PERTURBATION METHODS

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Local Approximation Methods

- Use information about $f : R \rightarrow R$ only at a point, $x_0 \in R$, to construct an approximation valid near x_0

- Taylor Series Approximation

$$
\begin{aligned}
f(x) &\doteq f(x_0) + (x - x_0)\, f'(x_0) + \frac{(x - x_0)^2}{2}\, f''(x_0) + \cdots + \frac{(x - x_0)^n}{n!}\, f^{(n)}(x_0) + \mathcal{O}(|x - x_0|^{n+1}) \\
&= p_n\,(x) + \mathcal{O}(|x - x_0|^{n+1})
\end{aligned}
$$

- Power series: $\sum_{n=0}^{\infty} a_n z^n$

  – The *radius of convergence* is

$$
r = \sup\{|z| : |\sum_{n=0}^{\infty} a_n z^n| < \infty\},
$$

  – $\sum_{n=0}^{\infty} a_n z^n$ converges for all $|z| < r$ and diverges for all $|z| > r$.

- Complex analysis

  – $f : \Omega \subset C \rightarrow C$ on the complex plane C is *analytic* on Ω iff

$$
\forall a \in \Omega \;\; \exists r, c_k \left(\forall \, \|z - a\| < r \left(f(z) = \sum_{k=0}^{\infty} c_k (z - a)^k \right) \right)
$$

  – A *singularity of* f is any a s. t. f is analytic on $\Omega - \{a\}$ but not on Ω.

  – If f or any derivative of f has a singularity at $z \in C$, then the radius of convergence in C of $\sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!}\, f^{(n)}(x_0)$, is bounded above by $\| \, x_0 - z \, \|$.

- Example: $f(x) = x^\alpha$ where $0 < \alpha < 1$.
  - One singularity at $x = 0$
  - Radius of convergence for power series around $x = 1$ is 1.
  - Taylor series coefficients decline slowly:

$$a_k = \frac{1}{k!}\frac{d^k}{dx^k}(x^\alpha)|_{x=1} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{1\cdot 2\cdot \cdots \cdot k}.$$

Table 6.1 (corrected): Taylor Series Approximation Errors for $x^{1/4}$

	Taylor series error				$x^{1/4}$
x N:	5	10	20	50	
3.0	5(−1)	8(1)	3(3)	1(12)	1.3161
2.0	1(−2)	5(−3)	2(−3)	8(−4)	1.1892
1.8	4(−3)	5(−4)	2(−4)	9(−9)	1.1583
1.5	2(−4)	3(−6)	1(−9)	0(−12)	1.1067
1.2	1(−6)	2(−10)	0(−12)	0(−12)	1.0466
.80	2(−6)	3(−10)	0(−12)	0(−12)	.9457
.50	6(−4)	9(−6)	4(−9)	0(−12)	.8409
.25	1(−2)	1(−3)	4(−5)	3(−9)	.7071
.10	6(−2)	2(−2)	4(−3)	6(−5)	.5623
.05	1(−1)	5(−2)	2(−2)	2(−3)	.4729

Implicit Function Theorem

- Suppose $h : R^n \to R^m$ is defined in $H(x, h(x)) = 0$, $H : R^n \times R^m \to R^m$, and $h(x_0) = y_0$.
  - Implicit differentiation shows
  $$H_x(x, h(x)) + H_y(x, h(x))h_x(x) = 0$$
  - At $x = x_0$, this implies
  $$h_x(x_0) = -H_y(x_0, y_0)^{-1} H_x(x_0, y_0)$$
  if $H_y(x_0, y_0)$ is nonsingular. More simply, we express this as
  $$h_x^0 = -\left(H_y^0\right)^{-1} H_x^0$$
  - Linear approximation for $h(x)$ is
  $$h^L(x) \doteq h(x_0) + h_x(x_0)(x - x_0)$$
- To check on quality, we compute
  $$E = \hat{H}(x, h^L(x))$$
  where $\hat{H}$ is a unit free equivalent of H. If $E < \varepsilon$, then we have an ε-solution.

- If $h^L(y)$ is not satisfactory, compute higher-order terms by repeated differentiation.
  - $D_{xx}H(x, h(x)) = 0$ implies
$$H_{xx} + 2H_{xy}h_x + H_{yy}h_x h_x + H_y h_{xx} = 0$$
  - At $x = x_0$, this implies
$$h_{xx}^0 = -\left(H_y^0\right)^{-1}\left(H_{xx}^0 + 2H_{xy}^0 h_x^0 + H_{yy}^0 h_x^0 h_x^0\right)$$
  - Construct the quadratic approximation
$$h^Q(x) \doteq h(x_0) + h_x^0(x - x_0) + \frac{1}{2}(x - x_0)^\top h_{xx}^0(x - x_0)$$
and check its quality by computing $E = H(x, h^Q(x))$.

Regular Perturbation: The Basic Idea

- Suppose x is an endogenous variable, ε a parameter
  - Want to find $x(\varepsilon)$ such that $f(x(\varepsilon), \varepsilon) = 0$
  - Suppose $x(0)$ known.
- Use Implicit Function Theorem
  - Apply implicit differentiation:

$$f_x(x(\varepsilon), \varepsilon)x'(\varepsilon) + f_\varepsilon(x(\varepsilon), \varepsilon) = 0 \tag{13.1.5}$$

  - At $\varepsilon = 0$, $x(0)$ is known and (13.1.5) is linear in $x'(0)$ with solution

$$x'(0) = -f_x(x(0), 0)^{-1} f_\varepsilon(x(0), 0)$$

  - Well-defined only if $f_x \neq 0$, a condition which can be checked at $x = x(0)$.
  - The linear approximation of $x(\varepsilon)$ for ε near zero is

$$x(\varepsilon) \doteq x^L(\varepsilon) \equiv x(0) - f_x(x(0), 0)^{-1} f_\varepsilon(x(0), 0)\varepsilon \tag{13.1.6}$$

- Can continue for higher-order derivatives of $x(\varepsilon)$.
  - Differentiate (13.1.5) w.r.t. ε

$$f_x x'' + f_{xx}(x')^2 + 2f_{x\varepsilon}x' + f_{\varepsilon\varepsilon} = 0 \tag{13.1.7}$$

  - At $\varepsilon = 0$, (13.1.7) implies that

$$\begin{aligned} x''(0) = & -f_x(x(0),0)^{-1} \left(f_{xx}(x(0),0)\ (x'(0))^2 \right. \\ & \left. +2f_{x\varepsilon}(x(0),0)\ x'(0) + f_{\varepsilon\varepsilon}(x(0),0) \right) \end{aligned}$$

  - Quadratic approximation is

$$x(\varepsilon) \doteq x^Q(\varepsilon) \equiv x(0) + \varepsilon x'(0) + \frac{1}{2}\varepsilon^2 x''(0) \tag{13.1.8}$$

- General Perturbation Strategy
  - Find special (likely degenerate, uninteresting) case where one knows solution
    - General relativity theory: begin with case of a universe with zero mass: ε is mass of universe
    - Quantum mechanics: begin with case where electrons do not repel each other: ε is force of repulsion
    - Business cycle analysis: begin with case where there are no shocks: ε is measure of exogenous shocks
  - Use local approximation theory to compute nearby cases
    - Standard implicit function may be applicable
    - Sometimes standard implicit function theorem will not apply; use appropriate bifurcation or singularity method.
  - Check to see if solution is good for problem of interest
    - Use unit-free formulation of problem
    - Go to higher-order terms until error is reduced to acceptable level
    - *Always* check solution for range of validity

Single-Sector, Deterministic Growth - canonical problem

- Consider dynamic programming problem

$$\max_{c(t)} \int_0^\infty e^{-\rho t} u(c) dt$$
$$\dot{k} = f(k) - c$$

- Ad-Hoc Method: Convert to a wrong LQ problem
  - McGrattan, JBES (1990)
    - Replace $u(c)$ and $f(k)$ with approximations around c^* and k^*
    - Solve linear-quadratic problem

$$\begin{array}{l} \max_c \int_0^\infty e^{-\rho t} \left(u(c^*) + u'(c^*)(c - c^*) + \frac{1}{2} u''(c^*)(c - c^*)^2 \right) \ dt \\ \text{s.t.} \quad \dot{k} = f(k^*) + f'(k^*)(k^* - k) - c \end{array}$$

    - Resulting approximate policy function is

$$C^{McG}(k) = f(k^*) + \rho(k - k^*) \neq C(k^*) + C'(k^*)(k - k^*)$$

    - Local approximate law of motion is $\dot{k} = 0$; add noise to get

$$dk = 0 \cdot dt + dz$$

    - Approximation is *random walk* when theory says solution is stationary
  - Fallacy of McGrattan noted in Judd (1986, 1988); point repeated in Benigno-Woodford (2004).

- Kydland-Prescott
  - Restate problem so that $\dot{k}$ is linear function of state and controls
  - Replace $u(c)$ with quadratic approximation
  - Note 1: such transformation may not be easy
  - Note 2: special case of Magill (JET 1977).
- Lesson
  - Kydland-Prescott, McGrattan provide no mathematical basis for method
  - Formal calculations based on appropriate IFT should be used.
  - Beware of *ad hoc* methods based on an intuitive story!

Perturbation Method for Dynamic Programming

- Formalize problem as a system of functional equations
  - Bellman equation:
$$\rho V(k) = \max_c \; u(c) + V'(k)(f(k) - c) \tag{1}$$
  - $C(k)$: policy function defined by
$$\begin{aligned} 0 &= u'(C(k)) - V'(k) \\ \rho V(k) &= u(C(k)) + V'(k)(f(k) - C(k)) \end{aligned} \tag{2}$$
  - Apply envelope theorem to (1) to get
$$\rho V'(k) = V''(k)(f(k) - C(k)) + V'(k)f'(k) \tag{1_k}$$
  - Steady-state equations
$$\begin{array}{ll} c^* = f(k^*) & \rho V(k^*) = u(c^*) + V'(k^*)(f(k^*) - c^*) \\ 0 = u'(c^*) - V'(k^*) & \rho V'(k) = V''(k)(f(k^*) - c^*) + V'(k)f'(k) \end{array}$$
  - Steady State: We know $k^*,\ V(k^*),\ C(k^*),\ f'(k^*), V'(k^*)$:
$$\rho = f'(k^*), \quad C(k^*) = f(k^*), \quad V(k^*) = \rho^{-1}u(c^*), \quad V'(k^*) = u'(c^*)$$
  - Want Taylor expansion:
$$\begin{aligned} C(k) &\doteq C(k^*) + C'(k^*)(k - k^*) + C''(k^*)(k - k^*)^2/2 + \ldots \\ V(k) &\doteq V(k^*) + V'(k^*)(k - k^*) + V''(k^*)(k - k^*)^2/2 + \ldots \end{aligned}$$

- Linear approximation around a steady state
  - Differentiate $(1_k, 2)$ w.r.t. k:

$$\rho V'' = V'''(f - C) + V''(f' - C') + V'' f' + V' f'' \tag{1_{kk}}$$

$$0 = u'' C' - V'' \tag{2_k}$$

  - At the steady state

$$0 = -V''(k^*)C'(k^*) + V''(k^*)f'(k^*) + V'(k^*)f''(k^*) \tag{1_k^*}$$

  - Substituting (2_k) into (1_k^*) yields

$$0 = -u''(C')^2 + u''C'f' + V'f''$$

  - Two solutions

$$C'(k^*) = \frac{\rho}{2}\left(1 \pm \sqrt{1 + \frac{4u'(C(k^*))f''(k^*)}{u''(C'(k^*))f'(k^*)f'(k^*)}}\right)$$

  - However, we know $C'(k^*) > 0$; hence, take positive solution

- Higher-Order Expansions
  - Conventional perception in macroeconomics: "perturbation methods of order higher than one are considerably more complicated than the traditional linear-quadratic case ..." – Marcet (1994, p. 111)
  - Mathematics literature: No problem (See, e.g., Bensoussan, Fleming, Souganides, etc.)
- Compute $C''(k^*)$ and $V'''(k^*)$.
  - Differentiate $(1_{kk}, 2_k)$:

$$\begin{aligned}\rho V''' &= V''''(f-C) + 2V'''(f'-C') + V''(f''-C'') \\ &\quad + V'''f' + 2V''f'' + V'f''' \end{aligned} \tag{1_{kkk}}$$

$$0 = u'''(C')^2 + u''C'' - V''' \tag{2_{kk}}$$

  - At k^*, (1_{kkk}) reduces to

$$0 = 2V'''(f'-C') + 3V''f'' - V''C'' + V'f''' \tag{1^*_{kkk}}$$

  - Equations $(1^*_{kkk}, 2^*_{kk})$ are *LINEAR* in unknowns $C''(k^*)$ and $V'''(k^*)$:

$$\begin{pmatrix} u'' & -1 \\ V'' & -2(f'-C') \end{pmatrix} \begin{pmatrix} C'' \\ V''' \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

  - Unique solution since determinant $-2u''(f'-C') + V'' < 0$.

- Compute $C^{(n)}(k^*)$ and $V^{(n+1)}(k^*)$.
  - Linear system for order n is, for some A_1 and A_2,

$$\begin{pmatrix} u'' & -1 \\ V'' & -n(f'-C') \end{pmatrix} \begin{pmatrix} C^{(n)} \\ V^{(n+1)} \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

  - Higher-order terms are produced by solving linear systems
  - The linear system is always determinate since $-nu''(f'-C') + V'' < 0$
- Conclusion:
  - Computing first-order terms involves solving quadratic equations
  - Computing higher-order terms involves solving linear equations
  - Computing higher-order terms is easier than computing the linear term.

Accuracy Measure

Consider the one-period relative Euler equation error:

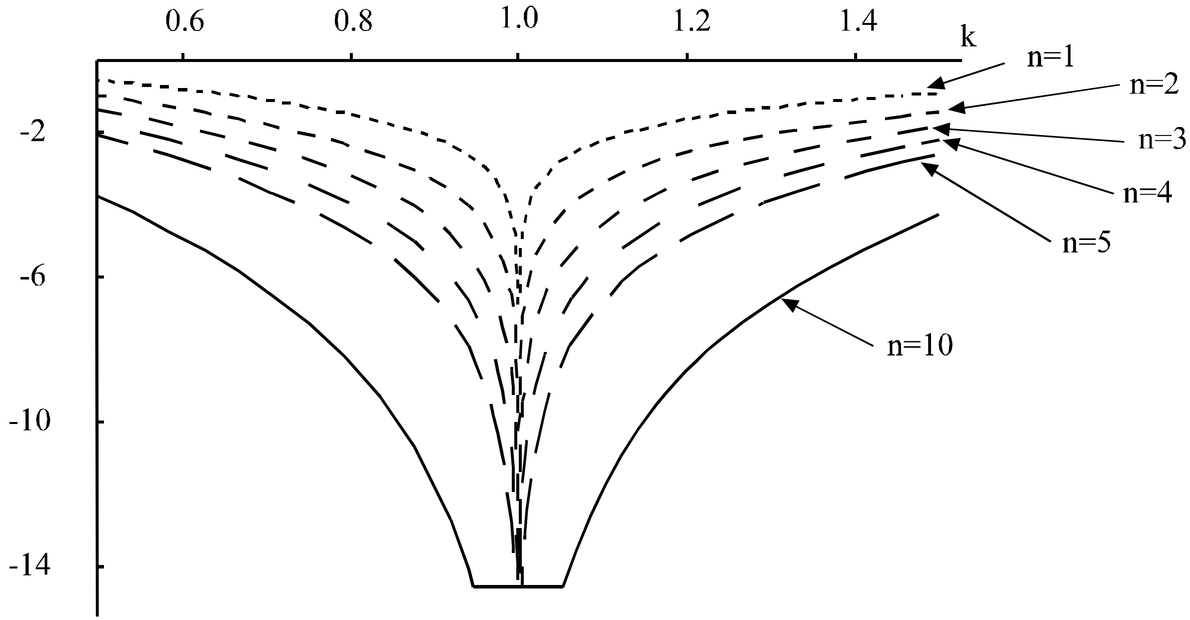
$$E(k) = 1 - \frac{V'(k)}{u'(C(k))}$$

- Equilibrium requires it to be zero.
- $E(k)$ is measure of optimization error
  - 1 is unacceptably large
  - Values such as .00001 is a limit for people.
  - $E(k)$ is unit-free.
- Define the L^p, $1 \leq p < \infty$, *bounded rationality accuracy* to be

$$\log_{10} \| E(k) \|_p$$

- The L^∞ error is the maximum value of $E(k)$.

Global Quality of Asymptotic Approximations



Graph of $\log_{10}|E(k|$

- Linear approximation is very poor even for k close to steady state
- Order 2 is better but still not acceptable for even $k = .9, 1.1$
- Order 10 is excellent for $k \in [.6, 1.4]$

Bifurcation Methods

- Suppose $H(h(\varepsilon),\varepsilon)=0$ but $H(x,0)=0$ for all x.
  - IFT says
$$h'(0)=-\frac{H_\varepsilon(x_0,0)}{H_x(x_0,0)}$$
  - $H(x,0)=0$ implies $H_x(x_0,0)=0$, and $h'(0)$ has the form $0/0$ at $x=x_0$.
  - l'Hospital's rule implies, if which is well-defined if $H_{\varepsilon x}(x_0,0)\neq 0$,
$$h'(0)=-\frac{H_{\varepsilon\varepsilon}(x_0,0)}{H_{\varepsilon x}(x_0,0)}.$$

Example: Portfolio Choices for Small Risks

- Simple asset demand model:
  - safe asset yields R per dollar invested and risky asset yields Z per dollar invested
  - If final value is $Y = W((1-\omega)R + \omega Z)$, then portfolio problem is

$$\max_{\omega} E\{u(Y)\}$$

- Small Risk Analysis
  - Parameterize cases

$$Z = R + \varepsilon z + \varepsilon^2 \pi \tag{1}$$

  - Compute $\omega(\varepsilon) \doteq \omega(0) + \varepsilon\omega'(0) + \frac{\varepsilon^2}{2}\omega''(0)$.around the deterministic case of $\varepsilon = 0$.
  - Failure of IFT: at $\varepsilon = 0$, $Z = R$, and $\omega(\ \varepsilon)$ is indeterminate, but we know that $\omega(\ \varepsilon)$ is unique for $\varepsilon \neq 0$

- Bifurcation analysis
  - The first-order condition for ω

$$0 = E\{u'\left(WR + \omega W(\varepsilon z + \varepsilon^2 \pi)\right)(z + \varepsilon\pi)\} \equiv G(\omega, \varepsilon) \tag{2}$$

$$0 = G(\omega, 0), \quad \forall \omega. \tag{3}$$

  - Solve for $\omega(\varepsilon) \doteq \omega(0) + \varepsilon\omega'(0) + \frac{\varepsilon^2}{2}\omega''(0)$. Implicit differentiation implies

$$0 = G_\omega \omega' + G_\varepsilon \tag{4}$$

$$G_\varepsilon = E\{u''(Y)W(\omega z + 2\omega\varepsilon\pi)W(z + \varepsilon\pi) + u'(Y)\pi\} \tag{5}$$

$$G_\omega = E\{u''(Y)(z + \varepsilon\pi)^2 \varepsilon\} \tag{6}$$

  - At $\varepsilon = 0$, $G(\omega, 0) = G_\omega(\omega, 0) = 0$ for all ω.
  - No point $(\omega, 0)$ for application of IFT to (3) to solve for $\omega'(0)$.

- We want $\omega_0 = \lim_{\varepsilon \to 0} \omega(\varepsilon)$.
  - Bifurcation theorem keys on ω_0 satisfying

$$\begin{aligned} 0 &= G_\varepsilon(\omega_0, 0) \\ &= u''(RW)\omega_0 \sigma_z^2 W + u'(RW)\pi \end{aligned} \tag{7}$$

    which implies

$$\omega_0 = -\frac{\pi}{\sigma_z^2} \frac{u'(WR)}{W u''(WR)} \tag{8}$$

  - (8) is asymptotic portfolio rule
    * same as mean-variance rule
    * ω_0 is product of risk tolerance and the risk premium per unit variance.
    * ω_0 is the limiting portfolio share as the variance vanishes.
    * ω_0 is not first-order approximation.

- To calculate $\omega'(0)$:
  - differentiate (2.4) with respect to ε

$$0 = G_{\omega\omega}\omega'\omega' + 2G_{\omega\varepsilon}\omega' + G_\omega\omega'' + G_{\varepsilon\varepsilon} \tag{9}$$

    where (without loss of generality, we assume $W = 1$)

$$\begin{aligned} G_{\varepsilon\varepsilon} = {} & E\{u'''(Y)(\omega z + 2\omega\varepsilon\pi)^2(z+\varepsilon\pi) + u''(Y)2\omega\pi(z+\varepsilon\pi) \\ & +2u''(Y)(\omega z + 2\omega\varepsilon\pi)\pi\} \\ G_{\omega\omega} = {} & E\{u'''(Y)(z+\varepsilon\pi)^3\varepsilon\} \\ G_{\omega\varepsilon} = {} & E\{u'''(Y)(\omega z + 2\omega\varepsilon\pi)(z+\varepsilon\pi)^2\varepsilon + u''(Y)(z+\varepsilon\pi)2\pi\varepsilon \\ & +u''(Y)(z+\varepsilon\pi)^2\} \end{aligned}$$

  - At $\varepsilon = 0$,

$$\begin{aligned} G_{\varepsilon\varepsilon} &= u'''(R)\omega_0^2 E\{z^3\} & \qquad G_{\omega\omega} &= 0 \\ G_{\omega\varepsilon} &= u''(R)E\{z^2\} \neq 0 & \qquad G_{\varepsilon\varepsilon\varepsilon} &\neq 0 \end{aligned}$$

  - Therefore,

$$\omega' = -\frac{1}{2}\frac{u'''(R)}{u''(R)}\frac{E\{z^3\}}{E\{z^2\}}\omega_0^2. \tag{10}$$

  - Equation (10) is a simple formula.
    * $\omega'(0)$ proportional to u'''/u''
    * $\omega'(0)$ proportional to ratio of skewness to variance.
    * If u is quadratic or z is symmetric, ω does not change to a first order.
  - We could continue this and compute more derivatives of $\omega(\varepsilon)$ as long as u is sufficiently differentiable.

- Other applications - see Judd and Guu (*ET*, 2001)
  - Equilibrium: add other agents, make π endogenous
  - Add assets
  - Produce a mean-variance-skewness-kurtosis-etc. theory of asset markets
  - More intuitive approach to market incompleteness then counting states and assets