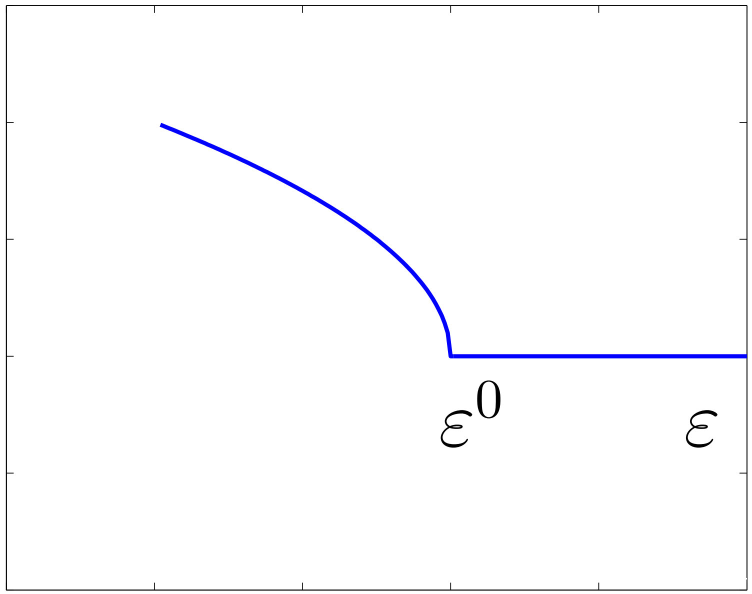
Modeling $r(\varepsilon)$

Under generic assumptions we get the expansion for $\varepsilon \leq \varepsilon^0$,

$$\begin{aligned} r(\varepsilon) &= \gamma\sqrt{\varepsilon^0 - \varepsilon} \\ &+ \mathcal{O}\big((\varepsilon^0 - \varepsilon)^{3/2}\big). \end{aligned}$$



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First order expansion

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Algorithm

Compute $r(\varepsilon)$ by solving the ODE and $dr(\varepsilon)/d\varepsilon$ by an exact inexpensive formula. Estimate γ and ε^0 and solve $r(\varepsilon) = \delta$. This yields a quadratically convergent method to ε^δ.

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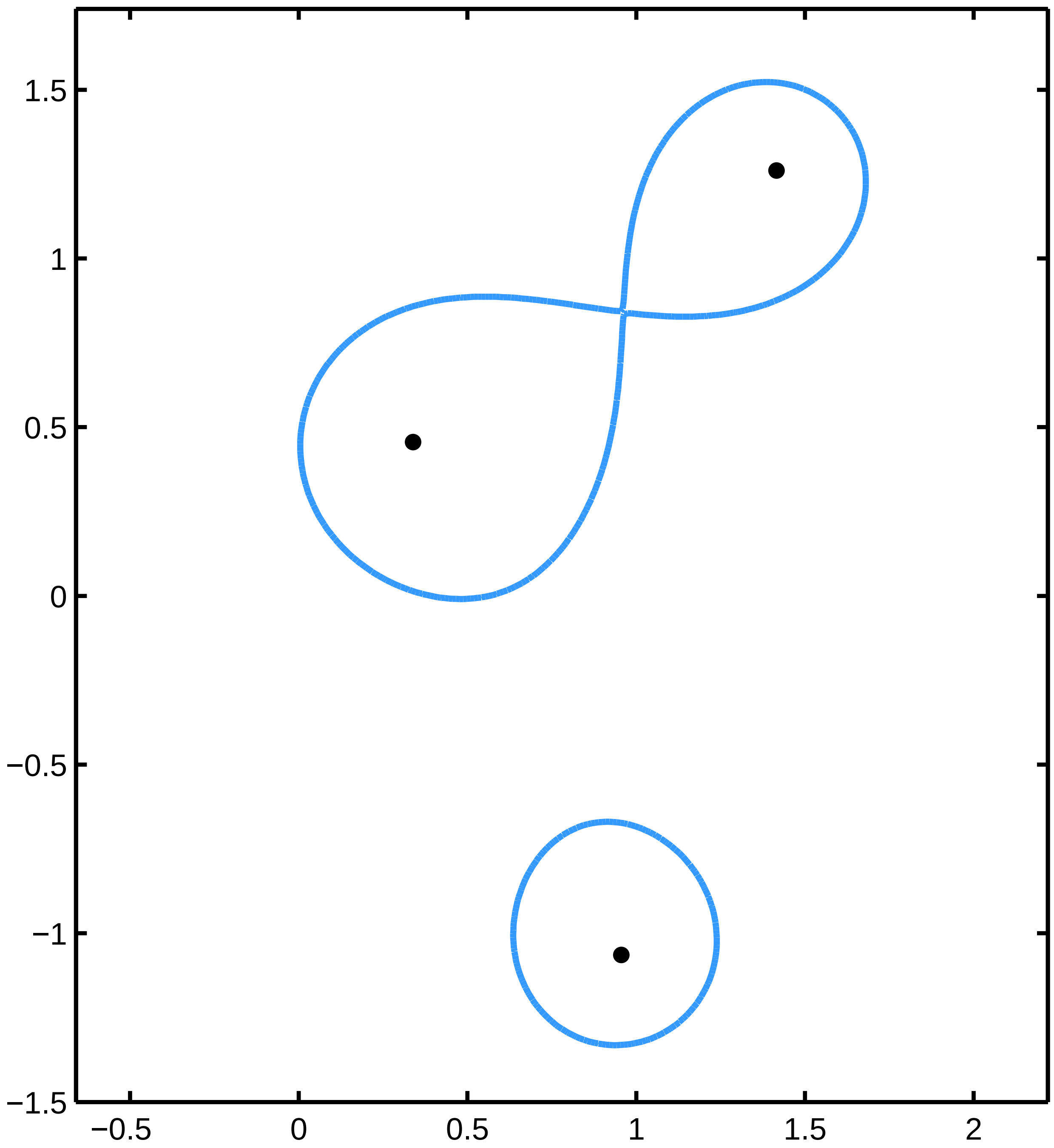
Example 1 ($\delta = 10^{-4}$)

k	ε_k^δ	$r(\varepsilon_k^\delta)$
7	0.082876946962636	0.000**9**10106101987
8	0.082876706789675	0.000**9999**89689847
9	0.082876706760826	0.000**9999999999**761

Example 1: ε^0-pseudospectrum



Real-structured distance

Step (ii) is unaltered. Step (i): the modified ODE

It is sufficient to replace S by $\mathrm{Re}\,(S)$ in the complex ODE and observe that stationary points are now real rank-4 matrices. We also prove that $\mathrm{Re}\,(S)$ does never vanish if A is not normal.

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Projected ODE

By F-orthogonal projection $\widetilde{\mathbf{P}}_E$ to tangent space $T_E\mathcal{M}_4$ of the manifold of real 4×4-matrices, we get

$$\dot{E} = -\widetilde{\mathbf{P}}_E\Big(\mathrm{Re}\,(S) - \mathrm{Re}\,\langle E, \mathrm{Re}\,(S)\rangle E\Big).$$

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Properties

(1) Monotonicity: $\dot{c} \leq 0$;

(2) Stationary points: same as unprojected ODE: $E \propto \mathrm{Re}\,(S)$.

Sparsity pattern ($\mathcal{P}$) structure

The sparsity preserving ODE

Is sufficient an F-orthogonal projecton of S onto $\mathcal{P}$ i.e. setting to zero all elements of S corresponding to zero elements of $\mathcal{P}$.

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Example 2 (Grcar matrix)

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Distances

$$w_{\mathbb{C}}(A) \approx 0.2151857$$

$$w_{\mathbb{R}}(A) \approx 0.3007253$$

$$w_{\mathbb{C},\mathcal{P}}(A) \approx 0.6845324$$

$$w_{\mathbb{R},\mathcal{P}}(A) \approx 0.9423366$$

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Large sparse problems may exploit the low rank-structure and computing efficiently the group-inverse (project with Michael).