

Taking Advantage of Degeneracy in Cone Optimization: with Applications to Sensor Network Localization

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The poster is for a workshop titled "Workshop on Numerical Linear Algebra and Optimization" held at the Pacific Institute for Mathematical Sciences from August 8-10, 2013, at the University of British Columbia. The poster features a blue header with the institute's logo and name. Below the title, it lists the dates and location. A central graphic shows a complex mathematical plot with several regions and axes. The text on the poster describes the workshop's focus on numerical linear algebra and optimization, mentioning topics like matrix eigenvalue problems, iterative methods, and applications in signal processing and control. It also lists the organizers: Inessa Danilova (University of Washington), Alan Goff (University of British Columbia), and Frank Wang (UBC). Contact information for the organizers is provided, along with a note that workshop participants are encouraged to submit abstracts for a poster abstract submission deadline on May 15, 2013. A small photo of Michael Overton is included at the bottom left, with a note that he will be in charge of the invited talks and the selected topics are representative of his research interests. The workshop will represent an opportunity to interact with many contributors on the occasion of the 1st birthday of the journal SIAM Review.

Pacific Institute for the Mathematical Sciences

Workshop on Numerical Linear Algebra and Optimization

August 8-10, 2013 University of British Columbia

The workshop will bring together researchers and practitioners in numerical linear algebra and optimization. Most topics are broad enough (e.g., iterative eigenvalue problems, iterative methods for sparse matrices, fast matrix-vector multiplication and fast algorithms for matrix-vector multiplication) to be of interest to a wide range of researchers in the field. The workshop will provide a unique opportunity for researchers to interact with many contributors on the occasion of the 1st birthday of the journal SIAM Review.

ORGANIZERS:
Inessa Danilova (University of Washington), Alan Goff (University of British Columbia) and Frank Wang (UBC)

CONTACT INFO:
Rosen 2013, 6470 University Building, 2104 Main Mall, Vancouver BC

POSTER DEADLINE:
Workshop participants are encouraged to submit abstracts for presenting a poster abstract submission deadline: May 15, 2013

FOR REGISTRATION AND ABSTRACTING INFORMATION:
<http://www.pims.math.ubc.ca/numerical-linear-algebra>

MICHAEL OVERTON is with Unicon for his research work in optimization and numerical linear algebra, and the selected topics are representative of his research interests. The workshop will represent an opportunity to interact with many contributors on the occasion of the 1st birthday of the journal SIAM Review.

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Motivation: Loss of Slater CQ/Facial reduction

- optimization algorithms rely on the KKT system;
and require that some constraint qualification (CQ) holds
(Slater's CQ/strict feasibility for convex conic optimization)
- However, surprisingly many conic opt, SDP relaxations,
instances arising from applications (QAP, GP, strengthened MC, SNL,
POP, Molecular Conformation)
do not satisfy Slater's CQ/are degenerate
- lack of Slater's CQ results in: unbounded dual solutions;
theoretical and numerical difficulties,
in particular for *primal-dual interior-point methods*.
- solution:
 - theoretical *facial reduction* (Borwein, W.'81)
 - preprocess for regularized smaller problem (Cheung, Schurr, W.'11)
 - take advantage of degeneracy (for SNL)
(Krislock, W.'10;)

Background/Abstract convex program

$$(ACP) \quad \inf_x f(x) \text{ s.t. } g(x) \preceq_K 0, x \in \Omega$$

where:

- $f: \mathbb{R}^n \rightarrow \mathbb{R}$ convex; $g: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is K -convex
 - $K \subset \mathbb{R}^m$ closed convex cone; $\Omega \subseteq \mathbb{R}^n$ convex set
 - $a \preceq_K b \iff b - a \in K$
 - $g(\alpha x + (1 - \alpha)y) \preceq_K \alpha g(x) + (1 - \alpha)g(y)$,
 $\forall x, y \in \mathbb{R}^n, \forall \alpha \in [0, 1]$

Slater's CQ: $\exists \hat{x} \in \Omega$ s.t. $g(\hat{x}) \in -\text{int } K$ ($g(\hat{x}) \prec_K 0$)

- guarantees strong duality
- essential for efficiency/stability in primal-dual interior-point methods
(near) loss of strict feasibility correlates with number of iterations and loss of accuracy

Case of Linear Programming, LP

Primal-Dual Pair: $A, m \times n / \mathcal{P} = \{1, \dots, n\}$ constr. matrix/set

$$\begin{array}{ll} \text{(LP-P)} & \max \quad b^\top y \\ & \text{s.t.} \quad A^\top y \leq c \end{array} \quad \begin{array}{ll} \text{(LP-D)} & \min \quad c^\top x \\ & \text{s.t.} \quad Ax = b, x \geq 0. \end{array}$$

Slater's CQ for (LP-P) / Theorem of alternative

$$\begin{aligned} \exists \hat{y} \text{ s.t. } c - A^\top \hat{y} > 0, \quad & ((c - A^\top \hat{y})_i > 0, \forall i \in \mathcal{P} =: \mathcal{P}^<) \\ \text{iff} & \\ Ad = 0, \quad c^\top d = 0, \quad d \geq 0 \implies & d = 0 \quad (*) \end{aligned}$$

implicit equality constraints: $i \in \mathcal{P}^=$

Finding solution $0 \neq d^*$ to (*) with max number of non-zeros determines (\mathcal{F}^y feasible set)

$$d_i^* > 0 \implies (c - A^\top y)_i = 0, \forall y \in \mathcal{F}^y \quad (i \in \mathcal{P}^=)$$

Facial Reduction: $A^T y \leq_f c$; minimal face $f \triangleq \mathbb{R}_+^n$

$$\begin{array}{ll}
 \text{(LP}_{\text{reg-P}}) & \max \quad b^T y \\
 & \text{s.t.} \quad (A^<)^T y \leq c^< \\
 & \quad \quad (A^=)^T y = c^=
 \end{array}$$

$$\begin{array}{ll}
 \text{(LP}_{\text{reg-D}}) & \min \quad (c^<)^T x^< + (c^=)^T x^= \\
 & \text{s.t.} \quad [A^< \quad A^=] \begin{pmatrix} x^< \\ x^= \end{pmatrix} = b \\
 & \quad \quad x^< \geq 0, x^= \text{ free}
 \end{array}$$

Mangasarian-Fromovitz CQ (MFCQ) holds

(after deleting redundant equality constraints!)

$$\left(\exists \hat{y} : \begin{array}{l} \underline{i \in \mathcal{P}^<} \\ (A^<)^T \hat{y} < c^< \end{array} \quad \begin{array}{l} \underline{i \in \mathcal{P}^=} \\ (A^=)^T \hat{y} = c^= \end{array} \right) \quad (A^=)^T \text{ is onto}$$

MFCQ holds iff dual optimal set is compact

Numerical difficulties if MFCQ fails; in particular for interior point methods! Modelling issue?

Facial Reduction/Preprocessing

Linear Programming Example, $x \in \mathbb{R}^2$

$$\begin{array}{ll} \max & (2 \ 6) y \\ \text{s.t.} & \begin{bmatrix} -1 & -1 \\ 1 & 1 \\ 1 & -1 \\ -2 & 2 \end{bmatrix} y \leq \begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix} \end{array}$$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ feasible; weighted last two rows $\begin{bmatrix} 1 & -1 & 1 \\ -2 & 2 & -2 \end{bmatrix}$ sum to zero.
 $\mathcal{P}^< = \{1, 2\}, \mathcal{P}^= = \{3, 4\}$

Facial reduction to 1 dim; substit. for y

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad -1 \leq t \leq \frac{1}{2}, \quad t^* = \frac{1}{2}, \quad \text{val}^* = 6$$

Case of ordinary convex programming, CP

$$(CP) \quad \sup_y b^\top y \text{ s.t. } g(y) \leq 0,$$

where

- $b \in \mathbb{R}^m$; $g(y) = (g_i(y)) \in \mathbb{R}^n$, $g_i : \mathbb{R}^m \rightarrow \mathbb{R}$ convex, $\forall i \in \mathcal{P}$
- Slater's CQ: $\exists \hat{y}$ s.t. $g_i(\hat{y}) < 0, \forall i$ (implies MFCQ)
- Slater's CQ fails implies implicit equality constraints exist,

i.e.:

$$\mathcal{P}^= := \{i \in \mathcal{P} : g(y) \leq 0 \implies g_i(y) = 0\} \neq \emptyset$$

Let $\mathcal{P}^< := \mathcal{P} \setminus \mathcal{P}^=$ and

$$g^< := (g_i)_{i \in \mathcal{P}^<}, g^= := (g_i)_{i \in \mathcal{P}^=}$$

(CP) is equivalent to $g(y) \leq_f 0$, f is minimal face

$$\begin{array}{ll}
 (\text{CP}_{\text{reg}}) & \sup \quad b^\top y \\
 & \text{s.t.} \quad g^<(y) \leq 0 \\
 & \quad \quad y \in \mathcal{F}^= \quad \text{or} \quad (g^=(y) = 0)
 \end{array}$$

where $\mathcal{F}^= := \{y : g^=(y) = 0\}$. Then

$\mathcal{F}^= = \{y : g^<(y) \leq 0\}$, so is a convex set!

Slater's CQ holds for (CP_{reg})

$$\exists \hat{y} \in \mathcal{F}^= : g^<(\hat{y}) < 0$$

modelling issue again?

Faithfully convex function f (Rockafellar'70)

f affine on a line segment only if affine on complete line containing the segment (e.g. analytic convex functions)

$\mathcal{F}^= = \{y : g^=(y) = 0\}$ is an affine set

Then:

$\mathcal{F}^= = \{y : Vy = V\hat{y}\}$ for some \hat{y} and full-row-rank matrix V .

Then MFCQ holds for

$$\begin{array}{ll} \text{(CP}_{\text{reg}}) & \sup \quad b^\top y \\ & \text{s.t.} \quad g^<(y) \leq 0 \\ & \quad \quad Vy = V\hat{y} \end{array}$$

$K = \mathcal{S}_+^n = K^*$ nonpolyhedral cone!

$$\text{(SDP-P)} \quad v_P = \sup_{y \in \mathbb{R}^m} b^\top y \text{ s.t. } g(y) := \mathcal{A}^* y - c \preceq_{\mathcal{S}_+^n} 0$$

$$\text{(SDP-D)} \quad v_D = \inf_{x \in \mathcal{S}^n} \langle c, x \rangle \text{ s.t. } \mathcal{A}x = b, x \succeq_{\mathcal{S}_+^n} 0$$

where:

- PSD cone $\mathcal{S}_+^n \subset \mathcal{S}^n$ symm. matrices
- $c \in \mathcal{S}^n$, $b \in \mathbb{R}^m$
- $\mathcal{A} : \mathcal{S}^n \rightarrow \mathbb{R}^m$ is a linear map, with adjoint \mathcal{A}^*
 $\mathcal{A}x = (\text{trace } A_j x) \in \mathbb{R}^m$
 $\mathcal{A}^* y = \sum_{i=1}^m A_i y_i \in \mathcal{S}^n$

(Assume feasibility: $\exists \tilde{y}$ s.t. $c - \mathcal{A}^* \tilde{y} \succeq 0$.)

$$\exists \hat{y} \text{ s.t. } s = c - \mathcal{A}^* \hat{y} \succ 0 \quad (\text{Slater})$$

iff

$$\mathcal{A}d = 0, \langle c, d \rangle = 0, d \succeq 0 \implies d = 0 \quad (*)$$

Face

A convex cone F is a **face** of K , denoted $F \trianglelefteq K$, if
 $x, y \in K$ and $x + y \in F \implies x, y \in F$
($F \triangleleft K$ proper face)

Minimal Faces

$f_P := \text{face } \mathcal{F}_P^S \trianglelefteq K$, \mathcal{F}_P^S is primal feasible set

$f_D := \text{face } \mathcal{F}_D^X \trianglelefteq K^*$, \mathcal{F}_D^X is dual feasible set

where: K^* denotes the dual (nonnegative polar) cone;
 face S denotes the smallest face containing S .

Regularization Using Minimal Face

Borwein-W.'81 , $f_P = \text{face } \mathcal{F}_P^S$

(SDP-P) is equivalent to the **regularized**

$$(\text{SDP}_{\text{reg-P}}) \quad v_{RP} := \sup_y \{ \langle b, y \rangle : \mathcal{A}^* y \preceq_{f_P} c \}$$

(slacks: $s = c - \mathcal{A}^* y \in f_P$)

Lagrangian Dual DRP Satisfies Strong Duality:

$$\begin{aligned} (\text{SDP}_{\text{reg-D}}) \quad v_{DRP} &:= \inf_x \{ \langle c, x \rangle : \mathcal{A}x = b, x \succeq_{f_P^*} 0 \} \\ &= v_P = v_{RP} \end{aligned}$$

and v_{DRP} is attained.

Alternative to Slater CQ

$$\mathcal{A}d = 0, \langle c, d \rangle = 0, 0 \neq d \succeq_{S_+^n} 0 \quad (*)$$

Determine a proper face $f \triangleleft S_+^n$

Let d solve (*) with $d = Pd_+P^T$, $d_+ \succ 0$, and $[P \ Q] \in \mathbb{R}^{n \times n}$ orthogonal. Then

$$\begin{aligned} c - \mathcal{A}^*y \succeq_{S_+^n} 0 &\implies \langle c - \mathcal{A}^*y, d^* \rangle = 0 \\ &\implies \mathcal{F}_P^s \subseteq S_+^n \cap \{d^*\}^\perp = QS_+^{\bar{n}}Q^T \triangleleft S_+^n \end{aligned}$$

(implicit rank reduction, $\bar{n} < n$)

- at most $n - 1$ iterations to satisfy Slater's CQ.
- to check [Theorem of Alternative](#)

$$\mathcal{A}d = 0, \langle c, d \rangle = 0, 0 \neq d \succeq_{S_+^n} 0, \quad (*)$$

use [stable](#) auxiliary problem

$$(AP) \quad \min_{\delta, d} \delta \quad \text{s.t.} \quad \left\| \begin{bmatrix} \mathcal{A}d \\ \langle c, d \rangle \end{bmatrix} \right\|_2 \leq \delta, \\ \text{trace}(d) = \sqrt{n}, \\ d \succeq 0.$$

- Both (AP) and its dual satisfy Slater's CQ.

$$(AP) \quad \min_{\delta, d} \delta \quad \text{s.t.} \quad \left\| \begin{bmatrix} \mathcal{A}d \\ \langle c, d \rangle \end{bmatrix} \right\|_2 \leq \delta, \\ \text{trace}(d) = \sqrt{n}, d \succeq 0.$$

Both (AP) and its dual satisfy Slater's CQ ... but ...

Cheung-Schurr-W'11, a $k = 1$ step CQ

Strict complementarity holds for (AP)

iff

$k = 1$ steps are needed to regularize (SDP-P).

Regularizing SDP

Minimal face containing $\mathcal{F}_P^S := \{s : s = c - \mathcal{A}^*y \succeq 0\}$

$$f_P = QS_+^{\bar{n}} Q^T$$

for some $n \times n$ orthogonal matrix $U = [P \ Q]$

(SPD-P) is equivalent to

$$\sup_y b^T y \text{ s.t. } g^<(y) \preceq 0, g^=(y) = 0,$$

where

$$g^<(y) := Q^T(\mathcal{A}^*y - c)Q$$

$$g^=(y) := \begin{bmatrix} P^T(\mathcal{A}^*y - c)P \\ P^T(\mathcal{A}^*y - c)Q + Q^T(\mathcal{A}^*y - c)P \end{bmatrix}.$$

(gen.) Slater CQ holds for the reduced program:

$\exists \hat{y}$ s.t. $g^<(y) \prec 0$ and $g^=(y) = 0$.

- Minimal representations of the data regularize (P);
use min. face f_P (and/or implicit rank reduction)
- goal: a backwards stable preprocessing algorithm to
handle (feasible) conic problems for which Slater's CQ
(almost) fails

Highly (implicit) degenerate/low-rank problem

- high (implicit) degeneracy translates to low rank solutions
- fast, high accuracy solutions

SNL - a Fundamental Problem of Distance Geometry; easy to describe - dates back to Grassmann 1886

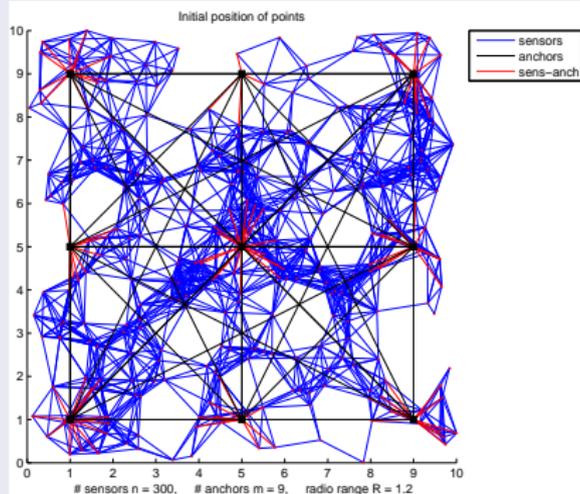
- r : embedding dimension
- n ad hoc wireless sensors $p_1, \dots, p_n \in \mathbb{R}^r$ to locate in \mathbb{R}^r ;
- m of the sensors p_{n-m+1}, \dots, p_n are anchors (positions known, using e.g. GPS)
- pairwise distances $D_{ij} = \|p_i - p_j\|^2, ij \in E$, are known within radio range $R > 0$



$$P^T = [p_1 \ \dots \ p_n] = [X^T \ A^T] \in \mathbb{R}^{r \times n}$$

Sensor Localization Problem/Partial EDM

Sensors \circ and Anchors \blacksquare



Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega)$

- node set $\mathcal{V} = \{1, \dots, n\}$
- edge set $(i, j) \in \mathcal{E}$; $\omega_{ij} = \|p_i - p_j\|^2$ known approximately
- The anchors form a clique (complete subgraph)
- **Realization of \mathcal{G} in \mathbb{R}^r** : a mapping of nodes $v_i \mapsto p_i \in \mathbb{R}^r$ with squared distances given by ω .

Corresponding Partial Euclidean Distance Matrix, EDM

$$D_{ij} = \begin{cases} d_{ij}^2 & \text{if } (i, j) \in \mathcal{E} \\ 0 & \text{otherwise (unknown distance),} \end{cases}$$

$d_{ij}^2 = \omega_{ij}$ are known squared Euclidean distances between sensors p_i, p_j ; anchors correspond to a **clique**.

Connections to Semidefinite Programming (SDP)

$D = \mathcal{K}(B) \in \mathcal{E}^n$, $B = \mathcal{K}^\dagger(D) \in \mathcal{S}^n \cap \mathcal{S}_C$ (centered $Be = 0$)

$P^\top = [p_1 \ p_2 \ \dots \ p_n] \in \mathcal{M}^{r \times n}$;

$B := PP^\top \in \mathcal{S}_+^n$ (Gram matrix of inner products);

$\text{rank } B = r$; let $D \in \mathcal{E}^n$ corresponding EDM; $e = (1 \ \dots \ 1)^\top$

$$\begin{aligned} \text{(to } D \in \mathcal{E}^n) \quad D &= (\|p_i - p_j\|_2^2)_{i,j=1}^n \\ &= (p_i^\top p_i + p_j^\top p_j - 2p_i^\top p_j)_{i,j=1}^n \\ &= \boxed{\text{diag}(B) e^\top + e \text{diag}(B)^\top - 2B} \\ &=: \mathcal{K}(B) \quad (\text{from } B \in \mathcal{S}_+^n). \end{aligned}$$

Nearest, Weighted, SDP Approx. (relax/discard rank B)

- $\min_{B \succeq 0} \|H \circ (\mathcal{K}(B) - D)\|$; rank $B = r$;
typical weights: $H_{ij} = 1/\sqrt{D_{ij}}$, if $ij \in E$, $H_{ij} = 0$ otherwise.
- with rank constraint: a non-convex, NP-hard program
- SDP relaxation is convex, **BUT**: expensive/low accuracy/implicitly highly degenerate (cliques restrict ranks of feasible B s)

Instead: (Shall) Take Advantage of Degeneracy!

clique α , $|\alpha| = k$ (corresp. $D[\alpha]$) with embed. dim. $= t \leq r < k$
 $\implies \text{rank } \mathcal{K}^\dagger(D[\alpha]) = t \leq r \implies \text{rank } B[\alpha] \leq \text{rank } \mathcal{K}^\dagger(D[\alpha]) + 1$
 $\implies \text{rank } B = \text{rank } \mathcal{K}^\dagger(D) \leq n - \boxed{(k - t - 1)} \implies$
Slater's CQ (strict feasibility) **fails**

BASIC THEOREM for Single Clique/Facial Reduction

Let:

- $\bar{D} := D[1:k] \in \mathcal{E}^k$, $k < n$, $\text{embdim}(\bar{D}) = t \leq r$ be given;
- $B := \mathcal{K}^\dagger(\bar{D}) = \bar{U}_B S \bar{U}_B^\top$, $\bar{U}_B \in \mathcal{M}^{k \times t}$, $\bar{U}_B^\top \bar{U}_B = I_t$, $S \in \mathcal{S}_{++}^t$ be full rank orthogonal decomposition of Gram matrix;
- $U_B := \begin{bmatrix} \bar{U}_B & \frac{1}{\sqrt{k}} \mathbf{e} \end{bmatrix} \in \mathcal{M}^{k \times (t+1)}$, $U := \begin{bmatrix} U_B & 0 \\ 0 & I_{n-k} \end{bmatrix}$, and $\begin{bmatrix} V & \frac{U^\top \mathbf{e}}{\|U^\top \mathbf{e}\|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1}$ be orthogonal.

Then the minimal face:

- $$\begin{aligned} \text{face } \mathcal{K}^\dagger(\mathcal{E}^n(1:k, \bar{D})) &= (US_+^{n-k+t+1}U^\top) \cap \mathcal{S}_C \\ &= (UV)S_+^{n-k+t}(UV)^\top \end{aligned}$$

The minimal face for single clique reduction

- $$\begin{aligned}\text{face } \mathcal{K}^\dagger(\mathcal{E}^n(1:k, \bar{D})) &= (US_+^{n-k+t+1}U^\top) \cap \mathcal{S}_C \\ &= (UV)S_+^{n-k+t}(UV)^\top\end{aligned}$$

Note that the minimal face is defined by the subspace $\mathcal{L} = \mathcal{R}(UV)$. We add $\frac{1}{\sqrt{k}}\mathbf{e}$ to represent $\mathcal{N}(\mathcal{K})$; then we use V to eliminate \mathbf{e} to recover a centered face.

Two (Intersecting) Clique Reduction/Subsp. Repres.

Let:

- $\alpha_1, \alpha_2 \subseteq 1:n; \quad k := |\alpha_1 \cup \alpha_2|$
- for $i = 1, 2: \bar{D}_i := D[\alpha_i] \in \mathcal{E}^{k_i}$, embedding dimension t_i ;
- $B_i := \mathcal{K}^\dagger(\bar{D}_i) = \bar{U}_i S_i \bar{U}_i^\top$, $\bar{U}_i \in \mathcal{M}^{k_i \times t_i}$, $\bar{U}_i^\top \bar{U}_i = I_{t_i}$, $S_i \in \mathcal{S}_{++}^{t_i}$;
- $U_i := \begin{bmatrix} \bar{U}_i & \frac{1}{\sqrt{k_i}} \mathbf{e} \end{bmatrix} \in \mathcal{M}^{k_i \times (t_i+1)}$; and $\bar{U} \in \mathcal{M}^{k \times (t+1)}$

satisfies $\mathcal{R}(\bar{U}) = \mathcal{R} \left(\begin{bmatrix} U_1 & 0 \\ 0 & I_{k_3} \end{bmatrix} \right) \cap \mathcal{R} \left(\begin{bmatrix} I_{k_1} & 0 \\ 0 & U_2 \end{bmatrix} \right)$, with $\bar{U}^\top \bar{U} = I_{t+1}$

- $U := \begin{bmatrix} \bar{U} & 0 \\ 0 & I_{n-k} \end{bmatrix} \in \mathcal{M}^{n \times (n-k+t+1)}$ and $\begin{bmatrix} v & \frac{U^\top \mathbf{e}}{\|U^\top \mathbf{e}\|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1}$
be orthogonal.

Then

$$\begin{aligned} \underline{\bigcap_{i=1}^2 \text{face } \mathcal{K}^\dagger(\mathcal{E}^n(\alpha_i, \bar{D}_i))} &= (US_+^{n-k+t+1}U^\top) \cap S_C \\ &= (UV)S_+^{n-k+t}(UV)^\top \end{aligned}$$

Expense/Work of (Two) Clique/Facial Reductions

Subspace Intersection for Two Intersecting Cliques/Faces

Suppose:

$$U_1 = \begin{bmatrix} U'_1 & 0 \\ U''_1 & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} I & 0 \\ 0 & U''_2 \\ 0 & U'_2 \end{bmatrix}$$

Then:

$$U := \begin{bmatrix} U'_1 \\ U''_1 \\ U'_2(U''_2)^\dagger U''_1 \end{bmatrix} \quad \text{or} \quad U := \begin{bmatrix} U'_1(U''_1)^\dagger U''_2 \\ U''_2 \\ U'_2 \end{bmatrix}$$

($Q_1 =: (U''_1)^\dagger U''_2$, $Q_2 =: (U''_2)^\dagger U''_1$ orthogonal/rotation)

(Efficiently) satisfies

$$\mathcal{R}(U) = \mathcal{R}(U_1) \cap \mathcal{R}(U_2)$$

Rotate to Align the Anchor Positions

- Given $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in \mathbb{R}^{n \times r}$ such that $D = \mathcal{K}(PP^T)$
- Solve the orthogonal Procrustes problem:

$$\begin{array}{ll} \min & \|A - P_2 Q\| \\ \text{s.t.} & Q^T Q = I \end{array}$$

$P_2^T A = U \Sigma V^T$ SVD decomposition; set $Q = UV^T$;
(Golub/Van Loan'79, Algorithm 12.4.1)

- Set $X := P_1 Q$

Summary: Facial Reduction for Cliques

- Using the basic theorem: each clique corresponds to a Gram matrix/corresponding subspace/corresponding face of SDP cone (implicit rank reduction)
- In the case where two cliques intersect, the union of the cliques correspond to the (efficiently computable) intersection of the corresponding faces/subspaces
- Finally, the positions are determined using a Procrustes problem

Results - Data for Random Noisless Problems

- 2.16 GHz Intel Core 2 Duo, 2 GB of RAM
- Dimension $r = 2$
- Square region: $[0, 1] \times [0, 1]$
- $m = 9$ anchors
- Using only Rigid Clique Union and Rigid Node Absorption
- Error measure: Root Mean Square Deviation

$$\text{RMSE} = \left(\frac{1}{n} \sum_{i=1}^n \|p_i - p_i^{\text{true}}\|^2 \right)^{1/2}$$

n # of Sensors Located

n # sensors \ R	0.07	0.06	0.05	0.04
2000	2000	2000	1956	1374
6000	6000	6000	6000	6000
10000	10000	10000	10000	10000

CPU Seconds

# sensors \ R	0.07	0.06	0.05	0.04
2000	1	1	1	3
6000	5	5	4	4
10000	10	10	9	8

RMSD (over located sensors)

n # sensors \ R	0.07	0.06	0.05	0.04
2000	$4e-16$	$5e-16$	$6e-16$	$3e-16$
6000	$4e-16$	$4e-16$	$3e-16$	$3e-16$
10000	$3e-16$	$5e-16$	$4e-16$	$4e-16$

Results - N Huge SDPs Solved

Large-Scale Problems

# sensors	# anchors	radio range	RMSD	Time
20000	9	.025	$5e-16$	25s
40000	9	.02	$8e-16$	1m 23s
60000	9	.015	$5e-16$	3m 13s
100000	9	.01	$6e-16$	9m 8s

Size of SDPs Solved: $N = \binom{n}{2}$ (# vrbls)

$\mathcal{E}_n(\text{density of } \mathcal{G}) = \pi R^2$; $M = \mathcal{E}_n(|E|) = \pi R^2 N$ (# constraints)

Size of SDP Problems:

$M = [3,078,915 \quad 12,315,351 \quad 27,709,309 \quad 76,969,790]$

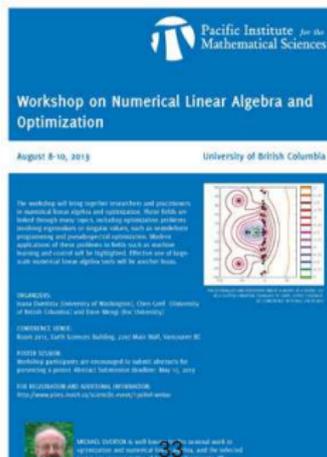
$N = 10^9 [0.2000 \quad 0.8000 \quad 1.8000 \quad 5.0000]$

Thanks for your attention!

Taking Advantage of Degeneracy in Cone Optimization: with Applications to Sensor Network Localization

Henry Wolkowicz

Dept. Combinatorics and Optimization, Univ. Waterloo



The poster is for a workshop titled "Workshop on Numerical Linear Algebra and Optimization" held at the University of British Columbia from August 8-10, 2013. It features the logo of the Pacific Institute for the Mathematical Sciences at the top. The main text describes the workshop's focus on bringing researchers and practitioners in numerical linear algebra and optimization together. It lists the organizers, Michael Overton and Gene Wynn, and provides contact information for the organizers. A small diagram of a network is also visible on the right side of the poster.

Pacific Institute for the Mathematical Sciences

Workshop on Numerical Linear Algebra and Optimization

August 8-10, 2013 University of British Columbia

The workshop will bring together researchers and practitioners in numerical linear algebra and optimization. Recent books are being published through many venues, including optimization, matrices, numerical algorithms for digital circuits, both in applications programming and management of optimization. Modern applications of these problems in both local and distributed learning and control will be highlighted. Effective use of large-scale numerical linear algebra tools will be another focus.

ORGANIZERS:
Michael Overton (University of Washington), Gene Wynn (University of British Columbia) and Gene Wynn (The University)

CONTACT INFO:
Michael Overton, Math Sciences Building, 1381 Main Mall, Vancouver BC

POSTER LOCUS:
Workshop participants are encouraged to submit abstracts for presentation to present. Abstract Submission Deadline: May 15, 2013

FOR INFORMATION AND SUBMISSIONS, VISIT US ONLINE:
<http://www.pims.math.ubc.ca/workshop>

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