Finding Equilibrium Points

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Further readings

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- Jofré, A., R.T. Rockafellar R.T & R. Wets. *Variational Inequalities and economic equilibrium*. To appear in Math. Operation Research (2006?)
- Jofré, A., RT. Rockafellar & R. Wets, *A variational inequality scheme for determining an economic equilibrium of classical or extended type*. In "Variational analysis and applications", 553--577, Nonconvex Optim. Appl., 79, Springer, New York, 2005.
- Jofré, A. & R. Wets, *Continuity properties of Walras equilibrium points. Stochastic equilibrium problems in economics and game theory*. Annals of Operations Research, 114 (2002), 229--243.
- S. P. Dirkse and M. C. Ferris. *The PATH solver: A non-monotone stabilization scheme for mixed complementarity problems*. Optimization Methods and Software, 5:123-156, 1995.

I. Walras Equilibrium Model

Deterministic

Pure Exchange model

Pure Exchange: Walras

agent's problem: Agents: $\text{a} \in \mathcal{A} \quad |\mathcal{A}|$ *finite* "large"

$$\bar{c}_a \in \arg\max u_a(c) \text{ so that } \langle p,c\rangle \le \langle p,e_a\rangle,\ \ c \in C_a$$

e_a: endowment of agent a, $e_a \in \operatorname{int} C_a$

u_a: utility of agent a, concave, usc

$C_a \to \mathbb{R}, \quad C_a \subset \mathbb{R}^n$ (survival set) convex

market clearing: $s(p) = \sum_{\text{a}\in\mathcal{A}} (e_a - \bar{c}_a)$ excess supply

equilibrium price: $\bar{p} \in \Delta$ such that $s(\bar{p}) \ge 0$

Δ unit simplex

The Walrasian

$$W(p,q) = \langle q, s(p) \rangle,\ \ W : \Delta \times \Delta \to \mathbb{R}$$

$\bar{p}$ equilibirum price

$$\Leftrightarrow \bar{p} \in \arg\max_p (\inf_q W(p,q))\ \&\ s(\bar{p}) \geq 0$$

Properties of W :

continuous in p ($e_a \in \operatorname{int} C_a$, '$a$-inf-compact') usc

linear in q, Δ compact convex

$W(p,p) \geq 0, \forall p \in \Delta$

i.e., W is a Ky Fan function

Ky Fan functions & inequality

- $K : B \times B \to \mathbb{R}$ is a Ky Fan function if

  (a) $\forall y : x \mapsto K(x, y)$ usc

  (b) $\forall x : y \mapsto K(x, y)$ convex

- Theorem. K Ky Fan fcn, dom $K = B \times B$, B compact

  $$\Rightarrow \text{ argmax-inf } K \neq \varnothing$$

  if $K(x, x) \geq 0$ on dom K, $\bar{x} \in$ argmax-inf K

  $$\Rightarrow \inf_y K(\bar{x}, y) \geq 0.$$

- The Walrasian is a Ky Fan function

  yields existence of equilibrium price.

II. Numerical Strategies

- Augmented Lagrangian
- Path Solver

Augmented Walrasian

- Augmented Walrasian:

$$\bar{p} \in \text{ argmax-inf W}$$

$$\cong \text{ saddle point } (\bar{p}, \bar{q}) \text{ of } \tilde{W}_r$$

$$\tilde{W}_r(p,q) = \inf_z \left\{ W(p,z) \,\middle|\, \|z - q\| \le r \right\},$$

$$\|\bullet\| \text{ an appropriate norm } (|\bullet|_\infty \text{ e.g.})$$

from Variational Analysis

- Properties: $\tilde{W}_r$ is usc in p, convex, lsc in (q,r)

  non-increasing in r

- Saddle-points: $\sup_{p\in\Delta}\left(\inf_{q\in\Delta} W(p,q)\right) =$

$$\sup_{p\in\Delta}\left(\inf_{q\in\Delta, r\in\mathbb{R}_+} \tilde{W}_r(p,q)\right) = \inf_{q\in\Delta, r\in\mathbb{R}_+}\left(\sup_{p\in\Delta} \tilde{W}_r(p,q)\right)$$

argument similar to augmented Lagrangian

Iterations

$$W(p,q) = \langle q, s(p) \rangle \text{ on } \Delta \times \Delta$$

$$\tilde{W}_r(p,q) = \inf_z \left\{ W(p,z) \middle| \, \|z-q\| \leq r \right\}$$

$$q^{k+1} = \arg\min_{q \in \Delta} \left[\min_z \left\langle z, s(p^k) \right\rangle \middle| \, \|z-q\| \leq r_k \right]$$

minimizing a linear form on a ball

reduces to finding the smallest element of $s(p^k)$

$$p^{k+1} = \arg\max_{p \in \Delta} \left[\min_z \left\langle z, s(p) \right\rangle \middle| \, \left\|z-q^{k+1}\right\| \leq r_{k+1} \right]$$

as $r_k \nearrow \infty, p^k \to \bar{p}$ (Walras equilibrium point)

Test: Demand functions

- Cobb-Douglas utility function (usc):

$$u_a(c) = \gamma_a \prod_{l=1}^{n} c_l^{\beta_a^l} \quad \text{with} \quad \sum_{l=1}^{n} \beta_a^l = 1,\ \beta_a^l \geq 0$$

- budget constraint: $\sum_l p_l c_l \leq \sum_l p_l e_a^l$

- demand: $\overline{c}_a^l(p) = (\beta_a^l / p_l)\left(\sum_k p_l e_a^k\right),\ l = 1,\ldots,n$

experiments: 10 agents, 150 goods (blink!)

Variational Inequality I

$$\max_c u_a(c) \text{ so that } \langle p, c\rangle \le \langle p, e_a\rangle, c \in C_a. \quad a \in \mathcal{A}$$

$$\sum_a (e_a - c_a) = s(p) \ge 0.$$

$$\Updownarrow$$

KKT-Optimality Conditions & Market Clearing:

$\bar{c}_a \in C_a$ optimal $\Leftrightarrow \exists \lambda_a \ge 0$ (linear constraint)

(a) $\langle p, e_a - \bar{c}_a\rangle \ge 0$ (feasibility)

(b) $\lambda_a \left(\langle p, e_a - \bar{c}_a\rangle\right) = 0$ (compl. slackness)

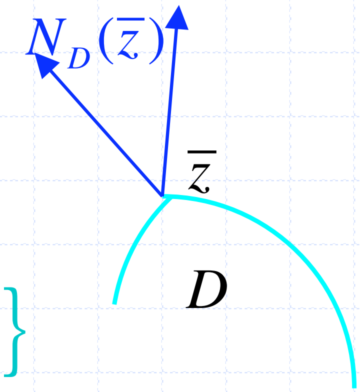
(c) $\nabla u_a(\bar{c}_a) - \lambda_a p = 0 \quad (e_a \in \text{ int } C_a)$

(d) $\sum_a (e_a - \bar{c}_a) = 0$ (market clearing)

Variational Inequality II

$$\max_c u_a(c) \text{ so that } \langle \mathrm{p,c} \rangle \le \langle p, e_a \rangle,\ c \in C_a, \forall a$$

$$\sum_a (e_a - c_a) = s(p) \ge 0.$$



$\Updownarrow$

$$N_D(\bar{z}) = \left\{ v \middle| \langle v, z - \bar{z} \rangle \le 0, \forall z \in D \right\}$$

$$G(p,(c_a),(\lambda_a)) = \left[\sum_a (e_a - c_a); \left(\lambda_a p - \nabla u_a(c_a) \right); \langle p, e_a - c_a \rangle \right]$$

$$D = \Delta \times \left(\prod_a C_a \right) \times \left(\prod_a \mathbb{R}_+ \right)$$

$$-G(\bar{p},(\bar{c}_a),(\bar{\lambda}_a)) \in N_D(\bar{p},(\bar{c}_a),(\bar{\lambda}_a))$$

D (unfortunately) is unbounded

bounding *D*: “solvable” V.I.

from D to $\hat{D}$ bounded with

explicit bounds derived via duality (finite # goods)

(global bound for C_a, κ_a depends on 'var'(u_a))

$$-G(\bar{p},(\bar{c}_a),(\bar{\lambda}_a)) \in N_{\hat{D}}(\bar{p},(\bar{c}_a),(\bar{\lambda}_a))$$

$$\hat{D} = \Delta \times \left(\prod_a \hat{C}_a\right) \times \left(\prod_a [0,\kappa_a]\right)$$

Polyhedral case: efficient algorithmic procedures

Path Solver .. *(by Ferris & all)*

$$-G(\bar{z}) \in N_D(\bar{z}), \quad \bar{z} = (\bar{p}, (\bar{c}_a), (\bar{\lambda}_a))$$

$$D = \Delta \times \left(\prod\nolimits_a C_a\right) \times \left(\prod\nolimits_a \mathbb{R}_+\right) = \{z | Az \geq b\}$$

Complementarity problem:

$$-G(z) = A^T y, \;\; y \geq 0, \;\; Az - b \perp y$$

with $K = \mathbb{R}^N \times \mathbb{R}_+^M$:

$$(z, y) \in K, \;\; H(z, y) \in -K^*, \;\; (z, y) \perp H(z, y)$$

$$H(z, y) = \begin{bmatrix} G(z) + A^T y \\ Az \end{bmatrix} - \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Equivalent non-smooth mapping

- $0 = H(\mathrm{prj}_K(z,y)) + (z,y) - \mathrm{prj}_K(z,y)$
- with simplified $K = \mathbb{R}_+^N$:

  (CP) $0 \le x \perp F(x) \ge 0$ Complementarity Problem

  (NS) $0 = F(x_+) + x - x_+$ Nonlinear system

- $\bar{x}$ sol'n (CP) $\Rightarrow$ $\tilde{x}$ sol'n (NS):

  $\tilde{x}_i = \bar{x}_i$ if $F_i(\bar{x}) = 0$, $\tilde{x}_i = -F_i(\bar{x}_i)$ if $F_i(\bar{x}) > 0$

  $\tilde{x}$ sol'n (NS) $\Rightarrow \tilde{x}_+$ sol'n (CP):

  $\tilde{x}_+ \ge 0,\ F(\tilde{x}_+) = \tilde{x}_+ - \tilde{x} \ge 0\ \&\ \tilde{x}_+ \perp \tilde{x}_+ - \tilde{x}$

PATH Solver: $x = (z, y),\ x_+ = \text{prj}_K(x, y)$

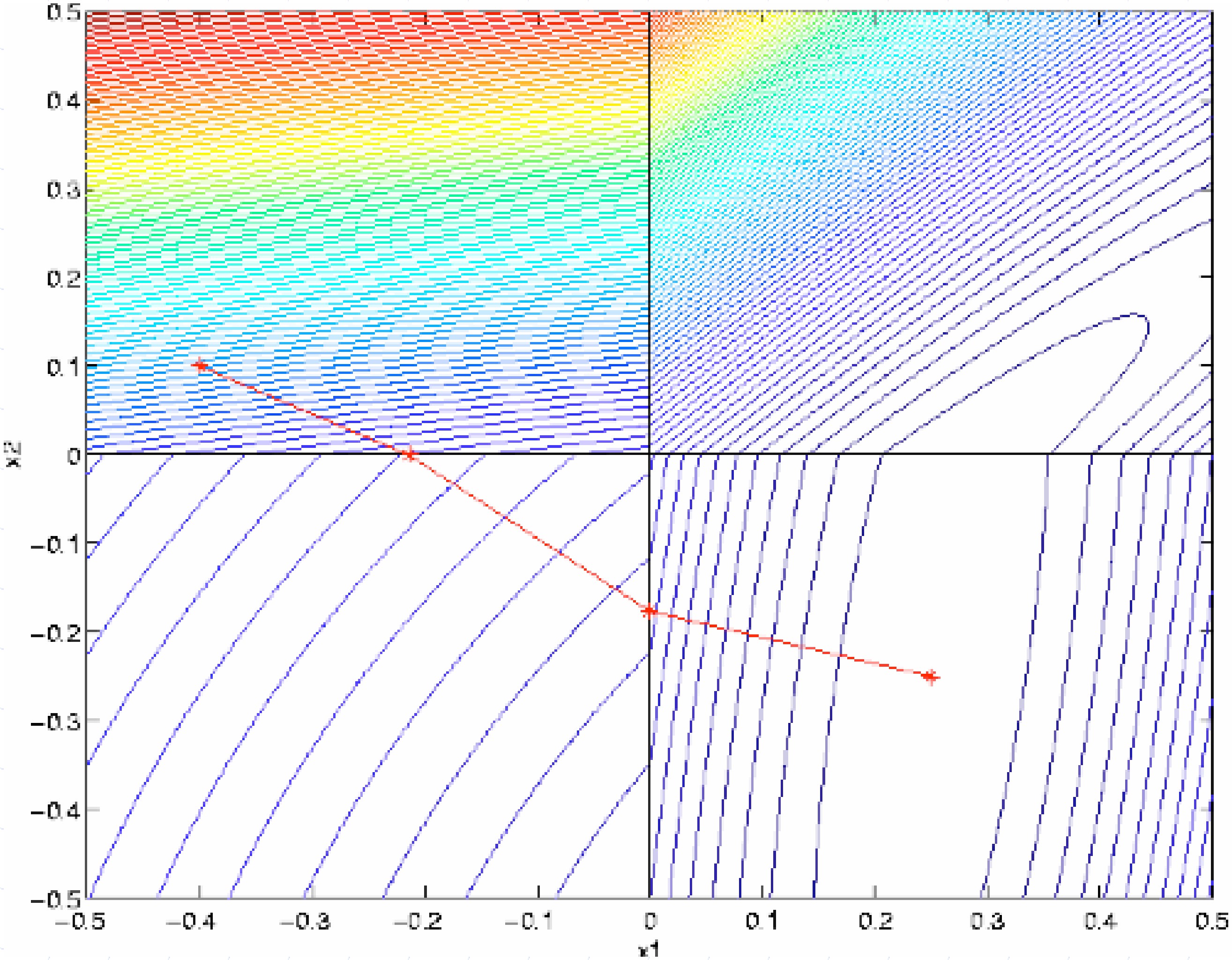
- PATH: Newton method based on non-smooth normal mapping:

$$H(x_+) + x - x_+$$

- Newton point: solution of piecewise linearization:

$$H(x_+^k) + \left\langle \nabla H(x_+^k), x_+ - x_+^k \right\rangle + x - x_+ = 0$$

The "Newton" step



with PATH Solver (experimental)

- Economy: (5 goods)
  - Skilled & unskilled workers
  - Businesses: Basic goods & leisure
  - Banker: bonds (riskless), 2 stocks
- 2-stages,   solved under # of scenarios
- utilities: CSE-functions (gen. Cobb-Douglas)
  - Utility in stage 2 assigned to financial instruments
  - only used for transfer in stage 1
- so far: mostly calibration

  numerically: `blink' (5000 iterations).

III. Continuity, Stability Issues

equilibrium points

solutions of V.I., …

Variational Convergence

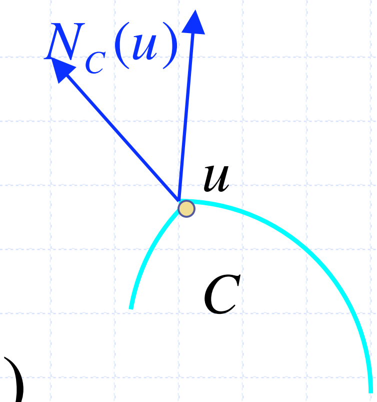
- solutions of optimization problems
  - $\arg\min f^\nu \to \arg\min f$: epi-convergence
  - $\arg\max f^\nu \to \arg\max f$: hypo-convergence
- stability of saddle points
  - saddle pts $K^\nu \to$ saddle pts K : epi/hypo-convergence
- stability of maxinf points
  - $\max\inf K^\nu \to \max\inf K$: lopsided convergence (tightly)

Walras Equilibrium points

- $\forall a \in \mathcal{A}$: $\bar{c}_a(p) \in \arg\max_{C_a} \left\{ u_a(c_a) \middle| \langle p, c_a \rangle \le \langle p, e_a \rangle \right\}$
- $s(p) = \sum_a (e_a - d_a(\bar{c}_a))$ excess supply
- find $\bar{p} \in \Delta$ (unit simplex) so that $s(\bar{p}) \ge 0$
- Walrasian: $W(p,q) = \langle q, s(p) \rangle$ Ky Fan fcn
- $\bar{p} \in \arg\max\text{–}\inf W \Leftrightarrow s(\bar{p}) \ge 0$
- conditions: $e_a \in \text{int dom } u_a$, "globally compact"
- Convergence: $u_a^v \to_{hypo} u_a,\ e_a^v \to e_a \Rightarrow$

  W^v converge lopsided tightly to W

Variational Inequalities



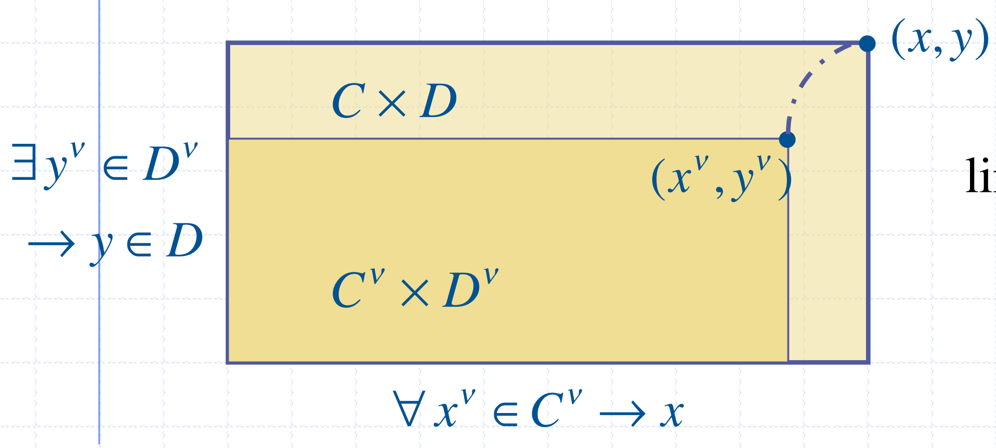
- $C \subset \mathbb{R}^n$ non-empty, convex
- $G : C \to \mathbb{R}^n$ continuous
- find $\bar{u} \in C$ such that $-G(\bar{u}) \in N_C(\bar{u})$

  where $v \in N_C(\bar{u}) \Leftrightarrow \langle v, u - \bar{u} \rangle \leq 0, \forall u \in C$
- with $K(u,v) = \langle G(u), v - u \rangle$ on $\text{dom}\, K = C \times C$

  ⇨ *K* is a Ky Fan function, K(u,u) ≥ 0.
- Find

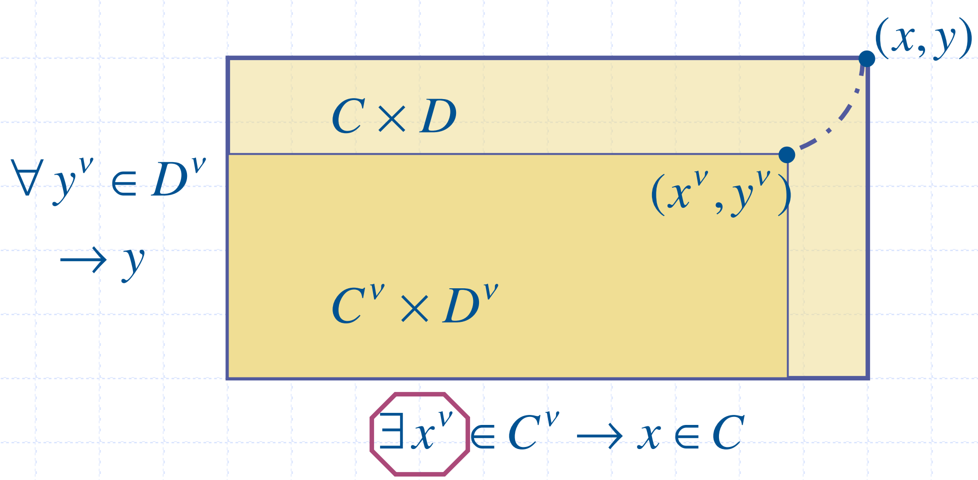
$$\bar{u} \in \arg\max\text{–}\inf K(\bullet,\bullet) \text{ so that } K(\bar{u},\bullet) \geq 0$$

Lopsided convergence: definition



$$\limsup_\nu K^\nu(x^\nu, y^\nu) \leq K(x, y) \text{ when } x \in C$$

$$K^\nu(x^\nu, y^\nu) \to -\infty \text{ when } x \notin C$$



$$\liminf_\nu K^\nu(x^\nu, y^\nu) \geq K(x, y) \text{ when } y \in D$$

$$K^\nu(x^\nu, y^\nu) \to \infty \text{ when } x \notin D$$

Lopsided tightly

- $K^\nu_{C^\nu \times D^\nu} \underset{lop-tightly}{\longrightarrow} K_{C\times D}$ if $K^\nu_{C^\nu \times D^\nu} \underset{lop}{\longrightarrow} K_{C\times D}$ &

  (b) $\forall x \in C, \exists x^\nu \to x, \forall y^\nu \in D^\nu$ and $y^\nu \to y$:

  $$\liminf K^\nu(x^\nu, y^\nu) \geq K(x,y) \quad \text{if } y \in D$$

  $$K^\nu(x^\nu, y^\nu) \to \infty \quad \text{if } y \notin D$$

  but also $\forall \varepsilon > 0$, $\exists B_\varepsilon$ compact (depends on $x^\nu \to x$):

  $$\inf_{B_\varepsilon \cap D^\nu} K^\nu(x^\nu, \bullet) \leq \inf_{D^\nu} K^\nu(x^\nu, \bullet) + \varepsilon, \ \forall \nu \geq \nu_\varepsilon$$

- THM. $K^\nu_{C^\nu \times D^\nu} \to K_{C\times D}$ lopsided tightly, $\bar{x}$ cluster point of $\{x^\nu \in \arg\max\text{–}\inf K^\nu_{C^\nu \times D^\nu}\}_{\nu\in\mathbb{N}} \Rightarrow \bar{x} \in \arg\max\text{–}\inf K_{C\times D}$

Proof ....

- $$K^{\nu}_{C^{\nu}\times D^{\nu}} \underset{lop-tightly}{\longrightarrow} K_{C\times D}$$

  Let $g^{\nu} = \inf_{y\in D^{\nu}} K^{\nu}(\bullet, y), \quad g = \inf_{y\in D} K(\bullet, y).$

  $$\Rightarrow g^{\nu} \underset{hypo}{\longrightarrow} g \quad \text{when} \quad \begin{cases} C^{\nu}_{g} = \left\{ x \in C^{\nu} \,\middle|\, g^{\nu}(x) > -\infty \right\} \\ C_{g} = \left\{ x \in C \,\middle|\, g(x) > -\infty \right\} \end{cases} \neq \varnothing$$

- then apply

  $$g^{\nu}_{C^{\nu}} \underset{hypo}{\longrightarrow} g_{C},\ x^{\nu} \in \arg\max_{C^{\nu}} g^{\nu},\ x^{\nu_k} \to \bar{x} \in C \Rightarrow \bar{x} \in \arg\max_{C} g$$

Extending Ky Fan's inequality

- $$K^\nu \to K \text{ lopsided}$$
  $$K^\nu \text{ Ky Fan } \Rightarrow K \text{ Ky Fan}$$
- $$\& \quad \text{when } \arg\max\text{–}\inf K^\nu \neq \emptyset$$
  $$\text{if } \bar{x} \in \text{ cluster-pts } \{\arg\max\text{–}\inf K^\nu\}$$
  $$\Rightarrow \bar{x} \in \arg\max\text{–}\inf K \ \& \ K(\bar{x}, \bullet) \geq 0$$
- Ky Fan fcns closed under tight-lopsided

  saddle fcns closed under hypo/epi-convergence

  usc fcns closed under hypo-convergence