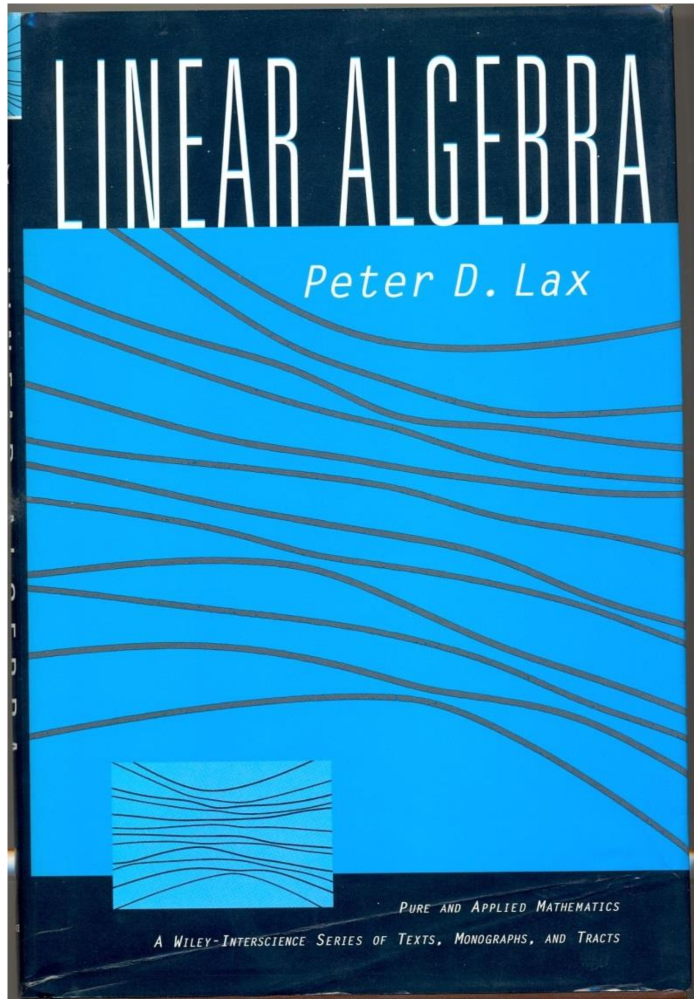
Eigenvalue avoided crossings

Nick Trefethen, Oxford U.


LINEAR ALGEBRA
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PURE AND APPLIED MATHEMATICS
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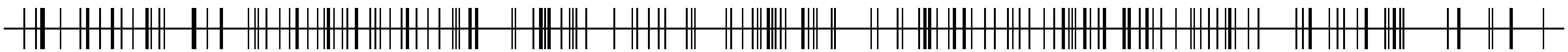
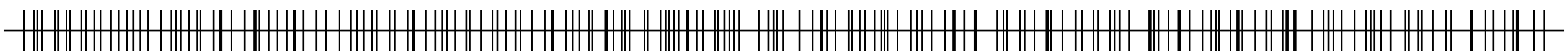
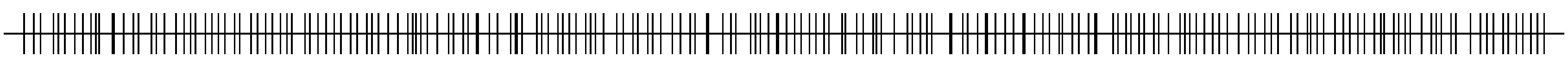
200 random points in an interval



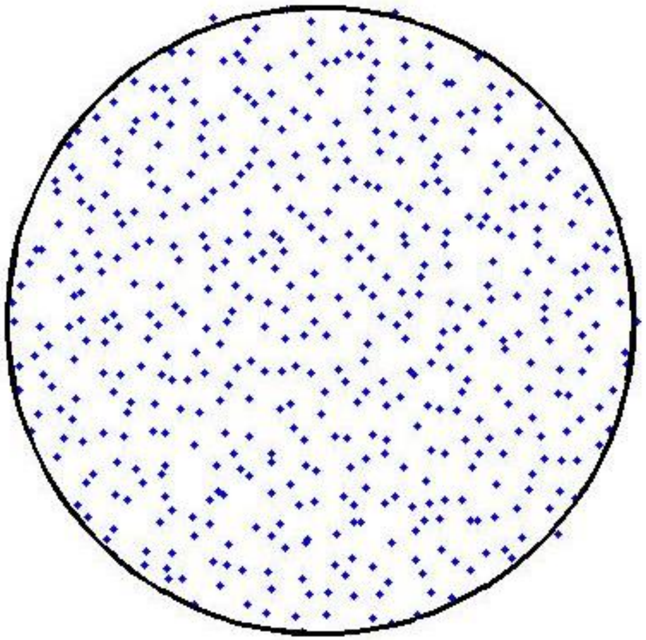
200 middle eigenvalues of a random real symmetric matrix



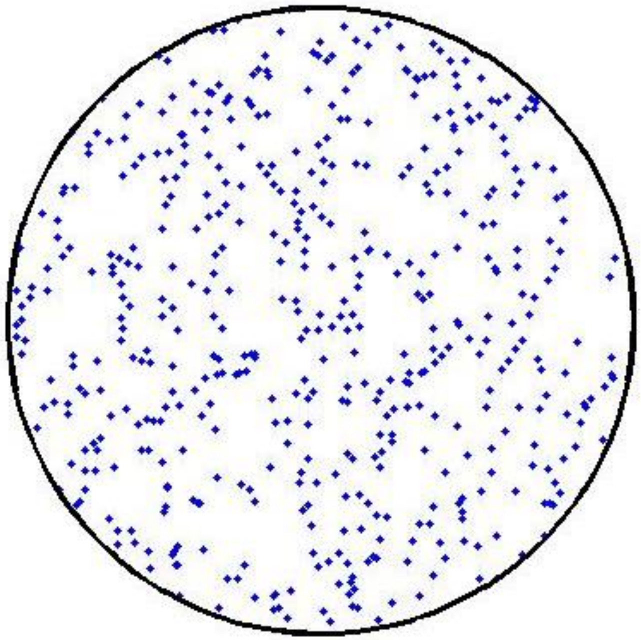
200 middle eigenvalues of a random complex hermitian matrix



nonsymmetric analogue



eigenvalues of random 400×400 nonsymmetric matrix



400 random points in a disk

Explanation

2x2 real symmetric $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ vs. $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

codimension 3–1 = 2

2x2 hermitian $\begin{pmatrix} a & b+ci \\ b-ci & d \end{pmatrix}$ vs. $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

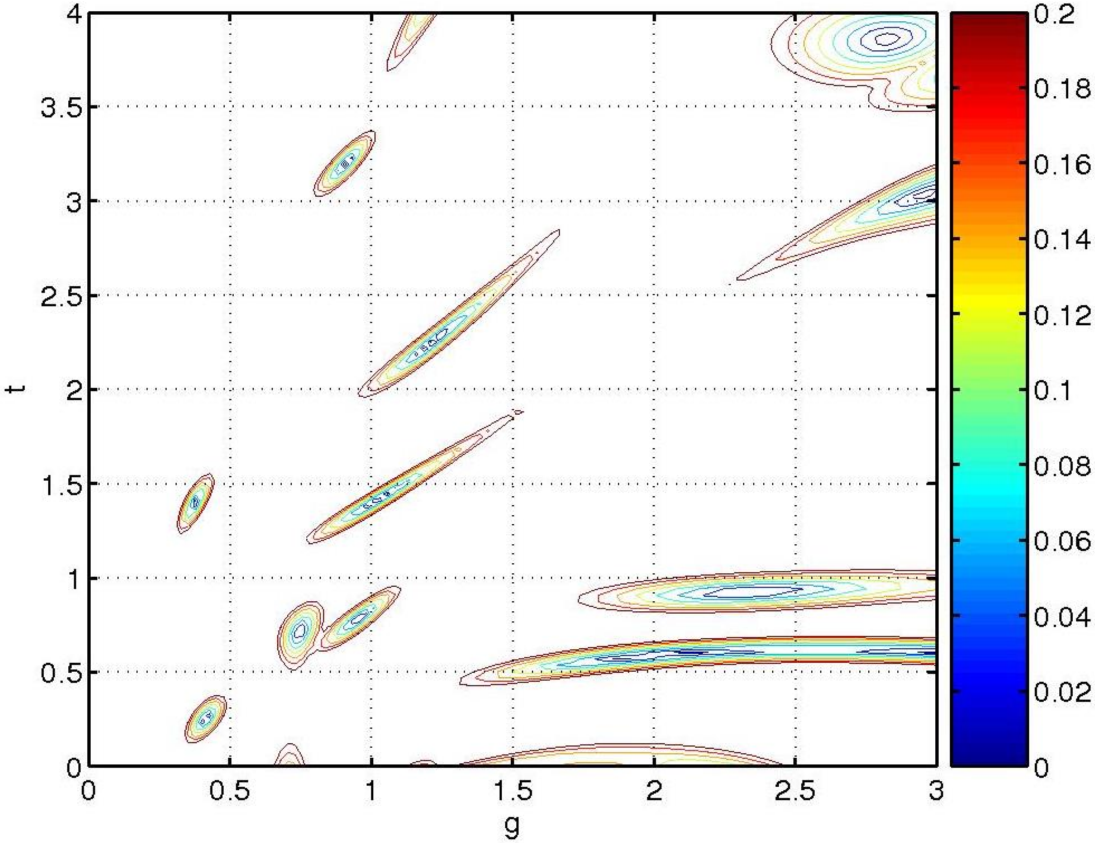
codimension 4–1 = 3

Hund 1927, Von Neumann & Wigner 1929, Teller 1937

From the Oxford Problem Solving Squad

Consider the 10×10 *matrices* $D = \mathrm{diag}(1, \ldots, 10)$, $T = \mathrm{tridiag}(1, 2, 1)$, $G = S^T S$, *where* $s_{i,j} = \sin(i \cdot j)$, *and define* $A(g,t) = D + gG + tT$. *What is the smallest* λ *which is a double eigenvalue of* $A(g,t)$ *for some* $g, t > 0$?

contour plot of min(diff(sort(eig(A(g,t)))))



Chebfun code for eigenvalues of (1–t)A+tB

This code computes an ∞ x 10 quasimatrix whose columns are the eigenvalues of (1–t)A+tB as functions of the parameter t .

```
n = 10;
A = randn(n); A = A+A';
B = randn(n); B = B+B';
ek = @(e,k) e(k);                  % returns kth element of vector e
eigA = @(A) sort(eig(A));          % returns sorted eigenvalues of matrix A
eigk = @(A,k) ek(eigA(A),k);       % returns kth eigenvalue of matrix A
E = chebfun;
for k = 1:n
   E(:,k) = chebfun(@(t) eigk((1-t)*A+t*B,k),[0 1],'vectorize');
end
plot(E)
```

repel.m

Riemann hypothesis

The zeros of ζ(s) on the critical line appear to be distributed like eigenvalues of random hermitian matrices.

A. M. ODLYZKO

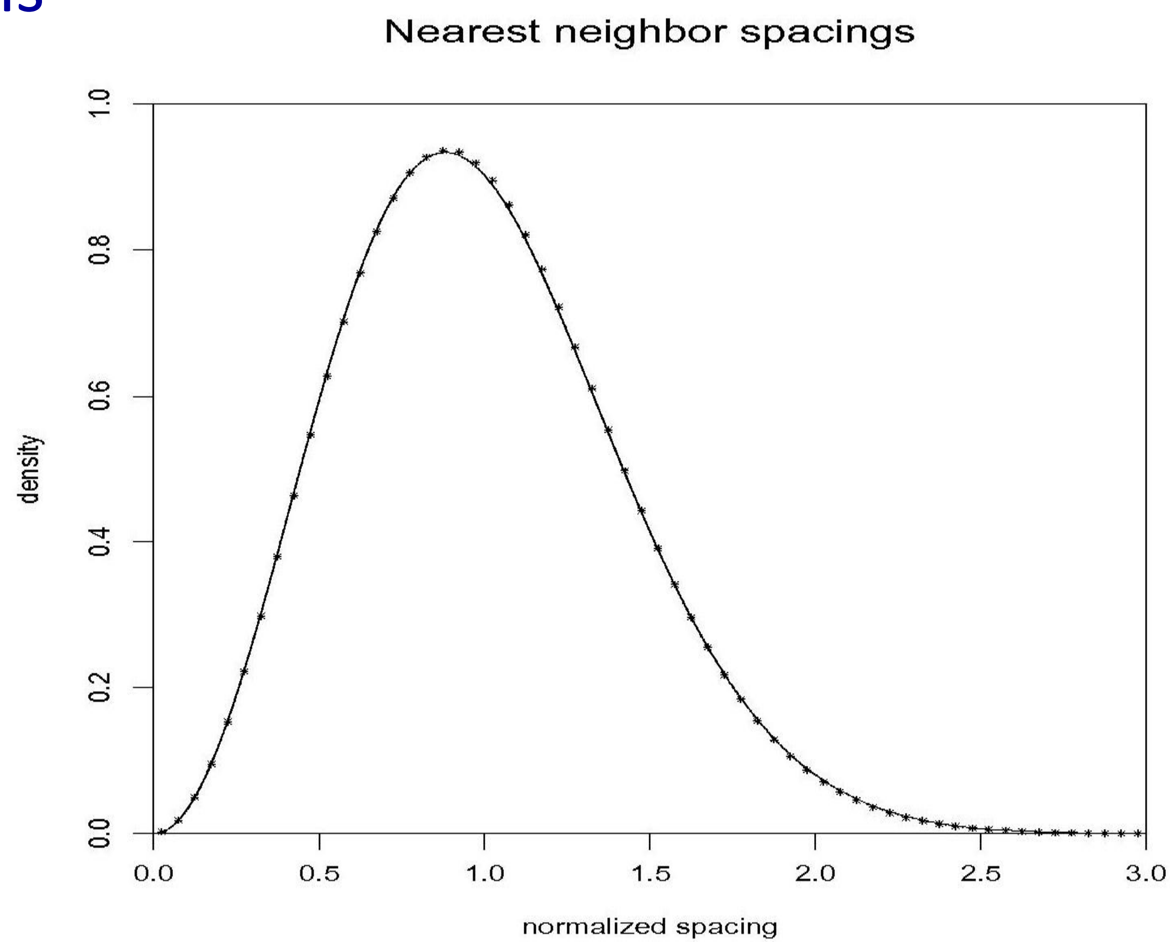


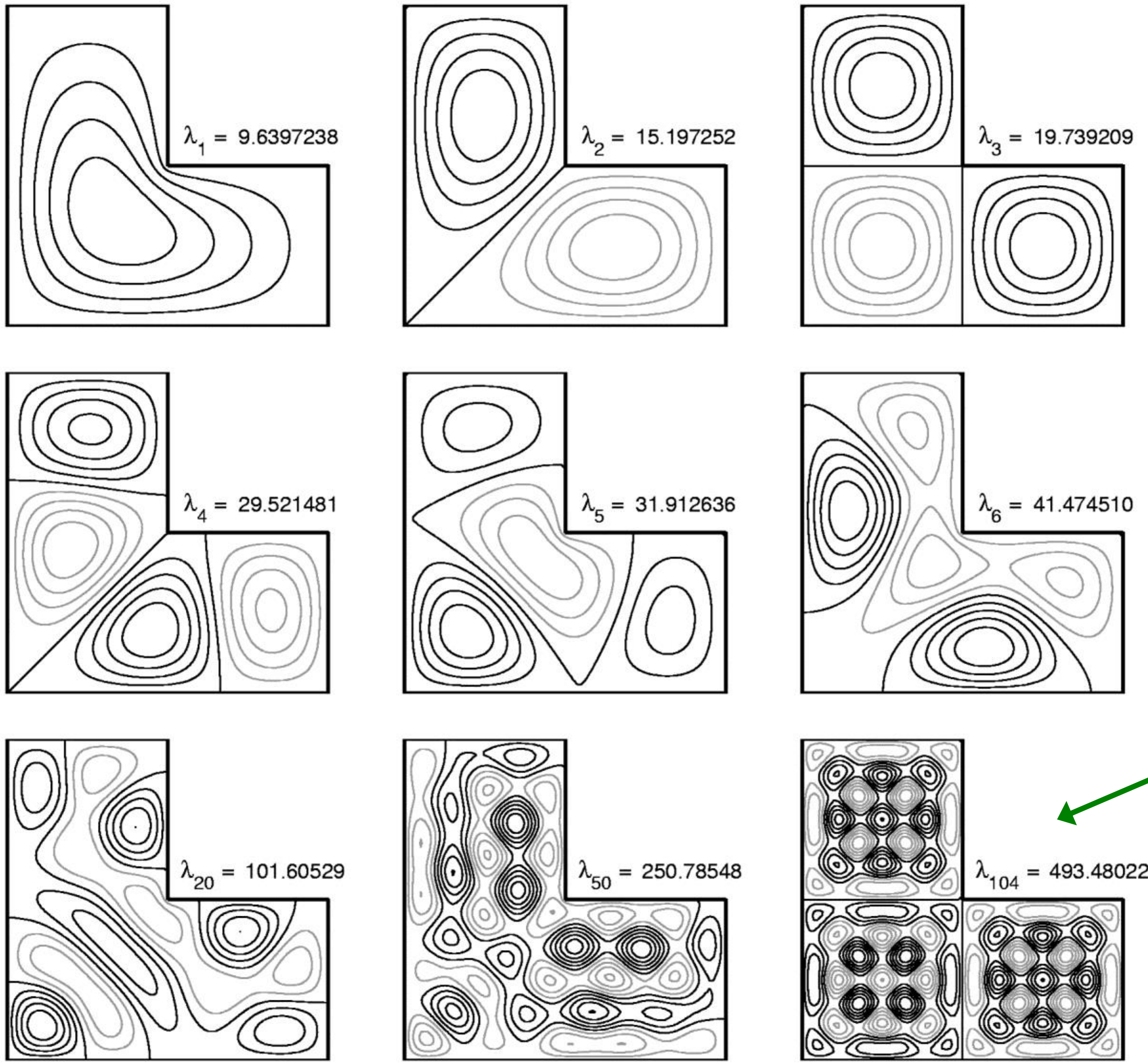
FIGURE 1. Probability density of the normalized spacings δ_n. Solid line: Gue prediction. Scatterplot: empirical data based on a billion zeros near zero # $1.3 \cdot 10^{16}$.

Planar drums

Eigenvalues of the Laplacian with Dirichlet BCs.
Analogous to real symmetric matrices.

from joint work with Timo Betcke

L shape — nongeneric

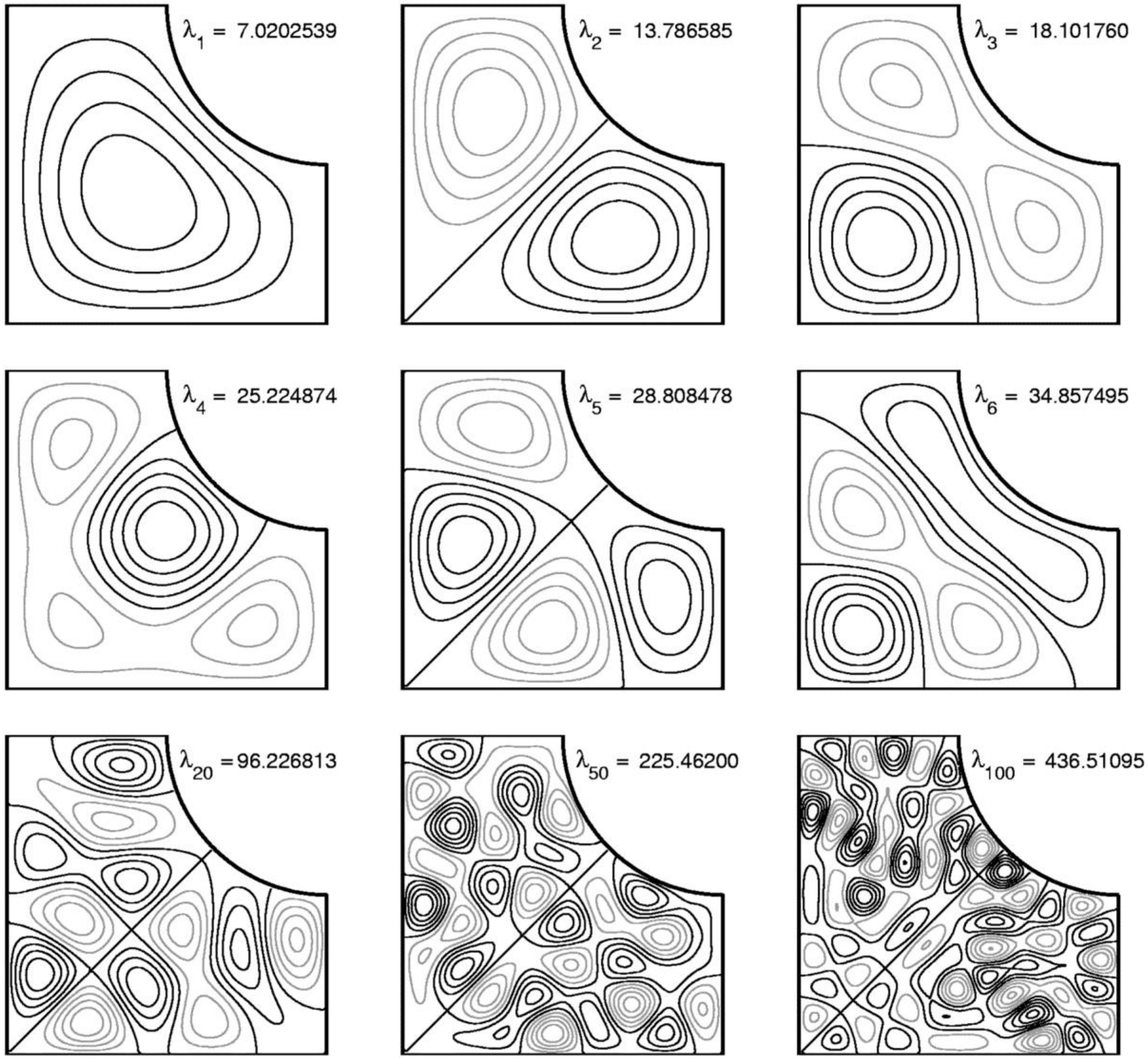


triple degeneracy:

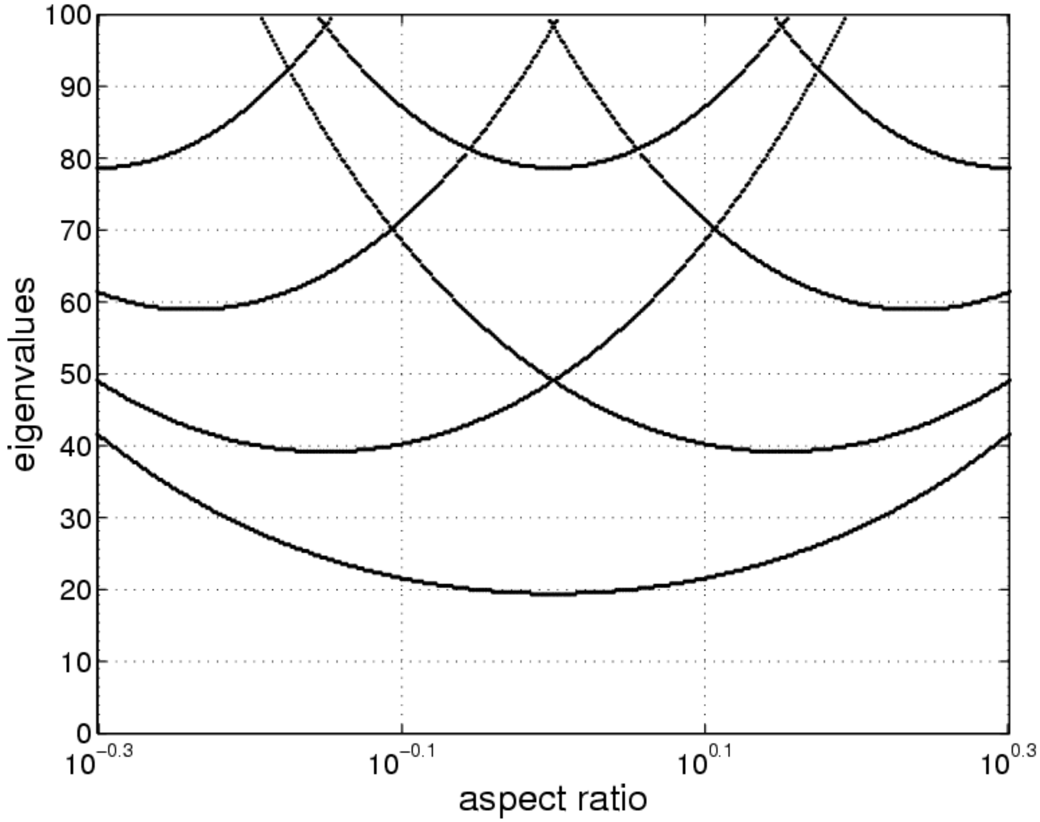
$5^2 + 5^2 =$
$1^2 + 7^2 =$
$7^2 + 1^2 = 50$

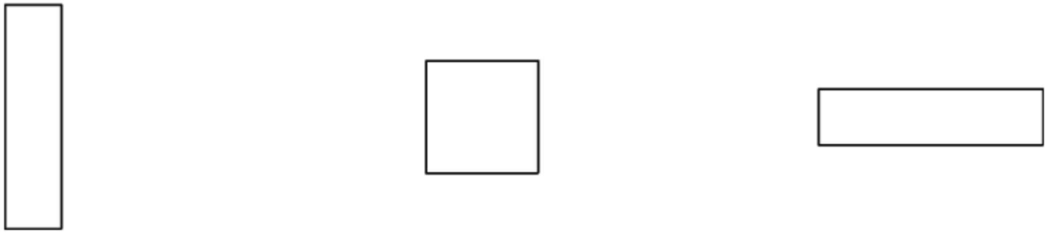
(2, 50, 325, 1105, 8125, ...)

Circular L shape — generic (presumably)

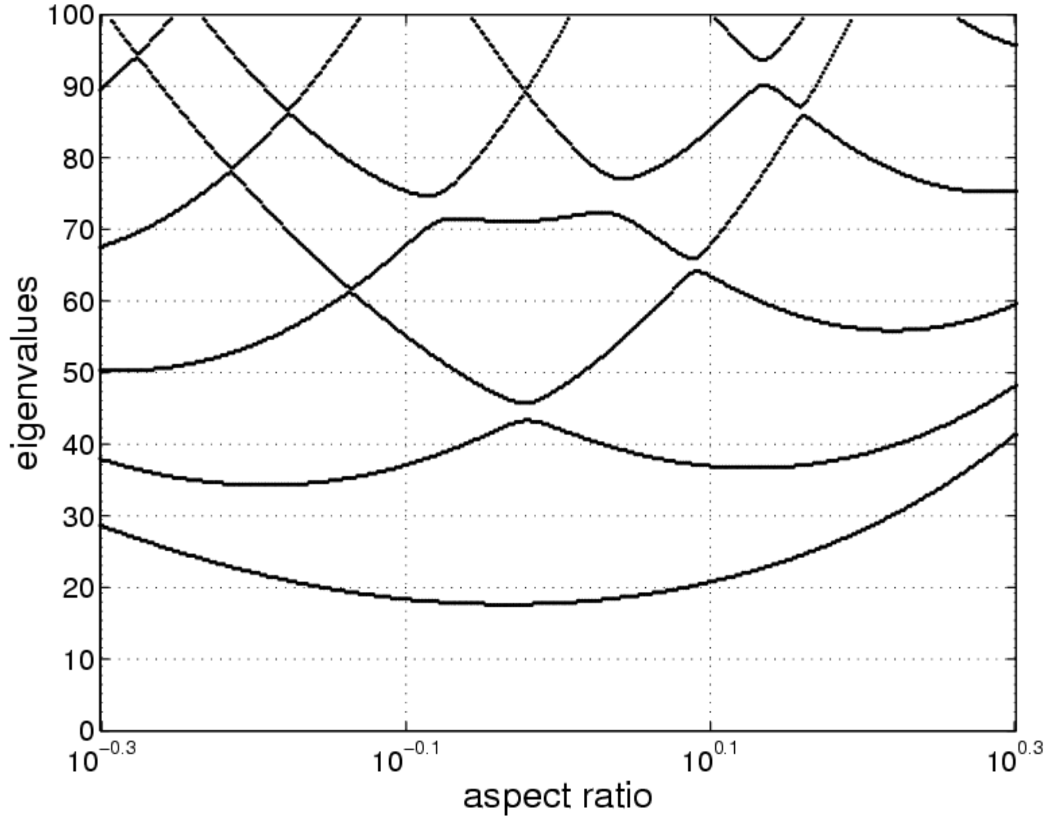


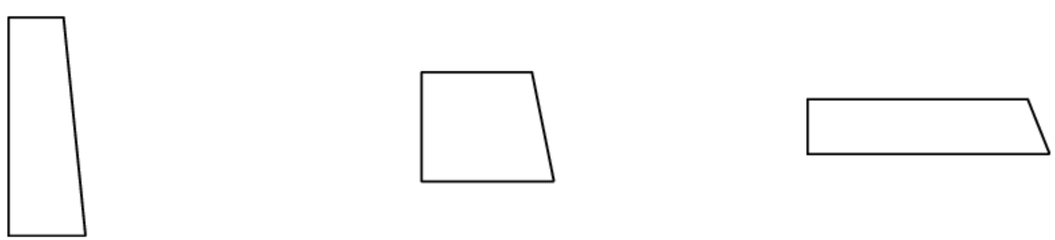
Eigenvalue crossings for rectangles



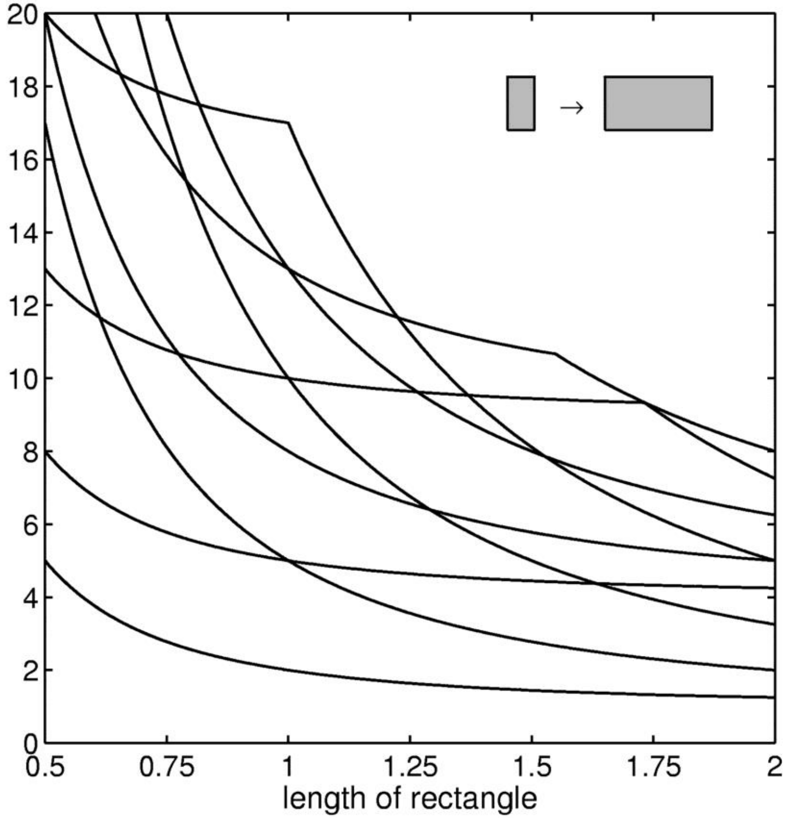


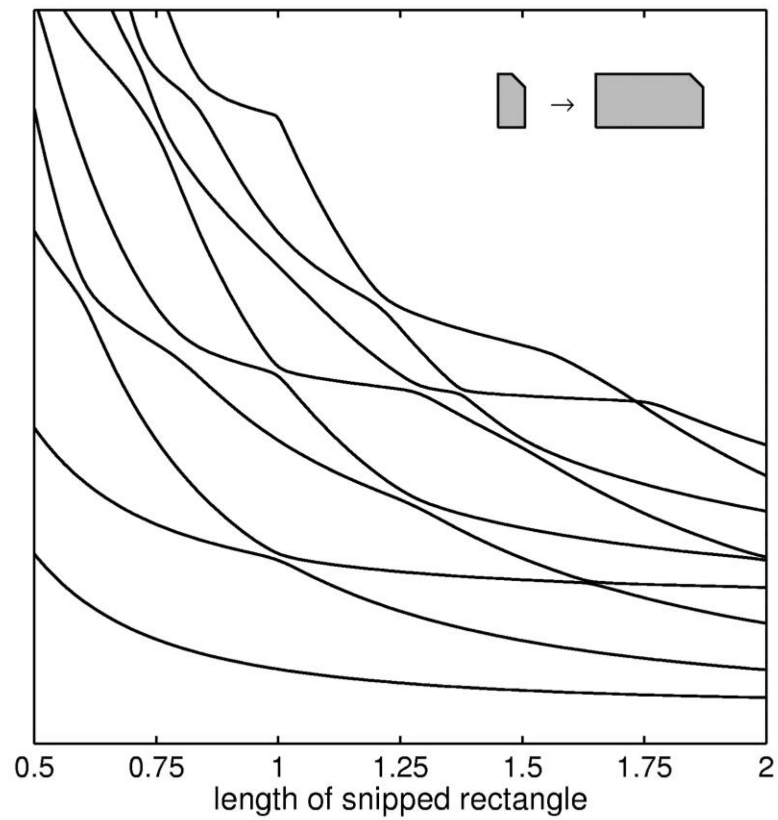
Eigenvalue avoided crossings for not-quite-rectangles





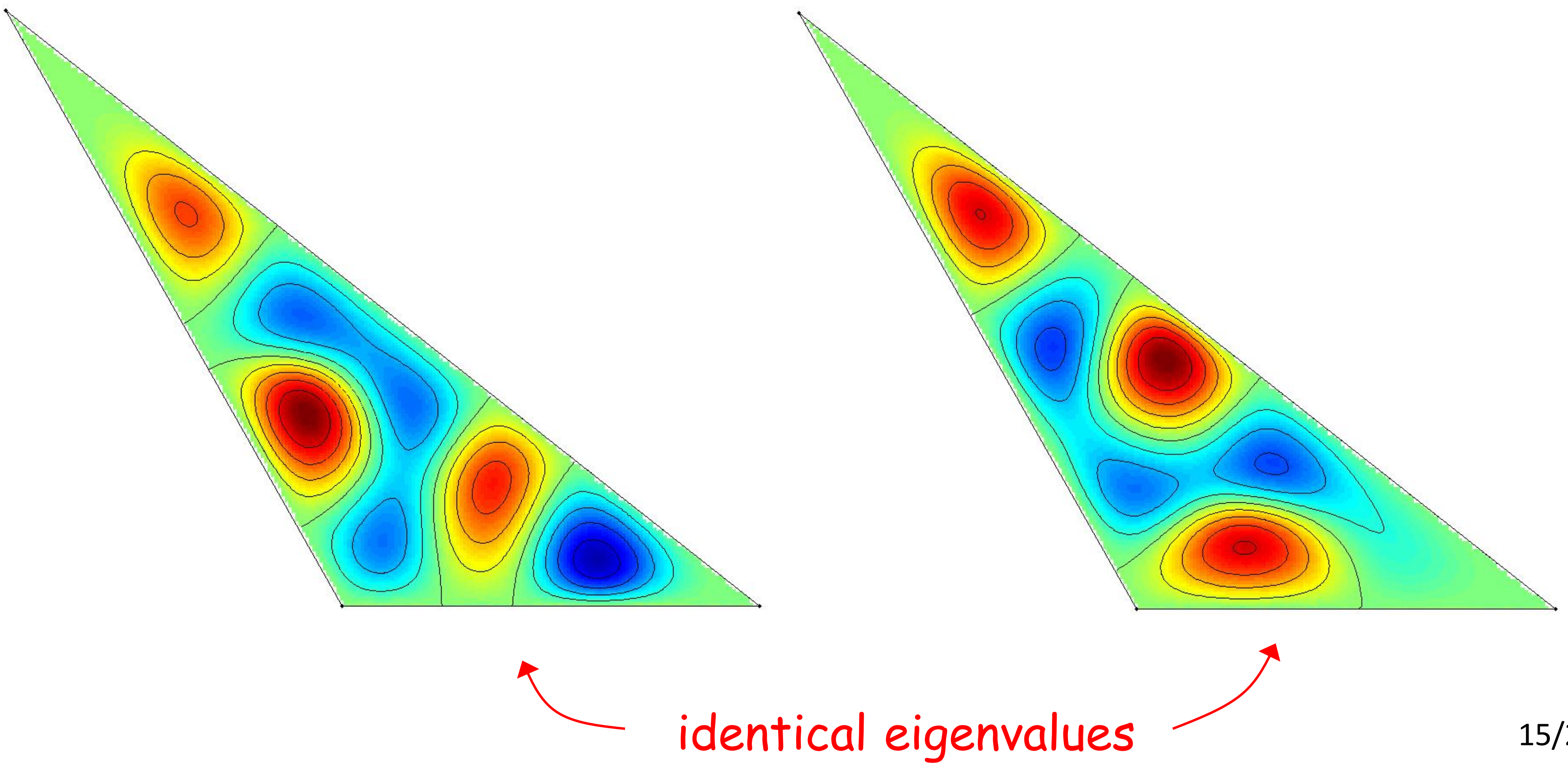
Another example of avoided crossings





Triangles with “accidental” degeneracy

The shape of a triangle is determined by two real parameters. You can tune them to find triangles with degenerate eigenvalues. Berry & Wilkinson, 1984.



Topology of these "diabolo" points

A fascinating subject related to Berry phase and many other things.

Von Neumann, Lax, Arnol'd, Berry, Overton....

We trace around a diabolo point in 2D parameter space. When we return where we started, the eigenfunction has changed sign.

From here it's a short step to the Pauli exclusion principle, the periodic table of the elements, and the structure of the universe.

32 M. V. Berry and M. Wilkinson

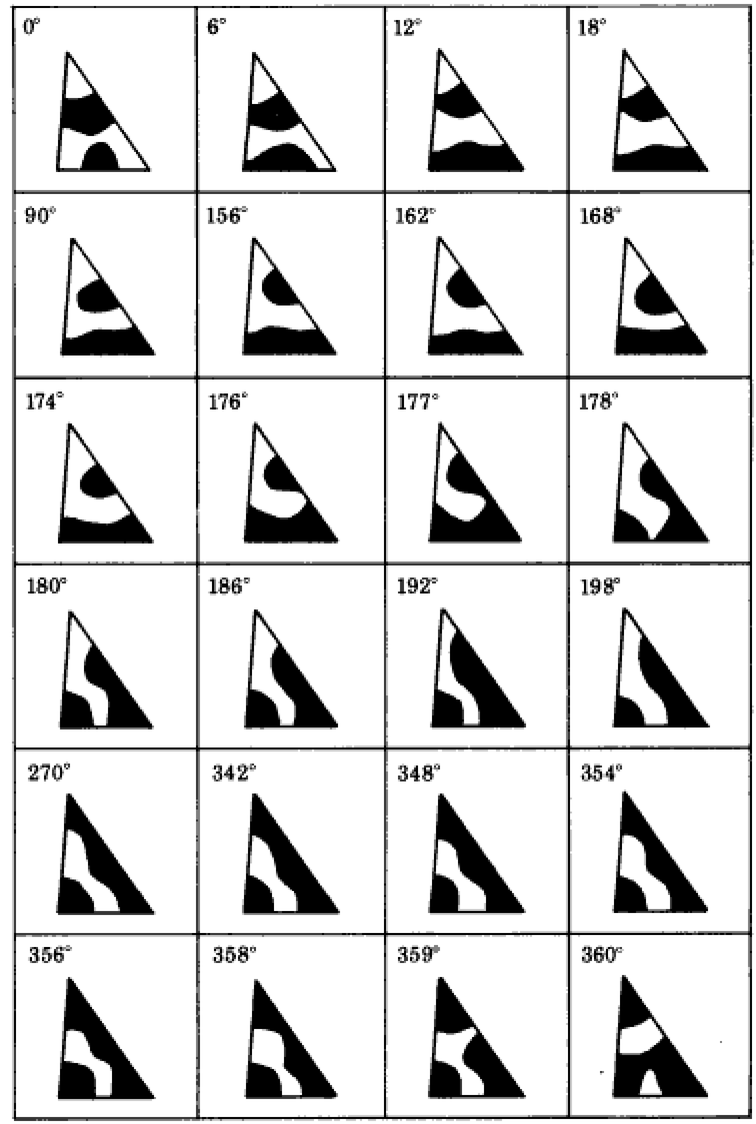


FIGURE 10. Positive (black) and negative (white) regions of a wavefunction during a circuit (in the X–Y plane) of the diabolical point between levels 6 and 7, for the angles indicated (measured from the X axis).

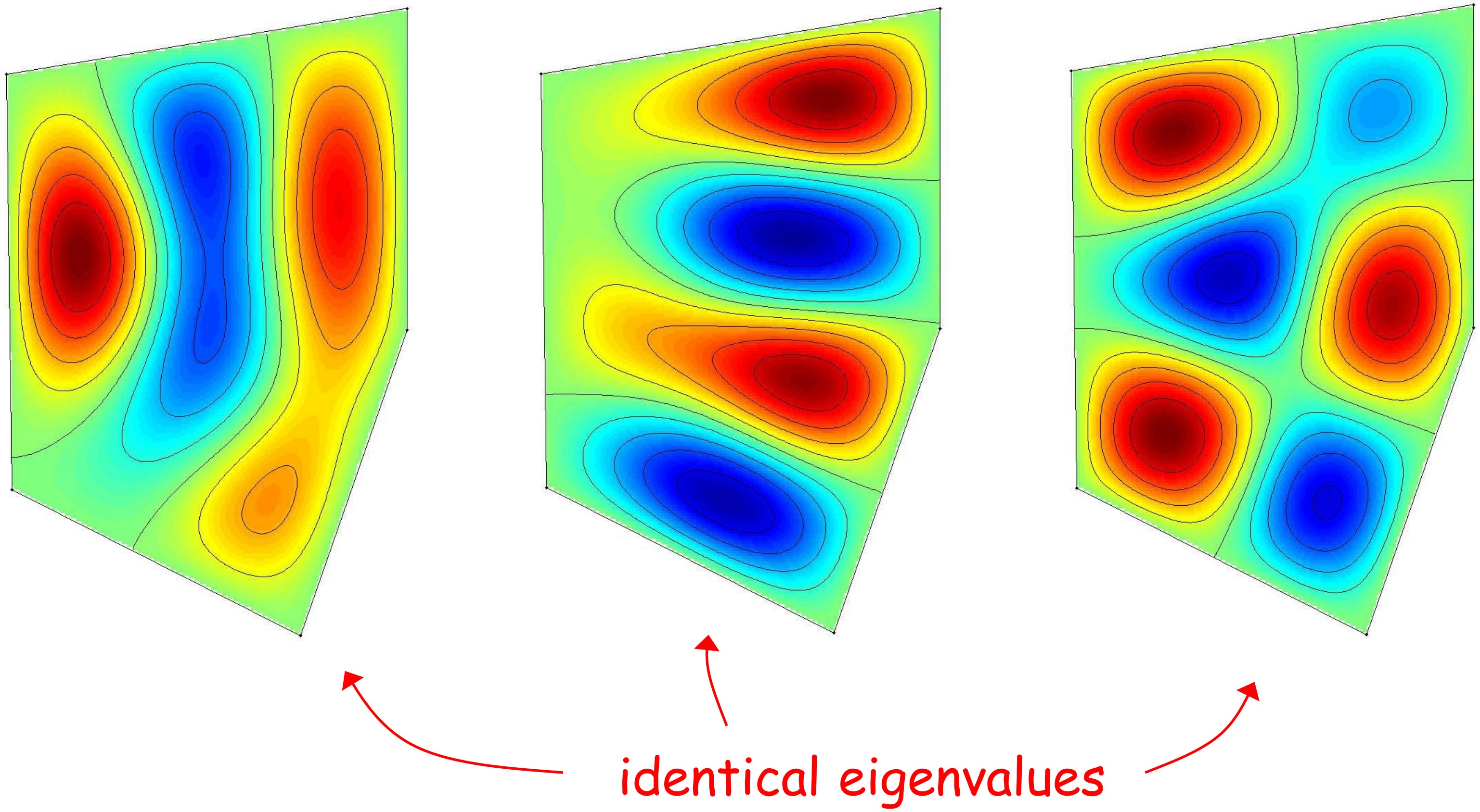
What about double degeneracy?

3x3 real symmetric $\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$ vs. $\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$

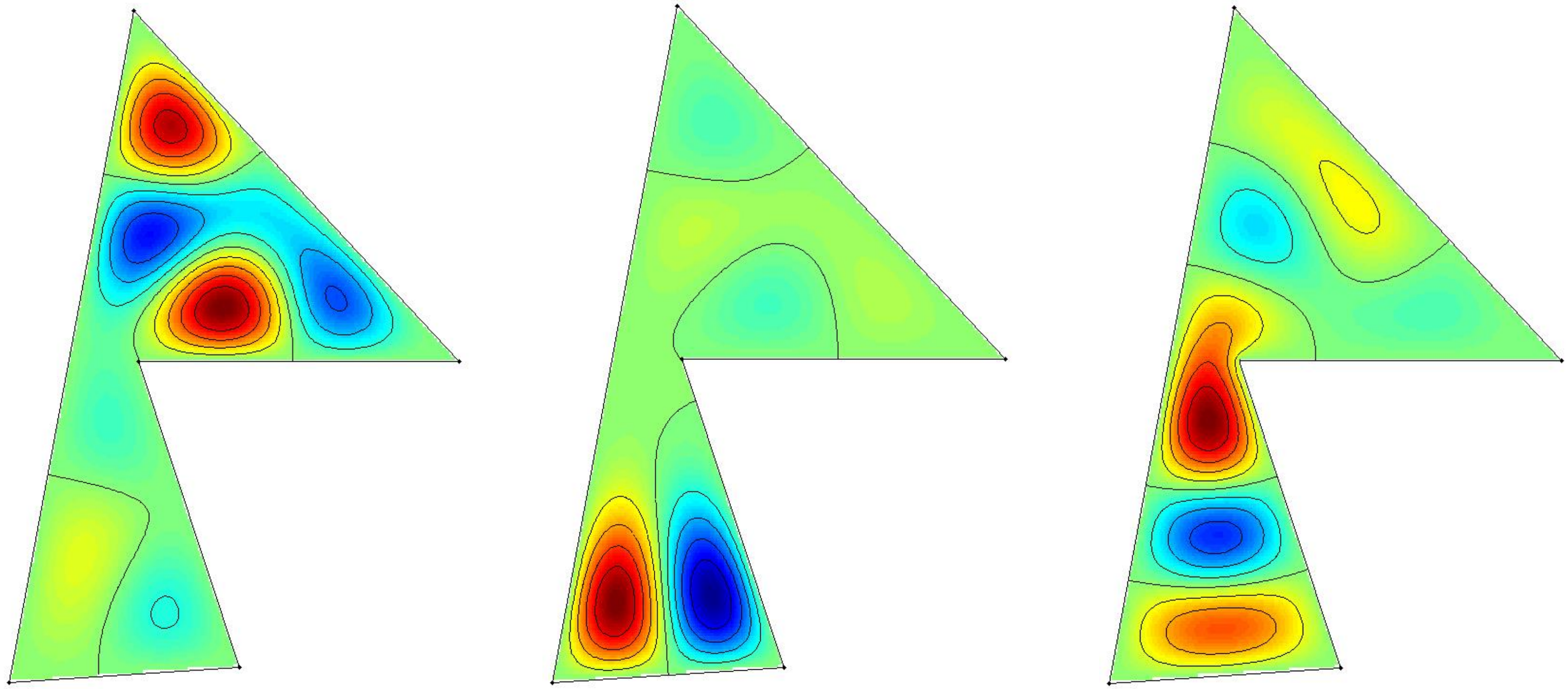
codimension 6–1 = 5

Quadrilaterals only have 4 shape parameters, but pentagons have 6.
So there should be pentagons with accidental triple eigenvalues.
Finding them is a nice problem of numerical optimization.

Simon Wojczyszyn: pentagon with triple eigenvalue



Another doubly degenerate pentagon



What can I say except...

There's nobody I enjoy discussing mathematical questions like these with more than Michael Overton.