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Projective modules in classical and quantum functional analysis

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Dedicated to I. M. Gel'fand on the occasion of his 90th birthday

Homological theory of the "algebras in analysis" exists in at least three different versions. First of all, there is the homological theory of Banach and more general locally convex algebras. This is about 40 years old. However, in the last decade of the previous century, a "homological section" appeared in a new branch of analysis, the so-called quantized functional analysis or, more prosaically, the theory of operator spaces. One of principal features of this theory, as is now widely realized, is the existence of different approaches to the proper quantum version of a bounded bilinear operator. In fact, two such versions are now thought to be most important; each of them has its own relevant tensor product with an appropriate universal property. Accordingly, there are two principal versions of quantized algebras and quantized modules, and this leads to two principal versions of quantized homology.

Thus we have now, in the first decade of the 21st century, three species of topological homology: the traditional (or "classical") one, and two "quantized" ones.

In these lectures, we shall restrict ourselves by studying, in the framework of these three theories, the fundamental concept of a projective module. This concept is "primus inter pares" among the three recognized pillars of the science of homology: projectivity, injectivity, and flatness. It is this notion that is the cornerstone for every sufficiently developed homological theory, let it be in algebra, topology, or, as in these notes, in functional analysis.

Our initial definitions of projectivity do not go far away from their prototypes in abstract algebra. However, the principal results concern essentially functional-analytic objects. As we shall see, they have, as a rule, no purely algebraic analogues. Moreover, some phenomena are strikingly different from what algebraists could expect, based on their experience.

0 On terminology and notation

Proofs (or what is held to be a proof) are placed between the signs "$\triangleleft$" and "$\triangleright$"; when placed side by side ($\triangleleft\triangleright$), these signs means "obviously" or "can be verified immediately". The sign $\iff$ replaces the words "if and only if". An equality by definition is often denoted by the sign ":=".

Throughout our presentation, the words "in-, sur-, or bijective" are applied to maps (in particular, to operators, morphisms of modules, etc.), and they have their standard *set-theoretical* meaning. On the other hand, the terms "mono-, epi- or isomorphism" are always used in the *theoretical-categorical* sense.

Now we must turn to more delicate matters. The problem is that the terminology and the notation of the subject, at least in its "quantum" part is still far from being established. Therefore we have to spend some time to explain what we shall mean by this word or sign.

Generally speaking, we shall freely use the treatise of E. Effros and Z.-J. Ruan [6] as the main source of knowledge about quantized spaces (called, in that book, operator spaces)[1]. However, by necessity we shall have to deviate considerably from the terms and notation of [6], and, of course, all differences must and will be explicitly stated. As to more traditional matters (Banach algebras and modules, ABC of category theory etc.), we shall follow the textbook [11] or the survey [15].

To begin with, we are afraid to use the overloaded adjective "operator" for spaces, algebras and modules of quantized functional analysis ("theory of operator spaces" as in [6] or [19]). Instead we shall use the term "*quantum space*" for the matrix-normed spaces, satisfying the now famous Effros–Ruan axioms [6, p. 20]. A matrix-norm on a linear space E is a collection $\|\cdot\|_n$ with $n = 1, 2, \ldots$, where the indicated symbol denotes a norm on the space $M_n(E)$ of square $n \times n$ matrices with the entries in E. We recall that in the case of a quantum space a matrix norm gives rise to a norm on any space $M_{m,n}$ of rectangular $m \times n$ matrices with the entries in E. (To introduce this norm, we transform a given rectangular matrix into a square matrix by adding zero entries; the resulting norm does not depend on the way we do it).

[1]Recently, another conspicuous book on the subject, written by G. Pisier [19], appeared. There the main concepts are exposed in the language of operators, not of $n \times n$ matrices, as in [6]. To be frank, this "coordinate free" approach and many other preferences in [19] more appeal to my personal taste. However, the majority of my readers are much more familiar with the book [6], written in a more elementary way, and with great accuracy and a high pedagogical skill.

If the first, and hence all, of the respective matrix norms are complete, we speak about a *quantum Banach space*.

At the same time, when we in these notes use the adjective "operator", we just mean that the respective space, algebra or module is a uniformly closed (= operator-norm closed) subspace of the space of all bounded operators on some Hilbert space H. This latter space, which is, in fact, an algebra, will always be denoted by $\mathcal{B}(H)$. (As an exception, in Section 3 we shall use the term "operator algebra" for a slightly wider class of algebras, but that will be specially mentioned in due time).

If $\varphi : E \to F$ is an operator between to quantum spaces, then the operator $\varphi_n : M_n(E) \to M_n(F), (x_{ij}) \mapsto (\varphi(x_{ij}))$, participating in the fundamental concept of a completely bounded operator, will be called the *n-th amplification of φ*. We recall that φ is called *completely bounded*, if $\sup\{\|\varphi_n\|; n = 1, 2, \ldots\} < \infty$. This supremum is called the *completely bounded norm* of φ, and it is usually denoted by $\| \cdot \|_{cb}$.

For a given quantum space E, the normed space $M_n(E)$ will be often called in these notes "the normed space in the n-th floor of E". In particular, the underlying Banach space, identified with the respective Banach space of 1×1 matrices, will be often called "the Banach space in the first floor of E". Rather often, speaking about E, we shall mean just the latter space; this will not lead to a misunderstanding.

To *quantize a Banach space* is to make it a quantum Banach space in such a way that the norm on the respective space in the first floor coincides with the given norm. We recall that every isometric operator from E into some $\mathcal{B}(H)$ gives rise to a quantization of E in this sense . (The standard procedure is the respective embedding of $M_n(E)$ into $\mathcal{B}(nH)$), and the subsequent endowing of $M_n(E)$ by the induced norm [6, pp. 20–21]. Here and after nH denotes the Hilbert sum of n copies of H). Conversely, by virtue of Ruan's Representation Theorem of [6, p. 33], each quantization of E can be obtained with the help of such an operator. In this connection, sometimes the term *quantization* will also mean the mentioned isometrical operator, implementing the relevant structure of a quantum Banach space in question; this will not cause confusion. If a space in question is already given as a subspace of some $\mathcal{B}(H)$, we shall always consider the quantization which is implemented by the respective natural embedding, and call this quantization *standard*.

As we shall see, each of the three homological theories, mentioned above, is intimately connected with "its own" distinguished type of tensor product, playing in this theory an outstanding role. As to the first one, it is the time-honored *projective tensor product of Banach spaces*, denoted, as usually, by "$\widehat{\otimes}$". The two others are two different tensor products of quantum Banach spaces: the completed version of the so-called *Haagerup tensor product* [6, p. 153], and the completed version of what was called in [6, p. 124] *operator-projective tensor product* (discovered by E. Effros and Z.-J. Ruan, and, simultaneously and independently, by D. Blecher and V. Paulsen). We shall denote the former by "$\overset{h}{\otimes}$" and the latter by "$\overset{o}{\otimes}$". (Naturally, we are forbidden to use the notation "$\widehat{\otimes}$" for this latter tensor product, as this is done in [6]). Accordingly, we shall often refer to these theories as *"$\widehat{\otimes}$-, $\overset{h}{\otimes}$- and $\overset{o}{\otimes}$"-theory.*

We emphasize that, for Banach spaces E and F, $E \widehat{\otimes} F$ is just a Banach space, not equipped with any quantization. At the same time, the objects $E \overset{h}{\otimes} F$ and $E \overset{o}{\otimes} F$ have meaning only for quantum Banach spaces E and F, and they are again quantum Banach spaces.

Let E and F be (just) Banach spaces or, according to the sense, quantum Banach spaces. We believe that our listener/reader knows fairly well what the norm on $E \widehat{\otimes} F$ is. As to the matrix-norms on $E \overset{h}{\otimes} F$ and $E \overset{o}{\otimes} F$, we recall, for his convenience, their definitions.

As usual, $E \otimes F$ denotes the algebraic tensor product of the relevant linear spaces. Each of the Banach spaces $M_n(E \overset{h}{\otimes} F)$ and $M_n(E \overset{o}{\otimes} F)$ is defined as the completion of $M_n(E \otimes F)$ in some specific norm, the so-called *Haagerup norm*, or $\overset{h}{\otimes}$*-norm* and, operator-projective norm, or $\overset{o}{\otimes}$-norm. These, in their turn, are defined as follows.

To speak about the $\overset{h}{\otimes}$-norm, we need at first a preparatory notion. Consider, for any $m, n \in \mathbb{N}$, two rectangular vector-valued matrices $\mathbf{v} = (v_{ij}) \in M_{n,m}(E)$ and $\mathbf{w} = (w_{ij}) \in M_{m,n}(F)$. Then the so-called *Effros product* (or matrix inner product) of $\mathbf{v}$ and $\mathbf{w}$, denoted by $\mathbf{v} \odot \mathbf{w}$, is, by definition, the matrix in $M_n(E \otimes F)$ with the entries $(\mathbf{v} \odot \mathbf{w})_{ij} := \sum_{k=1}^{n} v_{ik} \otimes w_{kj}$.

Now fix a matrix $\mathbf{u} \in M_n(E \otimes F)$. We take all possible representations of $\mathbf{u}$ in the form of the Effros product $\mathbf{v} \odot \mathbf{w}$ with $\mathbf{v} \in M_{n,m}(E)$, $\mathbf{w} \in M_{m,n}(F)$ with arbitrary $m \in \mathbb{N}$. (It is easy to see that such representations indeed exist). Then the Haagerup norm of $\mathbf{u}$ is defined as

$$\|\mathbf{u}\| = \inf\{\|\mathbf{v}\|\|\mathbf{w}\|\}, \tag{1}$$

where the infimum is taken over all indicated representations [6, p. 152].

The $\overset{o}{\otimes}$-norm needs another preparatory notion, the *tensor product* of vector-valued matrices. Now let $\mathbf{v} \in M_{k,l}(E)$ and $\mathbf{w} \in M_{m,n}(F)$ be two rectangular matrices for any $k, l, q, r \in \mathbb{N}$. We denote by $M_{k\times q, l\times r}(E \otimes F)$ the set of rectangular matrices with kq rows, indexed by double subscripts, say gh for $1 \le g \le k$ and $1 \le s \le q$, and with lr columns, indexed by double subscripts, say st for $1 \le g \le l$ and $1 \le t \le r$.[2] Then the tensor (or Kronecker) product of $\mathbf{v}$ and $\mathbf{w}$ is, by definition, the matrix $\mathbf{v} \otimes \mathbf{w} \in M_{k\times q, l\times r}(E \otimes F)$ with the entries $(v \otimes w)_{(gs)(ht)} := v_{gh} \otimes w_{st}$.

[2]These double subscripts can be ordered, say, in the lexicographical manner, but in fact there is no real need for this.

Now fix a matrix $\mathbf{u} \in M_n(E \otimes F)$. We take all possible representations of $\mathbf{u}$ in the form of the (usual) matrix product of three rectangular matrices $\alpha(\mathbf{v} \otimes \mathbf{w})\gamma$ with $\mathbf{v} \in M_{k,l}$, $\mathbf{w} \in M_{q,r}, \alpha \in M_{n,k\times q}$, and $\gamma \in M_{l\times r,n}$, with arbitrary $k, l, q, r \in \mathbb{N}$. Here, as you see, in the scalar matrix α the rows are indexed by single, and the columns by double subscripts, whereas γ has the "symmetric" structure. (It is not difficult to show that representations of the indicated form exist.) Then we have

$$\|\mathbf{u}\| = \inf\{\|\alpha\|\|\mathbf{v}\|\|\mathbf{w}\|\|\gamma\|\}, \tag{2}$$

where the infimum is taken over all these representations [6, p. 124].

The important fact that will be frequently used is that the $\overset{o}{\otimes}$-norm is bigger that the $\overset{h}{\otimes}$-norm, and, if we shall speak about the first floor, the $\widehat{\otimes}$-norm is still bigger. In equivalent terms, there exist a contractive operator j_1 and a completely contractive operator j_2 such that the diagram

$$\begin{tikzcd}
& E \otimes F \arrow[dl, "\mathbf{in}_1"'] \arrow[d, "\mathbf{in}_2"] \arrow[dr, "\mathbf{in}_3"] & \\
E \widehat{\otimes} F \arrow[r, "j_1"] & E \overset{o}{\otimes} F \arrow[r, "j_2"] & E \overset{h}{\otimes} F
\end{tikzcd}, \tag{3}$$

where $\mathbf{in}_k$, for $k = 1, 2, 3$, are the respective natural embeddings, is commutative.

The word "*bioperator*" is used as the abbreviation of "bilinear operator". As it was already mentioned, in the quantized functional analysis there are two principal versions of the concept of a bounded bioperator. The respective definitions, in their turn, depend on what to call the amplification of a given bioperator $\mathcal{R} : E \times F \to G$ between quantum spaces.

In the first approach (that was discovered earlier), the *n-th multiplicative amplification* or, briefly, *n-th $\overset{h}{\otimes}$-amplification* of $\mathcal{R}$ for $n = 1, 2, \dots$ is, by definition, the bioperator $\mathcal{R}_n^{(h)} : M_n(E) \times M_n(F) \to M_n(G)$, taking a pair $\mathbf{u} = (u_{ij})$, $\mathbf{v} = (v_{ij})$ to the $n \times n$ matrix $\mathbf{w}^{(h)}$ with the entries $w_{ij}^{(h)} := \sum_{k=1}^n \mathcal{R}(u_{ik}, v_{kj})$.

On the other hand, the *n,k-th complete amplification* or, briefly, *n,k-th $\overset{o}{\otimes}$-amplification* of $\mathcal{R}$ for $n, k = 1, 2, \dots$ is, by definition, the bioperator $\mathcal{R}_{n,k}^{(o)} : M_n(E) \times M_n(F) \to M_{n\times k}(G)$. This bioperator takes a pair $\mathbf{u} = (u_{gh})$, $\mathbf{v} = (v_{st})$ to the $nk \times nk$ matrix $\mathbf{w}^{(o)}$ with the entries $w_{(gs)(ht)}^{(o)} := \mathcal{R}(u_{gh}, v_{st})$.

If it happens that $\sup\{\|\mathcal{R}_n^{(h)}\|; n = 1, 2, \dots\} < \infty$, we say that the initial bioperator $\mathcal{R}$ is *multiplicatively bounded* and call this supremum the *multiplicatively bounded norm* $\mathcal{R}$. Replacing in this phrase the subscript n by n, k, and the superscript (h) by (o), we obtain the definitions of a *completely bounded* bioperator and of the *completely bounded norm* of such a bioperator. (Note that in the second definition we could, without changing the result, to restrict ourselves with the case $k = n$, but the presented form will happen to be more convenient.)

We recall that every multiplicatively bounded bioperator is automatically completely bounded, and its multiplicatively bounded norm is greater than or equal to its completely bounded norm. (One can deduce it from the easy observation that $\mathcal{R}^{(o)}_{n,n}(\mathbf{u},\mathbf{v}) = \mathcal{R}^{(h)}_{n\times n}(\mathbf{u}\otimes \mathbf{1}, \mathbf{1}\otimes\mathbf{v})$, where $\mathbf{1}$ is the identity matrix in $M_n(\mathbb{C}^n)$.)

Rather often in our exposition, we shall present the material in a parallel way (that is simultaneously) for all three theories. In the respective places we shall use the symbol $\widetilde{\otimes}$ ("unspecified tensor product") for each of the three mentioned types of tensor product. Accordingly, the reader can replace this symbol by any of the symbols $\widehat{\otimes}$, $\overset{h}{\otimes}$ or $\overset{o}{\otimes}$, but, of course, *he must be true to this chosen symbol throughout the whole text of our notes.*

To make such a device work smoothly, let us make some further agreements.

Throughout these notes, we shall frequently use the term "$\widehat{\otimes}$-space" for a classical Banach space, and the term "$\overset{h}{\otimes}$-space" as well as "$\overset{o}{\otimes}$-space" for a quantum Banach space. The term "$\widehat{\otimes}$*-bounded operator*" will mean (just) a bounded operator, whereas the terms "$\overset{h}{\otimes}$*-bounded*" as well as "$\overset{o}{\otimes}$-bounded operator" will equally mean a completely bounded operator. Often instead of "$\widetilde{\otimes}$*-bounded operator*" we shall say just "$\widetilde{\otimes}$*-operator*".

The category of Banach spaces and bounded operators will be denoted by **Ban**, and the category of quantum Banach spaces and completely bounded operators by **QBan**. The notation $\widetilde{\otimes}$-**Ban** will mean **Ban** for $\widetilde{\otimes} = \widehat{\otimes}$, and **QBan** for $\widetilde{\otimes} = \overset{h}{\otimes}$ or $\widetilde{\otimes} = \overset{o}{\otimes}$. It is obvious that isomorphisms in $\widetilde{\otimes}$-**Ban** are $\widetilde{\otimes}$-operators, possessing inverse $\widetilde{\otimes}$-operators; we shall call them $\widetilde{\otimes}$*-isomorphisms*.

Further, a "$\widehat{\otimes}$*-bounded bioperator*" is just a bounded bioperator, "$\overset{h}{\otimes}$*-bounded bioperator*" is a multiplicatively bounded bioperator, and "$\overset{o}{\otimes}$*-bounded bioperator*' is a completely bounded bioperator. Often instead of "$\widetilde{\otimes}$-bounded bioperator" we shall say just "$\widetilde{\otimes}$*-bioperator*".

When we speak about norms of operators and bioperators, we adhere to the following agreement. A $\widehat{\otimes}$*-norm* means just a classical norm of an operator, or, according to the sense, of a bioperator. At the same time, $\overset{h}{\otimes}$*-norm* is either the completely bounded norm of a completely bounded operator or it is the multiplicatively bounded norm of a multiplicatively bounded bilinear operator. Finally, $\overset{o}{\otimes}$*-norm* is either (again) the completely bounded norm of a completely bounded operator or it is the completely bounded norm of a completely bounded bilinear operator. These norms will be denoted respectively by $\|\cdot\|_{\widehat{\otimes}}$, $\|\cdot\|_{\overset{h}{\otimes}}$, and $\|\cdot\|_{\overset{o}{\otimes}}$. (There will be no confusion with what was called above $\overset{h}{\otimes}$- and $\overset{o}{\otimes}$-norm in the space $M_n(E\otimes F)$.)

An $\widetilde{\otimes}$-operator or $\widetilde{\otimes}$-bioperator φ is called $\widetilde{\otimes}$*-contractive* if $\|\varphi\|_{\widetilde{\otimes}} \leq 1$. An $\widetilde{\otimes}$-operator is called $\widetilde{\otimes}$*-isometric* if it is (just) isometric in the case $\widetilde{\otimes} = \widehat{\otimes}$ and completely isometric (that is, isometric in every floor) in the case where $\widetilde{\otimes} = \overset{h}{\otimes}$ or $\widetilde{\otimes} = \overset{o}{\otimes}$.

As a first occasion of such parallel presentation, we shall formulate the triple fundamental theorem, reflecting the very *raison d'être* of our tensor products.

Theorem 0 *Let $\mathcal{R} : E \times F \to G$ be a bioperator between $\widetilde{\otimes}$-spaces. Then $\mathcal{R}$ is $\widetilde{\otimes}$-bounded $\Longleftrightarrow$ there exists a unique linear $\widetilde{\otimes}$-bounded operator R such that the diagram*

$$\begin{tikzcd} E \times F \arrow[d, "\vartheta"'] \arrow[dr, "\mathcal{R}"] & \\ E \widetilde{\otimes} F \arrow[r, "R"] & G \end{tikzcd}$$

with $\vartheta : (x, y) \mapsto x \otimes y$, is commutative. Moreover, $\|R\|_{\widetilde{\otimes}} = \|\mathcal{R}\|_{\widetilde{\otimes}}$ holds.

$\triangleleft$ See [11, p. 32] in the "classical" case and [6, pp. 126 and 154] in the "quantum" cases. $\triangleright$

The property of the tensor product $\widetilde{\otimes}$, figuring in this theorem, is called the *universal property of this tensor product with respect to $\widetilde{\otimes}$-bounded bioperators*. The operator R is called *associated with the bioperator* $\mathcal{R}$ and, vice versa, the latter is called *associated with* R.

In what follows, we shall frequently use the functional-analytic versions of the tensor product functors. Namely, a fixed $\widetilde{\otimes}$-space E gives rise to the covariant functor denoted by

$$E\widetilde{\otimes}? : \widetilde{\otimes}\text{-}\mathbf{Ban} \to \widetilde{\otimes}\text{-}\mathbf{Ban}.$$

It sends a "running" $\widetilde{\otimes}$-space F to $E \widetilde{\otimes} F$ and an $\widetilde{\otimes}$-operator $\varphi : F \to G$ to $\mathbf{1}_E \widetilde{\otimes} \varphi : E \widetilde{\otimes} G \to E \widetilde{\otimes} G$; the latter operator is well defined (with the help of Theorem 0) by taking an elementary tensor $x \otimes y$ to $x \otimes \varphi(y)$. In the similar way, one can define the functor $? \widetilde{\otimes} E : \widetilde{\otimes}\text{-}\mathbf{Ban} \to \widetilde{\otimes}\text{-}\mathbf{Ban}$; $F \mapsto F \widetilde{\otimes} E$, $\varphi : F \to G \mapsto \varphi \widetilde{\otimes} \mathbf{1}_E : F \widetilde{\otimes} E \to G \widetilde{\otimes} E$.

We recall that, for any $E \in \widetilde{\otimes}\text{-}\mathbf{Ban}$, the $\widetilde{\otimes}$-spaces $E \widetilde{\otimes} \mathbb{C}$ and $\mathbb{C} \widetilde{\otimes} E$ can be identified with E by means of the $\widetilde{\otimes}$-isometrical isomorphism, taking $x \otimes \lambda$ (or $\lambda \otimes x$) to λx for $x \in E$ and $\lambda \in \mathbb{C}$. This simple fact will be frequently used.

Remark In the $\widehat{\otimes}$- and $\overset{o}{\otimes}$-theories, the introduced functors of "left" and "right" tensor multiplication have no practical difference; to speak exactly, they are naturally equivalent because of the commutativity of the operations $\widehat{\otimes}$ and $\overset{o}{\otimes}$. On the contrary, they are essentially different in the $\overset{h}{\otimes}$-theory, with its non-commutative tensor product. However, at the very beginning of the construction of the homological theory these differences still do not have an important impact.

Now we shall introduce, in our agreed simultaneous way, our three principal types of "algebras in analysis". Namely, by an $\widetilde{\otimes}$-*algebra* A we mean a $\widetilde{\otimes}$-space endowed by a $\widetilde{\otimes}$-bioperator of multiplication $m : A \times A \to A$ (which is, generally speaking, is not supposed to be $\widetilde{\otimes}$-contractive). Thus, a $\widehat{\otimes}$-algebra is just a "classical" Banach algebra. Any $\overset{h}{\otimes}$-algebra is automatically $\overset{o}{\otimes}$-algebra, but not vice versa. For both $\overset{h}{\otimes}$- and $\overset{h}{\otimes}$-algebras we shall use the generic name "*quantum algebras*".

The $\widetilde{\otimes}$-operator associated with m is called *the product operator for* A and denoted by $\pi : A \widetilde{\otimes} A \to A$. Clearly, it is uniquely determined by taking $a \otimes b$ to ab for $a, b \in A$.

Remark The striking theorem of Blecher [2] states that $\overset{h}{\otimes}$-algebras can be characterized, up to $\widetilde{\otimes}$-isomorphism, as operator norm-closed subalgebras of some $\mathcal{B}(H)$, with their standard quantization. (Therefore $\widetilde{\otimes}$-algebras are often referred — with the great danger of confusion, in our opinion — as "operator algebras".) But this characterization does not remain true in the larger class of $\overset{o}{\otimes}$-algebras.

From "algebras in analysis" we proceed to "modules in analysis". Let A be a fixed $\widetilde{\otimes}$-algebra. A *left* $\widetilde{\otimes}$*-module over* A or, in short, *left* A-$\widetilde{\otimes}$*-module* is a $\widetilde{\otimes}$-space X endowed by a $\widetilde{\otimes}$-bioperator of left outer multiplication $\dot{m} : A \times X \to X$. Since we consider almost exclusively left modules in these notes, the adjective "left" will be often omitted.

As in the case of algebras, speaking about quantum modules, we mean $\overset{h}{\otimes}$- and $\overset{o}{\otimes}$-modules together.

The $\widetilde{\otimes}$-operator, associated with $\dot{m}$, is called *the outer product operator for* X and denoted by $\pi_X : A \widetilde{\otimes} X \to X$. Clearly, it takes $a \otimes x$ to $a \cdot x$ for $a \in A$ and $x \in X$. (We often write just π, if X is fixed.)

As our final agreement, we take $\dot{\oplus}$ and $\dot{\otimes}$ as the symbols for the Hilbert direct sum and the Hilbert tensor product of Hilbert spaces. Sometimes, in order to avoid possible ambiguity, we shall use the latter symbol also for elementary tensors in the respective spaces (e.g., $x \dot{\otimes} y \in H \dot{\otimes} K$), and also for respective types of tensor products of operators (e.g., $a \dot{\otimes} b$, acting on $H \dot{\otimes} K$).

Other terms and notation will be fixed later.

1 General definitions and properties

All the basic homological definitions concerning the projectivity are parallel in all considered cases. They are special versions of the well-known definitions of relative homological algebra, however adjusted to the needs of functional analysis. Therefore, in order to avoid a tiresome repetition, we shall expound the general-categorical scheme, embracing all our functional-analytic constructions (and a lot of others).

Let $\mathcal{K}$ be a fixed additive category.

Definition 1.1 A *pre-relative structure* in $\mathcal{K}$ is a faithful additive functor $\square : \mathcal{K} \to \mathcal{L}$. (We recall that a functor is called *faithful*, if it takes different morphisms in the domain category to different morphisms in the range category.) A category equipped with a pre-relative structure is called a *pre-relative category*.

The manner in which we introduced this concept emphasized that in our pair of categories, $\mathcal{K}$ is a principal and $\mathcal{L}$ is an auxiliary subject of our consideration. For the same reason, we shall use the expression "a pre-relative category $(\mathcal{K}, \Box : \mathcal{K} \to \mathcal{L})$" (or just "$(\mathcal{K}, \Box)$"); the meaning is obvious.

Our main triple of examples (simultaneously presented) is as follows. Let A be a fixed $\widetilde{\otimes}$-algebra.

Definition 1.2 The *category of left $\widetilde{\otimes}$-modules over A*, denoted by $\mathbf{A}$-$\widetilde{\otimes}$-$\mathbf{mod}$, is the category with the indicated modules as objects and the maps that are morphisms of modules in the algebraic sense and at the same time $\widetilde{\otimes}$-bounded operators as morphisms. This category is made pre-relative by means of the functor $\Box : \mathbf{A}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod} \to \widetilde{\otimes}\text{-}\mathbf{Ban}$, taking modules to their underlying $\widetilde{\otimes}$-spaces and morphisms to the same maps, but considered (only) as $\widetilde{\otimes}$-operators .

So, we see that our functor (or, better, three functors) $\Box$ belong to the large family with the generic name "forgetful functors". Our particular functors "forget about the outer multiplication".

In what follows, we shall use the adjective "forgetful" meaning, according to the context, either the just mentioned triple of "concrete" functors or the "abstract" functor from Definition 1. This, as well as use of the same notation "$\Box$", will not cause a confusion.

Sometimes we shall need the unital versions of our categories, denoted, in our "unspecified way", by $\mathbf{UA}$-$\widetilde{\otimes}$-$\mathbf{mod}$. Now the relevant basic $\widetilde{\otimes}$-algebra A is supposed to be unital, and the respective left $\widetilde{\otimes}$-modules over A are also supposed to be unital. Evidently, in the case of a unital A the category $\mathbf{UA}$-$\widetilde{\otimes}$-$\mathbf{mod}$ is a full subcategory in $\mathbf{A}$-$\widetilde{\otimes}$-$\mathbf{mod}$. We also consider it as a pre-relative category with respect to similarly defined forgetful functor $\Box$.

Remark As it was already said, we consider in these lectures/notes only left modules. However, the general scheme presented below works equally well in the cases of other types of modules, notably bimodules (= two-sided modules). Regretfully, we have no space/time to speak about these interesting questions and their applications.

Let us return to our abstract pre-relative category $(\mathcal{K}, \Box : \mathcal{K} \to \mathcal{L})$. In what follows, it will be convenient to speak about morphisms, meaning only those in $\mathcal{K}$, and, taking into account our principal examples, refer to morphisms in $\mathcal{L}$ as "operators". A pre-relative structure enables us to distinguish in $\mathcal{K}$ a class of "best" (in fact, "becoming best after the oblivion") morphisms:

Definition 1.3 A morphism $\sigma : X \to Y$ in $\mathcal{K}$ is called *admissible* if $\Box(\sigma)$ is a retraction (i.e. it has a right inverse operator) in $\mathcal{L}$.

Remark It is easy to see that an admissible morphism in $(\mathcal{K}, \Box)$ is necessarily an epimorphism. In fact, it would be more precise to call such morphisms "admissible as epimorphisms"; soon we shall see that this notion leads to the concept of a projective object of a pre-relative category. However, the different notion of a morphism, admissible as a monomorphism, which would lead to the concept of an injective object, is left outside the scope of our lectures.

Let us turn, for a moment, to our principal examples. For all of them it is obvious that any admissible morphism is surjective, and that its kernel, as a subspace of the relevant Banach space, has a Banach complement. As to the "classical" category $\mathbf{A}$-$\widehat{\otimes}$-$\mathbf{mod}$, it is also obvious that the converse is true; however, it is not so for our "quantum" categories (give a simple counter-example!).

We now come to the principal definition in our whole course of lectures. Again, we have an abstract pre-relative category $(\mathcal{K}, \ \Box : \mathcal{K} \to \mathcal{L})$.

Definition 1.4 An object $P \in \mathcal{K}$ is called *projective* (or, more precise, *projective relative to* $\Box$), if, for any admissible morphism $\sigma : X \to Y$ in $\mathcal{K}$ and an arbitrary morphism $\varphi : P \to Y$ in $\mathcal{K}$, there exists a morphism $\psi : P \to X$ (called a *lifting of* φ) such that the diagram

$$\begin{tikzcd} & X \arrow[d, "\sigma"] \\ P \arrow[ur, "\psi"] \arrow[r, "\varphi"] & Y \end{tikzcd}$$

is commutative.

Unfortunately, we have no space/time to expound the virtues of projectivity. In fact, all homology theory is based on this concept. In particular, methods, based on projectivity, are very powerful in the computation of cohomology and homology groups of our algebras, with all the consequences for the structure theory of "algebras in analysis" and various areas where these algebras serve. (See, e.g., [10] or [15]). Now let us just believe that the concept is indeed worthy of the most intent study.

Proposition 1.1 *A retract (in* $\mathcal{K}$*) of a projective object is projective.*

$\triangleleft$ Obviously, we need to show that a diagram in $\mathcal{K}$ of the form

$$\begin{tikzcd} P \arrow[r, "\psi_1"] \arrow[d, shift right, "\tau"'] & X \arrow[d, "\sigma"] \\ Q \arrow[u, shift right, "\rho"'] \arrow[ur, "\psi"] \arrow[r, "\varphi"] & Y \end{tikzcd}$$

(considered initially without ψ and ψ_1), in which P is projective, $\tau \circ \rho = \mathbf{1}_Q$ and σ is admissible, may be made commutative by adding a morphism ψ. Since P is projective, the composition $\varphi \circ \tau$ (taken as φ in the previous definition) has a lifting, say ψ_1. It remains to set $\psi := \psi_1 \circ \rho$. $\triangleright$

Proposition 1.2 *If P is projective, then every admissible morphism $\sigma : X \to P$ is a retraction in $\mathcal{K}$.*

$\triangleleft$ All we have to do is to set $Y := P$ and $\varphi := \mathbf{1}_P$ in the diagram (3). $\triangleright$

Referring to projective objects in our principal categories **A-$\widetilde{\otimes}$-mod** and **UA-$\widetilde{\otimes}$-mod**, we shall, naturally, call them *projective left A-$\widetilde{\otimes}$-modules* and, respectively, *projective unital left A-$\widetilde{\otimes}$-modules*. Sometimes in the first case it will be more convenient to use the term "*$\widetilde{\otimes}$-projective left A-module*", and in the second case the term "*$\widetilde{\otimes}$-projective unital left A-module*".

Since we can consider unital modules (over a unital algebra) in both categories, there is an apparent danger of confusion. However, as a matter of fact, the "unital" version of projectivity is consistent with the "general" version (see our Corollary 3 below).

Where are we to look for projective objects?

Consider, apart from the "abstract forgetful functor" $\square$, a functor $\mathcal{F} : \mathcal{L} \to \mathcal{K}$, acting, as you see, in the opposite direction. Consider also a natural transformation of functors $\alpha : \mathbf{1}_{\mathcal{L}} \to \square\mathcal{F}$. (We recall that this means that for any $E \in \mathcal{L}$ an operator $\alpha_E : E \to \square(\mathcal{F}E)$ is given in such a way that, for any operator $\rho : E \to G$ in $\mathcal{L}$, the diagram

$$\begin{tikzcd} E \arrow[r, "\rho"] \arrow[d, "\alpha_E"'] & G \arrow[d, "\alpha_G"] \\ \square(\mathcal{F}E) \arrow[r, "\square(\mathcal{F}\rho)"] & \square(\mathcal{F}G) \end{tikzcd}$$

is commutative.)

Definition 1.5 A functor $\mathcal{F} : \mathcal{L} \to \mathcal{K}$ is called a *freedom functor* (to be precise: a *freedom functor with respect to the pre-relative structure* $\square$) if there exists α as above (called *associated to* $\mathcal{F}$) with the following property: for any pair $(X \in \mathcal{K}, E \in \mathcal{L})$ and an operator $\varphi_0 : E \to \square X$ there exists a unique morphism $\varphi : \mathcal{F}E \to X$ making the diagram

$$\begin{tikzcd} E \arrow[d, "\alpha_E"'] \arrow[dr, "\varphi_0"] & \\ \square(\mathcal{F}E) \arrow[r, "\square\varphi"] & \square X \end{tikzcd} \tag{4}$$

commutative. In this situation, an object in $\mathcal{K}$ of the form $\mathcal{F}E$ with $E \in \mathcal{L}$ is called a *free object* (or $\mathcal{F}$-free object, if there is a danger of a confusion) *with the base* E

Definition 1.6 A pre-relative category is called *relative* if it has a freedom functor.

In what follows we shall use the expression "the relative category $(\mathcal{K}, \Box, \mathcal{F}, \alpha)$"; its meaning is obvious. Let us fix such an aggregate.

Theorem 1.1 *Any free object in a relative category is projective.*

$\triangleleft$ Let, in the accepted notation, $\mathcal{F}E$ with $E \in \mathcal{L}$ be our free object, and let σ and φ be the "lifting data" in Definition 4; in particular, we suppose that $\Box\sigma$ has a right inverse operator ρ.

Consider, in $\mathcal{L}$. the diagram

$$\begin{tikzcd} E \arrow[r, "\psi_0"] \arrow[d, "\alpha_E"'] & \Box X \arrow[d, shift right, "\Box\sigma"'] \\ \Box(\mathcal{F}E) \arrow[r, "\Box\varphi"] & \Box Y \arrow[u, shift right, "\rho"'] \end{tikzcd}$$

with $\psi_0 = \rho \circ (\Box\varphi) \circ \alpha_E$; obviously, it is commutative. By virtue of Definition 5 (the "existence part"), there is a morphism ψ, making the diagram

$$\begin{tikzcd} E \arrow[d, "\alpha_E"'] \arrow[dr, "\psi_0"] & \\ \Box(\mathcal{F}E) \arrow[r, "\Box\psi"] & \Box X \end{tikzcd}$$

commutative. Then we have

$$\Box(\sigma\psi) \circ \alpha_E = (\Box\sigma) \circ (\Box\psi) \circ \alpha_E = (\Box\sigma) \circ \psi_0 = (\Box\varphi) \circ \alpha_E.$$

Therefore, putting $\varphi_0 := (\Box\varphi)\circ\alpha_E$, we see that the diagram (4), which is commutative by its construction, remains commutative, if we replace $\Box\varphi$ by $\Box(\sigma\psi)$. Using again Definition 5 (but now its "uniqueness part"), we obtain that $\varphi = \sigma \circ \psi$. The rest is clear. $\triangleright$

What does this "abstract nonsense" give for our concrete pre-relative categories of modules? It turns out that all of them are relative, and their free objects can be constructed with the help of the respective version of the tensor product.

Let A be an $\widetilde{\otimes}$-algebra, X a A-$\widetilde{\otimes}$-module, and E a $\widetilde{\otimes}$-space.

Proposition 1.3 *The $\widetilde{\otimes}$-space $X \widetilde{\otimes} E$ is an A-$\widetilde{\otimes}$-module with respect to the outer multiplication, well defined by $a \cdot (x \otimes y) := a \cdot x \otimes y$.*

$\triangleleft$ Consider the outer product $\widetilde{\otimes}$-operator $\pi_X : A \widetilde{\otimes} X \to X$ (see Section 0). Applying to it the functor $? \widetilde{\otimes} E$ (see Section 0), we obtain the $\widetilde{\otimes}$-bounded operator

$$\pi_X \widetilde{\otimes} \mathbf{1} : (A \widetilde{\otimes} X) \widetilde{\otimes} E \to X \widetilde{\otimes} E.$$

Now recall that all tensor products, denoted now by "$\widetilde{\otimes}$", are associative; this means there exists an $\widetilde{\otimes}$-isometry between $\widetilde{\otimes}$-spaces $A \widetilde{\otimes} (X \widetilde{\otimes} E)$ and $(A \widetilde{\otimes} X) \widetilde{\otimes} E$, well defined by identifying the elementary tensors $a \otimes (x \otimes y)$ and $(a \otimes x) \otimes y$ [11, p. 38] and [6, pp. 159 and 128]. Composing this isometry with $\pi_X \widetilde{\otimes} \mathbf{1}$, we see that there exists an $\widetilde{\otimes}$-operator from $A \widetilde{\otimes} (X \widetilde{\otimes} E)$ to $X \widetilde{\otimes} E$, well defined by taking the elementary tensor $a \otimes (x \otimes y)$ to the same $(a \cdot x) \otimes y$. Denote by $\dot{m} : A \times (X \widetilde{\otimes} E) \to X \widetilde{\otimes} E$ the respective associated $\widetilde{\otimes}$-bioperator. Using the associativity property of the outer multiplication in X and the density of the span of the elementary tensors in $X \widetilde{\otimes} E$, we easily see that $\dot{m}$ has the associativity property, required for a left outer multiplication in $X \widetilde{\otimes} E$. $\triangleright$

As a particular case, when we consider our basic algebra A as a left $\widetilde{\otimes}$-module over itself (with the inner multiplication taken as an outer one), this construction provides an A-$\widetilde{\otimes}$-module $A \widetilde{\otimes} E$. Further, if our E is itself an A-$\widetilde{\otimes}$-module, say Y, we easily see that the outer product operator $\pi_Y : A \widetilde{\otimes} Y \to Y$ is actually a morphism in $\mathbf{A}$-$\widetilde{\otimes}$-$\mathbf{mod}$. (Indeed, the relation $\pi_Y(a \cdot u) = a \cdot \pi_y$ is obvious when u is an elementary tensor in $A \widetilde{\otimes} Y$, and, since the span of elementary tensors is dense in the latter space, it is true for all $u \in A \widetilde{\otimes} Y$.) Accordingly, in what follows π_Y will be referred to as the *outer product morphism for* Y. A distinguished representative of this class of morphisms, is, of course, our familiar $\pi : A \widetilde{\otimes} A \to A$ (corresponding to the case where $X = Y = A$); this will be called from now on *the product morphism for* A.

Now consider a $\widetilde{\otimes}$-operator $\rho : E \to G$ between two $\widetilde{\otimes}$-spaces and the respective $\widetilde{\otimes}$-operator $\mathbf{1} \widetilde{\otimes} \rho : X \widetilde{\otimes} E \to X \widetilde{\otimes} G$. Using the density of the span of elementary tensors in $X \widetilde{\otimes} E$, we immediately see that this operator is in fact a morphism of A-modules. This shows that, as a matter of fact, the covariant functor $X \widetilde{\otimes} ?$, introduced in Section 0 as acting on $\widetilde{\otimes}$-$\mathbf{Ban}$, now takes this category to $\mathbf{A}$-$\widetilde{\otimes}$-$\mathbf{mod}$.

In order to use the functor $X \widetilde{\otimes} ? : \widetilde{\otimes}$-$\mathbf{Ban} \to \mathbf{A}$-$\widetilde{\otimes}$-$\mathbf{mod}$ for our aims, we need at the moment a very special case of a module X. Namely, take our basic $\widetilde{\otimes}$-algebra A and put $A_+ := A \oplus \mathbb{C}$; it has a distinguished element $(0, 1)$, denoted by e and called *the adjoined identity*. From pure algebra we know that A_+ is a unital algebra, the so-called *unitization of* A, with the multiplication $(a + \lambda e)(b + \mu e) := ab + \lambda b + \mu a + \lambda\mu$ for $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$ (extending the given one in A). The identity in A_+ is, of course, the adjoined identity. Apart from this, any left module Y over A becomes a left module over A_+, with the extended outer multiplication $(a+\lambda e)\cdot x := a\cdot x+\lambda x$ for $a\in A$, $y\in Y$, and $\lambda\in\mathbb{C}$. However, we shall include these constructions in the framework of functional analysis a minute later. Right now we shall only make A_+ a left $\widetilde{\otimes}$-module over A.

In fact, there are a lot of ways to do this (as well as other promised things) in such a manner that the natural embedding of A into A_+ is an $\widetilde{\otimes}$-isometry. For simplicity, we choose the following concrete device. Consider the isometric operator $i : A \to \mathcal{B}(H)$, coinciding with the given quantization in both "quantized" cases and arbitrarily taken in the "classical" case (i.e. when $\widetilde{\otimes}$ = is $\widehat{\otimes}$). After this, we proceed to the operator $i\dot{\oplus}\mathbf{1} : A_+ = A\oplus\mathbb{C} \to \mathcal{B}(H\dot{\oplus}\mathbb{C})$ and consider it as a quantization of A_+ in the "quantum" cases; as to the "classical" case, we just equip A_+ with the induced norm. (Here in fact we have, of course, $\|a+\lambda e\| = \max\{\|a\|,|\lambda|\}$).

Consider the operators $j : A_+ \to A, a+\lambda e\mapsto a$ and $k : A_+\to\mathbb{C}, a+\lambda e\mapsto\lambda$. It easily follows from the choice of the $\widetilde{\otimes}$-norm in A_+ that both are $\widetilde{\otimes}$-bounded.

Proposition 1.4 *The $\widetilde{\otimes}$-space A_+, equipped with the outer multiplication*

$$\dot{m} : A\times A_+ \to A_+, \quad (a, b+\lambda e)\mapsto ab+\lambda a$$

(that is, with the respective restriction of the inner multiplication in A_+) is an A-$\widetilde{\otimes}$-module.

◁ The operators j and k, indicated above, give rise to operators $\mathbf{1}\,\widetilde{\otimes}\, j : A\,\widetilde{\otimes}\, A_+ \to A\,\widetilde{\otimes}\, A$ and $\mathbf{1}\,\widetilde{\otimes}\, k : A\,\widetilde{\otimes}\, A_+ \to A\,\widetilde{\otimes}\,\mathbb{C}$, participating in the (non-commutative!) diagram

$$\begin{tikzcd}
& A\,\widetilde{\otimes}\, A \arrow[dr, "\pi"] & & \\
A\,\widetilde{\otimes}\, A_+ \arrow[ur, "\mathbf{1}\,\widetilde{\otimes}\, j"] \arrow[dr, "\mathbf{1}\,\widetilde{\otimes}\, k"'] & & A \arrow[r, "\mathbf{in}"] & A_+ \\
& A\,\widetilde{\otimes}\,\mathbb{C} \arrow[ur, "i"'] & &
\end{tikzcd}$$

where π is the product operator (see above), i the standard $\widetilde{\otimes}$-isometry between $A\,\widetilde{\otimes}\,\mathbb{C}$ and A (taking $\lambda\otimes y$ to λy), and **in** is the natural embedding. Put $\pi_+ := (\mathbf{in})\circ\pi\circ(\mathbf{1}\,\widetilde{\otimes}\, j)+(\mathbf{in})\circ i\circ(\mathbf{1}\,\widetilde{\otimes}\, k)$. Since all indicated operators are $\widetilde{\otimes}$-bounded, the same is true for π_+. Therefore the bioperator $\dot{m}$, being obviously associated to π_+ is, by virtue of Theorem 0, $\widetilde{\otimes}$-bounded. ▷

Now we concentrate on the functor $A_+\widetilde{\otimes}? : \widetilde{\otimes}$-**Ban** $\to$ **A**-$\widetilde{\otimes}$-**mod**, that is on the particular case of $X\widetilde{\otimes}?$ (see above) when $X := A_+$. Further, for any $E\in\widetilde{\otimes}$-**Ban**, consider the map $\alpha_E : E \to \square(A_+ \widetilde{\otimes} E), y \mapsto e \otimes y$. Obviously, it is an $\widetilde{\otimes}$-operator between E and the $\widetilde{\otimes}$-space $\square(A_+ \widetilde{\otimes} E)$, where, as we recall, $\square$ is the forgetful (about the outer multiplication) functor from $\widetilde{\otimes}$-**Ban** to **A**-$\widetilde{\otimes}$-**mod**. It is easy to see that the family of operators $\alpha = \{\alpha_E; E \in \widetilde{\otimes}\text{-}\mathbf{Ban}\}$ is a natural transformation between the two functors, acting on the category $\widetilde{\otimes}$-**Ban**, namely between the identity functor $\mathbf{1}$ and the functor $\square(A_+\widetilde{\otimes}?)$.

When it is clear from the context that we consider, for a moment, a given $\widetilde{\otimes}$-module X just as a $\widetilde{\otimes}$-space, we shall write either $\square X$ or simply X: this will not cause any confusion. In particular, the symbols $A_+ \widetilde{\otimes} X$ and $A_+ \widetilde{\otimes} \square X$ mean certainly the same object.

Proposition 1.5 *The functor $A_+\widetilde{\otimes}?$ is a freedom functor with respect to the pre-relative structure $\square : \mathbf{A}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod} \to \widetilde{\otimes}\text{-}\mathbf{Ban}$, and its associated natural transformation of functors is α. (Thus the pre-relative category $(\mathbf{A}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}, \square)$ is in fact relative.) Moreover, if $E \in \widetilde{\otimes}\text{-}\mathbf{Ban}$, $X \in \mathbf{A}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}$, and a $\widetilde{\otimes}$-bounded operator $\varphi_0 : E \to X$, then the morphism φ, appearing in Definition 5 and now acting from $A_+ \widetilde{\otimes} E$ to X, is well defined by taking $(a + \lambda e) \otimes y = a \cdot (e \otimes y) + \lambda(e \otimes y)$ to $a \cdot \varphi_0(y) + \lambda\varphi_0(y)$.*

$\triangleleft$ According to the definition of the freedom, we must take E, X and φ_0, indicated in the formulation, and show that there exists a unique morphism φ such that $(\square\varphi) \circ \alpha_E = \varphi_0$; in the present context the latter equality just means that $\varphi(e \otimes y) = \varphi_0(y)$ for all $y \in E$.

First of all, our desired φ, being a morphism, must take $(a+\lambda e)\otimes y = a\cdot(e\otimes y)+\lambda(e\otimes y)$ to $a \cdot \varphi_0(y) + \lambda\varphi_0(y)$, as it is indicated in the formulation. Therefore, φ is uniquely determined on elementary tensors and hence everywhere in $\varphi : A_+ \widetilde{\otimes} E$. Uniqueness has been proved; turn to the existence.

We now recall the operators $j : A_+ \to A$ and $k : A_+ \to \mathbb{C}$. They give rise to the $\widetilde{\otimes}$-bounded operators $j \widetilde{\otimes} \mathbf{1}_E : A_+ \widetilde{\otimes} E \to A \widetilde{\otimes} E$ and $k \widetilde{\otimes} \mathbf{1}_E : A_+ \widetilde{\otimes} E \to \mathbb{C} \widetilde{\otimes} E$, participating in the (non-commutative) diagram

$$\begin{tikzcd}
 & A \widetilde{\otimes} E \arrow[r, "\mathbf{1}\widetilde{\otimes}\varphi_0"] & A \widetilde{\otimes} X \arrow[dr, "\pi_X"] & \\
A_+ \widetilde{\otimes} E \arrow[ur, "j \widetilde{\otimes} \mathbf{1}"] \arrow[dr, "k \widetilde{\otimes} \mathbf{1}"'] & & & X \\
 & \mathbb{C} \widetilde{\otimes} E \arrow[r, "i"] & E \arrow[ur, "\varphi_0"'] &
\end{tikzcd}$$

where i is the standard $\widetilde{\otimes}$-isometry between $\mathbb{C}\widetilde{\otimes}X$ and X (taking $\lambda\otimes y$ to λy). Now define φ as $\pi \circ (\mathbf{1}_A \widetilde{\otimes} \varphi_0) \circ (j \widetilde{\otimes} \mathbf{1}_E) + \varphi_0 \circ i \circ (k \widetilde{\otimes} \mathbf{1}_E)$. Since all our operators are $\widetilde{\otimes}$-bounded, the same is true for φ. Further, the lower row of depicted operators immediately gives $\varphi(e \otimes y) = \varphi_0(y); y \in E$, as required. Finally, for an elementary tensor $u := (a + \lambda e) \otimes y$ we obviously have $\varphi(u) = a \cdot \varphi_0(y) + \lambda\varphi_0(y)$. On the other hand, for any $b \in A$ the upper row of depicted operators provides that $\varphi(b \cdot u)$, that is $\varphi((ba + \lambda b)\otimes$, is equal to $(ba + \lambda b) \cdot \varphi_0(y)$. Therefore we have $\varphi(b \cdot u) = b \cdot \varphi(u)$ for any elementary tensor u in $A_+ \widetilde{\otimes} E$. Since the density of the span of these elements in the latter space, such an equality holds for all elements u in $A_+ \widetilde{\otimes} E$. Thus φ is a morphism of left modules over A. $\triangleright$

Corollary 1.1 *The modules of the form $A_+\widetilde{\otimes}E$ with $E \in \widetilde{\otimes}$-**Ban** are projective objects in **A**-$\widetilde{\otimes}$-**mod**.*

In accordance with what was said above, the mentioned left A-$\widetilde{\otimes}$-modules will be called *free*.

The pre-relative category (**UA**-$\widetilde{\otimes}$-**mod**$, \Box)$ is also relative, and in this case the construction of the relevant freedom functor is a little bit simpler. It is the functor

$$A\widetilde{\otimes}? : \widetilde{\otimes}\text{-}\mathbf{Ban} \longrightarrow \mathbf{UA}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod},$$

and the respective natural transformation of functors is the family $\alpha := \{\alpha_E : E \to \Box(A_+ \widetilde{\otimes} E), y \mapsto e \otimes y\}$, where this time e denotes the "inner" (given) identity in A. (Now, of course, α connects the identity functor $\mathbf{1}$ and the functor $\Box(A\widetilde{\otimes}?)$.) We leave the details to the reader.

Compare, in the case of a general A, the modules $A_+ \widetilde{\otimes} E$ and $A \widetilde{\otimes} E$. Let **in** denote the natural embedding of A into A_+. Since the operator $\Box(\mathbf{in} \widetilde{\otimes} \mathbf{1}_E) : A \widetilde{\otimes} E \to A_+ \widetilde{\otimes} E$ has a right inverse, namely $j \widetilde{\otimes} \mathbf{1}_E$, it gives rise to a $\widetilde{\otimes}$-isomorphism of $A \widetilde{\otimes} \mathbf{1}_E$ onto its image. (In fact, it gives rise to an $\widetilde{\otimes}$-isometry, but we do not need this now.) Therefore we have a right to identify $A \widetilde{\otimes} E$ with this image, and consider the former as a closed submodule in $A_+\widetilde{\otimes}E$.

We return to the general scheme of homology in an abstract relative category $(\mathcal{K}, \Box : \mathcal{K} \to \mathcal{L})$ with a freedom functor $\mathcal{F}$ and the associate natural transformation of functors α. Now we shall show that the presence of a freedom functor actually gives more than just a class of projective objects. Namely, every object happens to be the range of an admissible morphism from a free object, and in terms of that morphism a workable equivalent definition of projectivity can be presented.

Take an arbitrary $X \in \mathcal{K}$, and consider the identity operator $\mathbf{1}_{\Box X}$ in $\mathcal{L}$. The latter, being taken as φ_0 in the Definition 5, gives rise to a morphism from $\mathcal{F}(\Box X)$ to X in $\mathcal{K}$ (denoted in that definition by φ). It deserves a special name and a special notation.

Definition 1.7 The indicated morphism is called *the canonical morphism for X* and is denoted by π_X^+.

Proposition 1.6 *The morphism π_X^+ is admissible.*

$\triangleleft$ Taking in Definition 5 $\varphi_0 := \mathbf{1}_{\Box X}$ and $\varphi := \pi_X^+$, we see that $(\Box\pi_X^+)\circ\alpha_{\Box X} = \mathbf{1}_{\Box X}$. The rest is clear. $\triangleright$

Theorem 1.2 *The following properties of an object $P \in \mathcal{K}$ are equivalent:*

(i) *P is projective;*

(ii) *P is a retract of a free object in $\mathcal{K}$;*

(iii) *the canonical morphism $\pi_P^+ : \mathcal{F}(\Box P) \to P$ is a retraction.*

$\triangleleft$ (i) $\Longrightarrow$ (iii). This follows from Theorem 1, and Propositions 2 and 6 combined.
(iii) $\Longrightarrow$ (ii) is clear.
(ii) $\Longrightarrow$ (i). This follows from Theorem 1 and Proposition 1 combined. $\triangleright$

As to our main example, that is the (triple) relative category $(\mathbf{A}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}, \Box)$, we can easily see the form of the canonical morphism in this case. Taking, for a given $X \in \mathbf{A}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}$, $\Box X$ as E, and $\alpha_{\Box X}$ as φ_0, we see that the morphism $\pi_X^+ : A \mathbin{\widetilde{\otimes}} X \to X$ is well defined by taking $(a+\lambda e)\otimes x$ for $a \in A$, $x \in X$, and $\lambda \in \mathbb{C}$ to $a\cdot x + \lambda x$. (Equivalently, if we remember from the pure algebra that X is a left module over A_+, we can take $b \otimes x$ for $b \in A_+$ and $x \in X$ to $b \cdot x$.) In the particular case $X := A_+$ (see Proposition 4), we get the morphism $\pi^+ : A_+ \mathbin{\widetilde{\otimes}} A_+ \to A_+$, well defined by taking $b \otimes c$ for $b, c \in A_+$ to bc. (Here we recall that A_+ is an algebra.) Observing that the multiplication in A_+ is associated with π^+ and that the outer multiplication in any $X \in \mathbf{A}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}$ is associated with π_X^+, we immediately get

Corollary 1.2 (i) *If the algebra A is an $\widetilde{\otimes}$-algebra, then the same is true for A_+.*

(ii) *If $X \in \mathbf{A}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}$, then X, as a left module over A_+, is a $\widetilde{\otimes}$-module.*

The previous theorem, being considered for the case of the categories $\mathbf{A}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}$ and $\mathbf{UA}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}$ acquires the following specific guise:

Theorem 1.2' *Let A be an arbitrary (respectively, a unital) $\widetilde{\otimes}$-algebra. The following properties of an arbitrary (respectively, a unital) A-$\widetilde{\otimes}$-module P are equivalent:*

(i) *P is projective in $\mathbf{A}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}$ (respectively, in $\mathbf{UA}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}$);*

(ii) *P is a retract of a module of a form $A_+ \mathbin{\widetilde{\otimes}} E$ (respectively, $A \mathbin{\widetilde{\otimes}} E$), where E is a $\widetilde{\otimes}$-space;*

(iii) *the canonical morphism $\pi_P^+ : A_+ \mathbin{\widetilde{\otimes}} P \to P$ (respectively, the outer product morphism $\pi_P : A \mathbin{\widetilde{\otimes}} P \to P$) is a retraction.* $\triangleleft\triangleright$

We conclude this section with the following general observation. Working with, generally speaking, non-unital modules, we can rather frequently check their projectivity with the help of the outer product morphisms instead of the somewhat more complicated canonical morphisms.

For any $X \in \mathbf{A}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}$, we denote the closure of $span\{a \cdot x; a \in A, x \in X\}$ by $A \cdot X$. Obviously, $A \cdot X$ is a submodule in X, called the *essential submodule of* X. In particular, the essential submodule of $A_+ \widetilde{\otimes} E$ for $E \in \widetilde{\otimes}\text{-}\mathbf{Ban}$ is evidently $A \widetilde{\otimes} E$ (identified, as we recall, with the image of $\mathbf{in} \widetilde{\otimes} \mathbf{1}_E$). A module X is called *non-degenerate (or stable, or essential)* if $A \cdot X = X$. Also observe that, for any morphism $\varphi : X \to Y$ in $\mathbf{A}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}$, we obviously have $\varphi(A \cdot X) \subseteq A \cdot Y$.

Theorem 1.3 *For the projectivity of $P \in \mathbf{A}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}$ it is sufficient, and if P is non-degenerate, it is also necessary that the outer product morphism $\pi : A \widetilde{\otimes} P \to P$ is a retraction.*

$\triangleleft$ The composition of a right inverse morphism to π with the morphism $\mathbf{in} \widetilde{\otimes} \mathbf{1}_P$ is obviously a right inverse morphism to π^+. By virtue of the previous theorem, this proves the sufficiency.

Further, let P be projective; then the previous theorem gives a morphism $\rho : P \to A_+ \widetilde{\otimes} P$, right inverse to π^+. Since $P = A \cdot P$, it follows from the mentioned properties of essential submodules that the image of ρ lays in $A \widetilde{\otimes} P$. Evidently, the respective corestriction of ρ is a right inverse to π. $\triangleright$

Since unital modules over unital algebras are certainly non-degenerate, we have an immediate

Corollary 1.3 *A unital left $\widetilde{\otimes}$-module over a unital $\widetilde{\otimes}$-algebra A is projective as an object of* $\mathbf{UA}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}$ $\Longleftrightarrow$ *it is projective as an object in* $\mathbf{A}\text{-}\widetilde{\otimes}\text{-}\mathbf{mod}$.

It can well happen, as it will be demonstrated by examples, that the same module, being not projective in the classical theory, becomes projective after some natural quantization. Examples of the opposite meaning also exist. However, if we shall compare the projectivity in the two quantum theories, there is a certain "one-way" connection:

Proposition 1.7 *Let X be a $\widetilde{\otimes}$-module over an $\widetilde{\otimes}$-algebra A for the case $\widetilde{\otimes} = \overset{h}{\otimes}$ and hence $\widetilde{\otimes} = \overset{o}{\otimes}$. Suppose that X is $\overset{o}{\otimes}$-projective. Then it is $\overset{h}{\otimes}$-projective.*

$\triangleleft$ Let $\rho : X \to A_+ \overset{o}{\otimes} X$ be a right inverse to the canonical morphism for X in $\mathbf{A}\text{-}\overset{o}{\otimes}\text{-}\mathbf{mod}$. Then the operator composition $j_2 \circ \rho : X \to A_+ \overset{h}{\otimes} X$, where j_2 is an operator, indicated in the introductory section (see diagram (3)), is obviously a right inverse to the canonical morphism for X in $\mathbf{A}\text{-}\overset{h}{\otimes}\text{-}\mathbf{mod}$. The rest is clear. $\triangleright$

2 Projective and non-projective ideals

From now on, we proceed to one of most typical problem in topological homology: *Which modules, belonging to this or that well known and "popular" class, are projective?* In fact, this means clarifying the connections between projectivity and the properties of algebras and modules expressed in traditional terms of algebra, analysis and topology.

It seems natural to begin with such an important class of modules as ideals, proper and non-proper alike.

Let A be an $\widetilde{\otimes}$-algebra, and I be its left closed ideal. Obviously, I as a left A-$\widetilde{\otimes}$-module with respect to the outer multiplication, defined by $a \cdot x := ax; a \in A, x \in I$ (that is, as the inner multiplication in A).

When is such a module projective?

In particular, this question concerns the non-proper ideal of A, that is A itself. If the latter, as a left A-$\widetilde{\otimes}$-module, is projective, we shall say that A is a *left projective* $\widetilde{\otimes}$*-algebra*.

First of all, we distinguish a simple sufficient condition.

Proposition 2.1 *If I has, as an algebra, a right identity, then it is projective.*

$\triangleleft$ Consider the map $\tau : A_+ \to I, a \mapsto ap$, where p is the mentioned right identity. Since the multiplication in A_+ is a $\widetilde{\otimes}$-bioperator, τ is an $\widetilde{\otimes}$-operator. ("The joint continuity implies the separate continuity".) Moreover, it is evidently a morphism of A-modules. Finally, it is obvious that $\tau \circ (\mathbf{in}) = \mathbf{1}_I$, where **in** is the natural embedding of I into A_+. We see that I is a retract in $\mathbf{A}$-$\widetilde{\otimes}$-$\mathbf{mod}$ of the free module $A_+ = A_+ \widetilde{\otimes} \mathbb{C}$, and therefore, by virtue of Theorem 1.2'(ii), it is projective. $\triangleright$

One can easily guess that the indicated sufficient condition of the projectivity is far away from being necessary. Before presenting the simplest relevant example, we shall recall a natural way of quantization of a very important class of Banach algebras.

Let Ω be a locally compact topological space. As usual, $C_0(\Omega)$ denotes the Banach $(= \widehat{\otimes}$-)algebra of all continuous functions on Ω vanishing the infinity with the pointwise operations and the sup-norm. This algebra (more accurately, its underlying Banach space) is equipped with the following special quantization. Consider the isometric operator from $C_0(\Omega)$ into $\mathcal{B}(l_2(\Omega))$, taking a function $a(t)$ to the "diagonal" operator of the coordinate-wise multiplication $g(t) \mapsto a(t)g(t)$ for $g \in l_2(\Omega)$. It is easy to see that the norm in the space $M_n(C_0(\Omega))$, corresponding to such a quantization, is given by

$$\|a\|_n = \sup\{\|a(t)\|; t \in \Omega\};$$

here $a = (a_{ij}) \in M_n(C_0(\Omega))$ and $\|a(t)\|$ is the standard norm of the scalar matrix $(a_{ij}(t)) \in M_n$. (We recall that M_n is identified with $\mathcal{B}(\mathbb{C})$.) The indicated quantization of $C_0(\Omega)$ will be called *standard*.

We recall that a *uniform algebra* is, by definition, a closed subalgebra in some $C_0(\Omega)$. Such an algebra is considered as a Banach algebra and as a quantized space with the inherited norm and quantization; the latter quantization will be also called *standard*[3]. It is well known (and easy to check) that every uniform algebra is an $\overset{h}{\otimes}$- and hence an $\overset{o}{\otimes}$-algebra with respect to the standard quantization.

Example 2.1 Let $\mathbb{D}$ be the closed unit disc in $\mathbb{C}$, and $\mathcal{A}(\mathbb{D})$ be the disc-algebra. (Recall that it is a closed subalgebra in $C(\mathbb{D})$, consisting of functions, analytic in the interior of the disc). Consider the maximal ideal $\mathcal{A}_0(\mathbb{D}) = \{w : w(0) = 0\}$ in $\mathcal{A}(\mathbb{D})$.

From what was said about the norm and the quantization of a uniform algebra, it follows easily that the map from $\mathcal{A}(\mathbb{D})$ to $\mathcal{A}_0(\mathbb{D})$, taking $w(z)$ to $zw(z)$, is an $\widetilde{\otimes}$-isomorphism between left $\mathcal{A}(\mathbb{D})$-$\widetilde{\otimes}$-modules $\mathcal{A}(\mathbb{D})$ and $\mathcal{A}_0(\mathbb{D})$. Since the algebra $\mathcal{A}(\mathbb{D})$ is unital, it is a free unital module over itself and hence projective. Therefore the ideal $\mathcal{A}_0(\mathbb{D})$ is a projective module as well.

This example shows that a projective ideal in a $\widetilde{\otimes}$-algebra is by no means obliged to have a right identity.

Exercise 2.1 Show that $\mathcal{A}_0(\mathbb{D})$ is also a projective module over itself, i.e. it is as a left projective $\widetilde{\otimes}$-algebra. And this is despite the product morphism π_A for such an A is not a surjective map. (Thus we see, in particular, that the condition on X to be non-degenerate in Theorem 1.3 can not be omitted.)

Now we concentrate on ideals of the $\widetilde{\otimes}$-algebras $C_0(\Omega)$. As it is well known, every closed ideal I of such an algebra has the form $I = \{a \in C_0(\Omega) : a(t) = 0 \text{ for all } t \in \Delta\}$, where Δ is a closed subset in Ω, the so-called *hull* of the ideal I. We recall also that the open set $\Omega_I := \Omega \setminus \Delta$ coincides, up to a homeomorphism, with the Gel'fand spectrum of I as of a commutative Banach algebra.

It turns out that the property of an ideal I in $C_0(\Omega)$ to be or not to be projective is completely determined by the topology of Ω_I. We recall that a topological space T is called *paracompact* if every open cover σ of T has a locally finite open refinement σ_0. Here "refinement" means that every set, belonging to σ_0, is contained in some set of σ, whereas "locally finite" means the following: every point in T has such a neighborhood that has non-empty intersections only with finite number of sets, belonging to σ_0.

Theorem 2.1 (i) *A closed ideal in the $\widetilde{\otimes}$-algebra $C_0(\Omega)$ is a projective $C_0(\Omega)$-$\widetilde{\otimes}$-module $\Longleftrightarrow$ its Gel'fand spectrum is paracompact. In particular, $C_0(\Omega)$ is a left projective $\widetilde{\otimes}$-algebra $\Longleftrightarrow$ Ω is a paracompact space.*

[3]Actually, this quantization is the particular case of the so-called minimal quantization, the procedure that can be applied to an arbitrary Banach space; cf. [6]. But we do not need this fact in our notes.

(ii) (the generalization of the "$\Longrightarrow$" part of (i)) *If a closed ideal I in an arbitrary commutative $\widetilde{\otimes}$-algebra A is a projective A-$\widetilde{\otimes}$-module, then the Gel'fand spectrum of I is paracompact. In particular, if a commutative $\widetilde{\otimes}$-algebra is left projective, then its Gel'fand spectrum is paracompact.*

We shall not give here the complete proof of this theorem which is rather long and in some parts technical; it is presented, in the traditional framework (that is for $\widehat{\otimes}$-algebras) in [10]. Instead, we shall consider in details two illuminating particular cases. The first one will illustrate the "sufficiency", and the second one will deal with the "necessity".

Theorem 2.2 *If Ω is a metrizable compact space, then every closed ideal in the $\widetilde{\otimes}$-algebra $C_0(\Omega)$ is projective.*

(Since the well-known theorem of A. Stone asserts that every metrizable topological space is paracompact, this result immediately follows from Theorem 1. However, we shall give an independent proof.)

$\triangleleft$ Let I be a given ideal. By virtue of Theorem 1.3 (or, if you prefer, Theorem 1.2'(ii) and Corollary 1.3), it is sufficient to show that the canonical morphism $\pi := \pi_I : A \mathbin{\widetilde{\otimes}} I \to I$ has a right inverse in $C(\Omega)$-$\widetilde{\otimes}$-**mod**.

Let $\Delta \subseteq \Omega$ be the hull of I. Since Ω is metrizable, there exists a base U_n with $n = 1, 2, \ldots$ of neighborhoods of Δ, such that the closure of U_{n+1} lies in U_n. Take $e_n \in I$ such that $0 \leq e_n \leq 1, e_n(t) = 1$ for $t \notin U_n$, and $e_n(t) = 0$ for $t \in U_{n+1}$ with $n = 1, 2, \ldots$. Obviously, this sequence of functions is an approximate identity for I. Further, we set $y_1 := e_1$ and $y_n := e_n - e_{n-1}$ for $n > 1$. Finally, we set $z_n := \sqrt{y_n}$ for all n.

Now, for an arbitrary $x \in I$, we consider the formal series $\sum_{n=1}^{\infty} xz_n \otimes z_n$ of elementary tensors in $A \otimes I$. We remember, that the latter linear space is dense in the Banach space $A \mathbin{\widetilde{\otimes}} I$. (Here, of course, by $A \overset{h}{\otimes} I$ and $A \overset{o}{\otimes} I$ we mean the "usual" Banach space in the first floor of the respective quantized Banach space.) We want to prove that this series converges in the $A \mathbin{\widetilde{\otimes}} I$, and simultaneously to estimate the norm of its sum. For this aim we need

Lemma *Consider, for any $a \in C(\Omega)$ and natural m, n with $m < n$, the sum*

$$u := \sum_{k=m+1}^{n} az_n \otimes z_n.$$

Then the norm of this element in the Banach space $A \mathbin{\widetilde{\otimes}} I$ satisfies $\|u\| \leq 2C$, where $C := \max\{|a(t)| : t \in U_{m-1}\}$, if $m > 1$, and $C := \|a\|$, if $m = 1$.

$\triangleleft$ The idea of the proof is to represent u in the form, more convenient to our needs. Namely, consider in $A \mathbin{\widetilde{\otimes}} I$ the element

$$v = \frac{1}{n-m} \sum_{k=1}^{n-m} \left[\left(\sum_{q=1}^{n-m} \zeta^{qk} a z_{m+q} \right) \otimes \left(\sum_{r=1}^{n-m} \zeta^{-rk} z_{m+r} \right) \right],$$

where ζ is $e^{2\pi i/(n-m)}$ (the primitive root of the degree $n-m$ from 1). Since the norm in $A \mathbin{\widetilde{\otimes}} I$ is a crossnorm (that is, the norm of any elementary tensor is the product of norms of its components), we have

$$\|v\| \leq \frac{1}{n-m} \sum_{k=1}^{n-m} \left\| \sum_{q=1}^{n-m} \zeta^{qk} a z_{m+q} \right\| \left\| \sum_{r=1}^{n-m} \zeta^{-rk} z_{m+r}) \right\|,$$

where the norms in the right side of the inequality are, as we remember, sup-norms in $C(\Omega)$. Further, for every $t \in \Omega$ we have

$$\left| \sum_{q=1}^{n-m} \zeta^{qk} a(t) z_{m+q}(t) \right| \leq |a(t)| \sum_{q=1}^{n-m} z_{m+q}(t).$$

Functions $z_n(t)$ are chosen in such a way that $\sum_{n=1}^{\infty} z_n^2(t) \leq 1$, and no more than two summands in this sum are not zero; therefore $\sum_{q=1}^{n-m} z_{n+q}(t) \leq \sqrt{2}$. Apart from this, if $t \notin U_n$, then obviously $z_{n+k} = 0$ holds for all $k = 1, 2, \ldots$. Therefore $\left| \sum_{q=1}^{n-m} \zeta^{qk} a(t) z_{m+q}(t) \right| \leq C\sqrt{2}$ and $\left| \sum_{r=1}^{n-m} \zeta^{qk} z_{m+r}(t) \right| \leq \sqrt{2}$ hold for all $t \in \Omega$ and $k = 1, \ldots, n-m$. Combining this with the estimate for $\|v\|$ indicated above, we immediately get $\|v\| \leq 2C$.

But what a bird is this v? Using algebraic properties of the operation "$\otimes$" and collecting similar terms, we see that

$$v = \frac{1}{n-m} \sum_{q,r=1}^{n-m} \lambda_{q,r} a z_{m+q} \otimes z_{m+r},$$

where $\lambda_{q,r} = \sum_{k+1}^{n} \zeta^{k(q-r)}$. But then evidently $\lambda_{q,r} = n-m$ if $q = r$ and $\lambda_{qr} = 0$ otherwise. Thus v is not other thing than our initial u. The rest is clear. $\triangleright$

The end of the proof Return to the formal series $\sum_{n=1}^{\infty} x z_n \otimes z_n$ with $x \in I$. Let $\varepsilon > 0$ be given. Since $x = 0$ on Δ, there exists $m \in \mathbb{N}$ such that $|x(t)| < \varepsilon$ whenever $t \in U_m$. Therefore by virtue of Lemma we have, for any $n > m$,

$$\left\| \sum_{k=m+1}^{n} x z_n \otimes z_n \right\| < 2\varepsilon.$$

Therefore our series fulfills the Cauchy criterion and hence converges in the Banach space $C(\Omega) \mathbin{\widetilde{\otimes}} I$. Denote the sum of this series by $\rho_{\widetilde{\otimes}}(x)$.

Now consider the map $\rho_{\widetilde{\otimes}} : I \to C(\Omega) \widetilde{\otimes} I, x \mapsto \rho_{\widetilde{\otimes}}(x)$. Evidently, it is a linear operator and, moreover, a morphism of left $C(\Omega)$-modules (to begin with, in the sense of pure algebra). Moreover,

$$\pi \circ \rho_{\widetilde{\otimes}}(x) = \pi \left(\lim_{n\to\infty} \sum_1^n x z_k \otimes z_k \right)$$
$$= \lim_{n\to\infty} \pi \left(\sum_1^n x z_k \otimes z_k \right) = \lim_{n\to\infty} \sum_1^n x z_k^2 = \lim_{n\to\infty} x e_n = x.$$

Thus $\pi \circ \rho_{\widetilde{\otimes}} = \mathbf{1}_I$, and it remains, of all conditions of Theorem 1.3, only to show that $\rho_{\widetilde{\otimes}}$ is an $\widetilde{\otimes}$-operator.

Our Lemma, this time considered for $m = 1$, gives $\|\rho_{\widetilde{\otimes}}(x)\| \leq 2\|x\|$. Thus $\rho_{\widetilde{\otimes}}$ is bounded. Of course, this observation completes the proof in the "classical" case $\widetilde{\otimes} = \widehat{\otimes}$. As to the two "quantized" cases, it is still not the end of the story: we must show that, furthermore, the operator $\rho_{\overset{o}{\otimes}}$ is *completely* bounded. It will imply that I is $\overset{o}{\otimes}$-projective, and this, by Proposition 1.7, will guarantee that it is $\overset{h}{\otimes}$-projective as well. In what follows, we shall denote the map $\rho_{\overset{o}{\otimes}}$ just by ρ.

We fix $n \in \mathbb{N}$ (choosing the size of relevant matrices) and consider the operator

$$\rho_n : M_n(I) \to M_n(C(\Omega) \overset{o}{\otimes} I),$$

the respective amplification of ρ. Then for the matrix $\mathbf{x} = (x_{ij}) \in M_n(I)$ we have $\rho_n(\mathbf{x}) = (\rho(x_{ij})) = \lim_{l\to\infty} \rho_n^{(l)}\mathbf{x})$, where $\rho_n^{(l)}(\mathbf{x}) \in M_n(C(\Omega) \overset{o}{\otimes} I)$ is the matrix with the entries $u_{ij}^{(l)} := \sum_{k=1}^l x_{ij} z_k \otimes z_k$. In other words, if $\dot{m} : C(\Omega) \times (C(\Omega) \widetilde{\otimes} I)$ is the bioperator of the outer multiplication, and $\mathbf{z}_l$ is the short notation for $\sum_{k=1}^l z_k \otimes z_k$, then

$$\rho_n^{(l)}(\mathbf{x}) = \dot{m}_{n,1}(\mathbf{x}, \mathbf{z}_l).$$

Here, $\dot{m}_{n,1} : M_n(C(\Omega)) \times (C(\Omega) \overset{o}{\otimes} I) \to M_n(C(\Omega) \overset{o}{\otimes} I)$ is the $n,1$-th complete amplification of $\dot{m}$, discussed in the introductory section. (We identify, of course, $C(\Omega) \overset{o}{\otimes} I$ with $M_{1\times 1}(C(\Omega) \overset{o}{\otimes} I)$ and $M_n(C(\Omega) \overset{o}{\otimes} I)$ with $M_{n\times 1}(C(\Omega) \overset{o}{\otimes} I)$.)

Finally, recall that the bioperator $\dot{m}$ is, by virtue of Proposition 1.3, completely bounded. (In fact, in our concrete situation it is even completely contractive, but we do not need it here.) Since the completely bounded norm of $\dot{m}$ is, by definition, the upper bound of the (usual) norms of all complete amplifications of this bioperator, and these include, of course, $\dot{m}_{n,1}$, we immediately have $\|\dot{m}_{n,1}\| \leq \|\dot{m}\|_{\overset{o}{\otimes}}$, and hence $\|\rho_n^{(l)}(\mathbf{x})\| \leq \|\dot{m}\|_{\overset{o}{\otimes}}\|\|\mathbf{x}\|\|\mathbf{z}_l\|$. Using our Lemma again, now in the simplest case $a \equiv 1$ and $m = 1$, we see that $\|\mathbf{z}_l\| \leq 2$, and hence $\|\rho_n^{(l)}(\mathbf{x})\| \leq 2\|\dot{m}\|_{\overset{o}{\otimes}}\|\|\mathbf{x}\|$. Since this happens for any l, we have the limit equality $\|\rho_n(\mathbf{x})\| \leq 2\|\dot{m}\|_{\overset{o}{\otimes}}\|\|\mathbf{x}\|$ and hence $\|\rho_n\| \leq 2\|\dot{m}\|_{\overset{o}{\otimes}}$ for all n. Thus the operator ρ is completely bounded (and, moreover, satisfies$\|\rho\|_{\overset{o}{\otimes}} \leq 2\|\dot{m}_{\overset{o}{\otimes}}\|$). ▷

We proceed to an example of an ideal with "topologically bad" spectrum.

Denote by Φ the segment of the transfinite line, ending with the first uncountable cardinal $\aleph_1$. Recall that Φ is a compact topological space with respect to the order topology (see, e.g., [7, p. 82]). It is well known (and easy to prove) that the topological space Φ has the following rather exotic property: *If $a(\aleph_1) = 0$, then $a(t) = 0$ for all sufficiently large countable ordinals t.* In other words, a vanishes in some neighbourhood of $\aleph_1$ in Φ.

We concentrate on the $\widetilde{\otimes}$-algebra $C(\Phi)$ and on its maximal ideal $I := \{a \in C(\Phi) : a(\aleph_1) = 0\}$.

Theorem 2.3 *The $C(\Phi)$-$\widetilde{\otimes}$-module I is not projective.*

(Since the Gel'fand spectrum $\Phi \setminus \{\aleph_1\}$ of is one of best known examples of non-paracompact topological spaces, this result also directly follows from Theorem 1. However, again we think that it is rather instructive to present an independent proof.)

$\triangleleft$ Suppose that, on the contrary, I is projective. Then Theorem 1.3 provides a morphism $\rho : I \to C(\Phi) \widetilde{\otimes} I$ in $C(\Phi)$-$\widetilde{\otimes}$-**mod**, right inverse to $\pi := \pi_I : C(\Phi) \widetilde{\otimes} I \to I$. Our aim is to show that the existence of this hypothetical ρ will lead us to a contradiction.

Introduce the bioperator $\mathcal{R} : C(\Phi) \times I \to C(\Phi \times \Phi)$, taking a pair $a \in C(\Phi)$, $x \in I$ to the function $u(s,t) := a(s)x(t)$ with $s, t \in \Phi$. Of course, this bioperator is contractive, but we want to show that, moreover, it is multiplicatively contractive (with respect to the standard quantization of participating Banach spaces that was discussed above). For this aim we consider, for every $n = 1, 2, \ldots$, the n-th multiplicative amplification of $\mathcal{R}$ (see Section 0). In the present context it is the bioperator $\mathcal{R}_n : M_n(C(\Phi)) \otimes M_n(I) \to M_n(C(\Phi \otimes \Phi))$, taking a pair $\mathbf{a} = (a_{ij}) \in M_n(C(\Phi))$, $\mathbf{x} = (x_{ij}) \in M_n(I)$ to the matrix $\mathbf{u} \in M_n(C(\Phi \otimes \Phi))$ with the entries $u_{ij} := \sum_{k=1}^n \mathcal{R}(a_{ik}, x_{kj})$.

For all $s, t \in \Phi$, $1 \leq i, j \leq n$ we have $u_{ij}(s,t) = \sum_{k=1}^n a_{ik}(s)x_{kj}(t)$. Consequently, the scalar matrix $\mathbf{u}(s,t) = (u_{ij}(s,t))$ with fixed s, t is the product of the scalar matrices $\mathbf{a}(s) = (a_{ij}(s))$ and $\mathbf{x}(t) = (x_{ij}(t))$. Therefore, taking into account the concrete quantization of our spaces, we have

$$\begin{aligned}
&\|\mathcal{R}_n(\mathbf{a}, \mathbf{x})\| = \|\mathbf{u}\| = \sup\{\|\mathbf{u}(s,t)\|; s, t \in \Phi\} \\
&\quad \leq \sup\{\|\mathbf{a}(s)\|\|\mathbf{x}(t)\|; s, t \in \Phi\} \leq (\sup\{\|\mathbf{a}(s)\| : s \in \Phi\})(\sup\{\|\mathbf{x}(t)\|; t \in \Phi\}) = \|\mathbf{a}\|\|\mathbf{x}\|.
\end{aligned}$$

It follows that $\|\mathcal{R}_n\| \leq 1$ for all n. This means that the bioperator $\mathcal{R}$ is multiplicatively contractive, and hence (see Section 0) completely contractive. Therefore, by virtue of Theorem 0, there exists, for any choice of $\widetilde{\otimes}$, the $\widetilde{\otimes}$-bounded operator $R : C(\Phi) \widetilde{\otimes} I \to C(\Phi \times \Phi)$, uniquely determined by taking an elementary tensor $a \otimes x$ to $u(s,t) := a(s)x(t)$.

Lemma 1 *Let $v \in C(\Phi) \widetilde{\otimes} I$ and $s \in \Phi$. Then the function $a(t) := [R(v)](s,t)$ belongs to I.*

$\triangleleft$ The assertion is obvious for elementary tensors and hence it is true for their span. But the latter is dense in $C(\Phi) \mathbin{\widetilde{\otimes}} I$, R is continuous, and I is closed. The rest is clear. $\triangleright$

The continuation of the proof Recall that $C(\Phi) \mathbin{\widetilde{\otimes}} I$ has the structure of a left $C(\Phi)$-module. The space $C(\Phi \times \Phi)$ is also a left $C(\Phi)$-module with the outer multiplication given by $[a \cdot u](s,t) := a(s)u(s,t)$. Taking elementary tensors in $C(\Phi) \mathbin{\widetilde{\otimes}} I$ and using the density of their span in the latter space, we immediately see that the constructed operator R is a morphism of $C(\Phi)$-modules. The same is obviously true for the operator $\Delta : C(\Phi \times \Phi) \to C(\Phi); u(s,t) \mapsto b(s) := u(s,s)$ (acting by the restriction on the diagonal in $\Phi \times \Phi$).

Lemma 2 *The diagram*

$$\begin{tikzcd} C(\Phi) \mathbin{\widetilde{\otimes}} I \arrow[r, "R"] \arrow[d, shift right, "\pi"'] & C(\Phi \times \Phi) \arrow[d, "\Delta"] \\ I \arrow[u, shift right, "\rho"'] \arrow[r, "\mathbf{in}"] & C(\Phi) \end{tikzcd}$$

where **in** *is the natural embedding, is commutative (in other words, $\Delta \circ R = (\mathbf{in}) \circ \pi$ and $\Delta \circ R \circ \rho = \mathbf{in}$).*

$\triangleleft$ Obviously, we have $\Delta(R(v)) = (\mathbf{in})(\pi(v))$ provided v is an elementary tensor in $C(\Phi) \mathbin{\widetilde{\otimes}} I$. Combining this observation with the density of the span of elementary tensor in the latter space and using the continuity of all involved maps, we immediately obtain the first desired equality. The second equality follows from the first one, combined with the relation $\pi \circ \rho = \mathbf{1}_I$. $\triangleright$

The end of the proof For any countable ordinal α we denote by $e_\alpha \in I$ the function such that $e_\alpha(t) = 1$, if $t \leq \alpha$, and $e_\alpha(t) = 0$, if $t > \alpha$. Denote by E_α the function $R \circ \rho \in C(\Phi \otimes \Phi)$. If $\alpha, \beta; \alpha < \beta$ are two countable ordinals, we have $e_\alpha \cdot e_\beta = e_\alpha e_\beta = e_\alpha$, and therefore

$$E_\alpha = [R \circ \rho](e_\alpha \cdot e_\beta) = e_\alpha \cdot ([R \circ \rho](e_\beta)) = e_\alpha \cdot E_\beta.$$

It follows, in particular, that

$$\alpha < \beta \quad \text{implies} \quad E_\alpha(\alpha, \beta) = E_\beta(\alpha, \beta). \tag{5}$$

Further, for all $\alpha, s \in \Phi$, the relation $E_\alpha(s,s) = [\Delta \circ R \circ \rho](s)$, combined with Lemma 2, gives $E_\alpha(s,s) = e_\alpha(s)$. In particular, this implies that

$$E_\alpha(s,s) = 1 \qquad \text{if } s \leq \alpha. \tag{6}$$

Now we proceed to the construction of a certain increasing sequence $\alpha_k; k = 1, 2, \dots$ of countable ordinals. As to α_1, we choose it arbitrarily. Suppose that $\alpha_1, \dots, \alpha_k$ are already chosen. Consider the function $a_k(t) := E_{\alpha_k}(\alpha_k, t); t \in \Phi$. By virtue of Lemma 1, $a_k \in I$, and therefore, as it was mentioned above, there exists a countable ordinal α_{k+1} such that $E_{\alpha_k}(\alpha_k, t) = 0$ whenever $t \geq \alpha_{k+1}$. Since $E_{\alpha_k}(\alpha_k, \alpha_k) = 1$ (see (6)), we have $\alpha_{k+1} > \alpha_k$.

Thus, by induction, the sequence α_k is constructed. It is easy to see that it converges, and its limit, say ω, is still a countable ordinal. This observation happens to be crucial.

What is $E_\omega(\omega, \omega)$? On one hand, by (6) (with ω as α), $E_\omega(\omega, \omega) = 1$. On the other hand, combining the continuity of the function $E_\omega(s, t)$, the equality (5) (with α_k as α and ω as β) and, finally, the very construction of our ordinals α_k, we have

$$E_\omega(\omega, \omega) = \lim_{k\to\infty} E_\omega(\alpha_k, \omega) = \lim_{k\to\infty} E_{\alpha_k}(\alpha_k, \omega) = 0.$$

We came to a contradiction. $\triangleright$

Apart from good topological properties of their spectrum, indicated in Theorem 1(ii), projective ideals in commutative Banach and quantized Banach algebras possess a number of special analytic and geometric properties. Let us concentrate on the most transparent case of a maximal ideal I in a unital commutative $\widetilde{\otimes}$-algebra A. In what follows, Ω is the Gel'fand spectrum of A, $\partial\Omega$ is the Shilov boundary of Ω, s is a point in Ω, representing I, and I^2 is the topological square of I (that is, the closure of $span\{xy; x, y \in I\}$).

Theorem 2.4 (i) *If $s \in \partial\Omega$, then $I^2 = I$. Besides, there is a constant $C > 0$ such that, for any $t \in \Omega$, there exists $x \in I$ (that is, $x \in A$ with $x(s) = 0$) such that $a(t) = 1$ and $\|x\| < C$.*

(ii) (L. I. Pugach) *If $s \notin \partial\Omega$, than $\dim I/I^2 = 1$. Besides, (the main assertion) there exists a neighborhood U of s in the Gel'fand topology of Ω that is an analytic disc.*

(Recall that the latter means that there is a homeomorphism $\omega : \mathbb{D}_0 \to U$, where $\mathbb{D}_0$ is the open unit disc in $\mathbb{C}$, such that for every $a \in A$ the function $z \mapsto a(\omega(z))$ where $z \in \mathbb{D}_0$ is holomorphic.)

The proof see in Pugach's paper [23] and in [11].

Exercise 2.2 Prove, using the previous theorem, that maximal ideals in the Banach algebra $C_n[0, 1]$ of n times smooth functions on $[0, 1]$ are not projective.

Now we turn to algebras of sequences. The next example is the algebra of summable sequences l_1 (also denoted by $l_1(\mathbb{N})$) with the coordinatewise multiplication. Denote by $l_1(\mathbb{N} \times \mathbb{N})$, or, for brevity, by l_1^2 the Banach space of double summable sequences. We consider it as a left Banach l_1-module with the outer multiplication $[a \cdot x](m, n) := a(m)x(m, n)$ for $a \in l_1$ and $x \in l_1^2$. (Here and below we denote the terms of our sequences by $a(m)$ and $x(m, n)$ instead of a_m and x_{mn}.)

Example 2.2 We want to show that l_1 is left $\widehat{\otimes}$-projective. For this aim, consider the diagram

$$\begin{tikzcd} l_1(\mathbb{N}) \widehat{\otimes} l_1(\mathbb{N}) \arrow[rr, "i"] \arrow[dr, "\pi"'] & & l_1(\mathbb{N}\times\mathbb{N}) \arrow[dl, "\nabla"] \\ & l_1(\mathbb{N}) & \end{tikzcd} \tag{7}$$

where i is the standard isometric isomorphism, taking $a \otimes b$ to the double sequence $x(m,n) := a(m)b(n))$, π is the product operator, and ∇ "restricts to the diagonal", taking x to $a; a(n) := x(n,n)$. It is clear that all these maps are morphisms in $\mathbf{l_1} - \widehat{\otimes} - \mathbf{mod}$, and the diagram is commutative.

Set $\Delta : l_1 \to l_1^2, a \mapsto x$, where $x(n,n) := a(n)$ and $x(m,n) := 0$ if $m \neq n$. Obviously, this is also a morphism in the same category, right inverse to ∇. It follows that $\rho := i^{-1} \circ \Delta$ is a morphism in l_1-$\widehat{\otimes}$-**mod**, right inverse to π. Since the algebra is obviously non-degenerate, Theorem 1.3 closes the matter.

Remark We have considered only the "$\widehat{\otimes}$-case". However, l_1 can be made an $\overset{h}{\otimes}$ and hence an $\overset{o}{\otimes}$-algebra with respect to the so-called maximal quantization that will be discussed a little bit later. A similar argument could show that our algebra is also projective in both "quantum" senses. In fact, in all three theories l_1 has a much stronger property: it is, as they say, biprojective. The traditional version of this statement is proved in [10], and the quantum versions in [1].

So far our ideals in $\widetilde{\otimes}$-algebras behaved, in the question of their projectivity, in the same way for all three theories. In other words, our results did not depend on which kind of the three tensor products we choose. This is, however, not a universal phenomenon. Now we shall present the apparently simplest example of an algebra that behaves differently in the traditional and the quantum settings.

Consider, instead of l_1, the Banach algebra $l_2 = l_2(\mathbb{N})$, again with the coordinate-wise multiplication. Denote by $l_2(\mathbb{N}\times\mathbb{N})$, and also, for brevity, by l_2^2 the Hilbert space of double square-summable sequences. Similarly to the "l_1-case", it is a left Banach l_2-module with the outer multiplication $[a \cdot x](m,n) := a(m)x(m,n)$. In what follows, we shall use the standard identification of the Hilbert space l_2^2 with the Hilbert tensor square $l_2 \dot{\otimes} l_2$ (We recall that the respective isometric isomorphism is uniquely determined by taking the double sequence $a(k)b(m)$ with $a, b \in l_2$ and $k, m = 1, 2, \dots$ to $a \dot{\otimes} b$). In particular, the ort (= element of the natural orthonormal basis) $e_{mk} \in l_2^2$, defined by $e_{mk}(s,t) = 1$ if $s = m$ and $t = k$ and $e_{mk}(s,t) = 0$ otherwise, is identified with $e_m \dot{\otimes} e_k$, the tensor product of two orts in l_2. We shall alternately use both notation e_{mk} and $e_m \dot{\otimes} e_k$.

We proceed to a quantization of l_2, and also of l_2^2. Among many possible ways to do it we choose the so-called *column quantization*. Later we shall discuss such a quantization for an arbitrary Hilbert space, but now we need only these two particular cases. The column quantization of l_2 is the isometric operator $l_2 \to \mathcal{B}(l_2)$, taking a to the operator $\breve{a}$ that sends e_1 to a and other orts to 0. (Thus we take a given square-summable sequence to the operator depicted by the infinite matrix that has our sequence as its first column and zeroes in remaining places; hence, the word "column".) Similarly, the column quantization of l_2^2 is the isometric operator from this space to $\mathcal{B}(l_2^2)$, taking x to the operator $\breve{x} : e_{11} \mapsto x$ and $e_{mk} \mapsto 0$ for other double indexes.

We can identify a vector $a \in l_2$ with the operator $\mathbb{C} \to l_2, 1 \mapsto a$. Then we can treat a matrix, say $\mathbf{a}$, in $M_n(l_2)$ as a matrix with operator entries. Such a matrix depicts an operator from $\mathbb{C}^n$ into nl_2, and it is easy to see that $\|\mathbf{a}\|$ is exactly the norm of that operator. Thus $M_n(l_2)$ can be identified with $\mathcal{B}(l_2)$ and, similarly, $M_n(l_2^2)$ with $\mathcal{B}(l_2^2)$.

Proposition 2.2 *There exists a completely isometric isomorphism*

$$R : l_2(\mathbb{N}) \overset{h}{\otimes} l_2(\mathbb{N}) \to l_2(\mathbb{N} \times \mathbb{N}),$$

well defined by taking $e_k \otimes e_m$ to e_{km}.

$\triangleleft$ At first we introduce the bioperator $\mathcal{R} : l_2 \times l_2 \to l_2^2, (a, b) \mapsto x$ with $x(k, m) := a(k)b(m)$ and wish to show that it is multiplicatively contractive.

Fix n and consider the multiplicative amplification $\mathcal{R}_n : M_n(l_2) \times M_n(l_2) \to M_n(l_2^2)$. Take arbitrary $\mathbf{a} = (a_{ij}),\mathbf{b} = (b_{ij}) \in M_n(l_2)$. Then, by definition, the matrix $\mathcal{R}_n(\mathbf{a}, \mathbf{b})$ has the entries $c_{ij} := \sum_{k=1}^n \mathcal{R}(a_{ik}, b_{kj}) = \sum_{k=1}^n a_{ik} \dot{\otimes} b_{kj}$. (We remember and will remember the identification of l_2^2 with $l_2 \dot{\otimes} l_2$.)

Consider the operators$\breve{\mathbf{a}}, \breve{\mathbf{b}} \in \mathcal{B}(nl_2)$, depicted by the matrices with the operator entries $\breve{a}_{ij} : l_2 \to l_2$, respectively $\breve{b}_{ij} : l_2 \to l_2$, and such that $\breve{a}_{ij}$ takes e_1 to a_{ij}, $\breve{b}_{ij}$ takes e_1 to b_{ij}, and both take other orts to 0. Similarly, consider $\hat{\mathbf{c}} \in \mathcal{B}(nl_2^2)$, depicted by the matrix with the operator entries $\hat{c}_{ij} : l_2^2 \to l_2^2$ such that $\hat{c}_{ij}$ takes $e_{11} = e_1 \dot{\otimes} e_1$ to c_{ij} and takes other orts to 0. We notice that $\hat{c}_{ij}$ is exactly $\sum_{k=1}^n \breve{a}_{ik} \dot{\otimes} \breve{b}_{kj}$. Further, by the definition of the column quantization, we have $\|\mathbf{a}\| = \|\breve{\mathbf{a}}\|$, $\|\mathbf{b}\| = \|\breve{\mathbf{b}}\|$ and $\|\mathcal{R}_n(\mathbf{a}, \mathbf{b})\| = \|\breve{\mathbf{c}}\|$.

Now we introduce two more operators, denoted by $\hat{\mathbf{a}}, \hat{\mathbf{b}} \in \mathcal{B}(nl_2)^2$. They are depicted by the following matrices: the first one has the operator entries $\hat{a}_{ij} := \breve{a}_{ij} \dot{\otimes} \mathbf{1} : l_2 \dot{\otimes} l_2 \to l_2 \dot{\otimes} l_2$, and the second one $\hat{b}_{ij} := \mathbf{1} \dot{\otimes} \breve{b}_{ij} : l_2 \dot{\otimes} l_2 \to l_2 \dot{\otimes} l_2$. Look at the composition $\hat{\mathbf{a}} \circ \hat{\mathbf{b}}$ of these operators. The ij-th entry of its matrix is $\sum_{k=1}^n \hat{a}_{ik} \hat{b}_{kj} = \sum_{k=1}^n (\breve{a}_{ik} \dot{\otimes} \mathbf{1})(\mathbf{1} \dot{\otimes} \breve{b}_{kj}) = \sum_{k=1}^n \breve{a}_{ik} \dot{\otimes} \breve{b}_{kj} = \hat{c}_{ij}$. Thus $\hat{\mathbf{a}} \circ \hat{\mathbf{b}}$ is not other thing than $\hat{\mathbf{c}}$.

Since $\hat{\mathbf{a}}$ can be represented as $\breve{\mathbf{a}} \dot{\otimes} \mathbf{1} \in \mathcal{B}(nl_2 \dot{\otimes} l_2)$, we have $\|\hat{\mathbf{a}}\| = \|\breve{\mathbf{a}}\| = \|\mathbf{a}\|$. Similarly, the representation of $\hat{\mathbf{b}}$ as $\mathbf{1} \dot{\otimes} \breve{\mathbf{b}} \in \mathcal{B}(l_2 \dot{\otimes} nl_2)$ gives $\|\hat{\mathbf{b}}\| = \|\breve{\mathbf{b}}\| = \|\mathbf{b}\|$. Consequently, we have

$$\|\mathcal{R}_n(\mathbf{a}, \mathbf{b})\| = \|\hat{\mathbf{c}}\| \leq \|\hat{\mathbf{a}}\| \|\hat{\mathbf{b}}\| = \|\breve{\mathbf{a}}\| \|\breve{\mathbf{b}}\| = \|\mathbf{a}\| \|\mathbf{b}\|.$$

Thus $\mathcal{R}$ is indeed multiplicatively contractive.

If so, then, by Theorem 0, $\mathcal{R}$ gives rise to the completely contractive operator $R : l_2 \overset{h}{\otimes} l_2 \to l_2^2$, acting on elementary tensors in the prescribed way. Our aim is to show that R is a completely isometric isomorphism.

Take, for any $n \in \mathbb{N}$, the respective amplification $R_n : M_n(l_2 \overset{h}{\otimes} l_2) \to M_n(l_2^2)$.

Obviously, elements of the form $\sum_{k=1}^{N} e_k \otimes b_k$ for all possible $b_k \in l_2$ and $N \in \mathbb{N}$ constitute a dense subspace in $l_2 \otimes l_2$. It easily follows that matrices of the form $\mathbf{u} = (u_{ij})$, where $u_{ij} = \sum_{k=1}^{N} e_k \otimes b_{ij}^{(k)}$ for some $b_{ij}^{(k)} \in l_2$ and $N \in \mathbb{N}$, constitute a dense subspace in $M_n(l_2) \overset{h}{\otimes} M_n(l_2)$. Therefore it is sufficient to fix such a $\mathbf{u}$ and show that $\|\mathbf{u}\| \le \|R_n(\mathbf{u})\|$.

Let us consider two rectangular matrices $\mathbf{v} \in M_{n,Nn}(l_2)$ and $\mathbf{w} \in M_{Nn,n}(l_2)$. The first one has the form of the "block-row"

$$(\ldots, \mathbf{v}_k, \ldots),$$

where, for any $k = 1, \ldots, N$, $\mathbf{v}_k = (v_{k,ij}) \in M_n(l_2)$ is the diagonal matrix with entries e_k in the main diagonal and 0 in other places. The second one has the form of the "block-column"

$$\begin{pmatrix} \vdots \\ \mathbf{w}_k \\ \vdots \end{pmatrix},$$

where, for any $k = 1, \ldots, N$, $\mathbf{w}_k \in M_n(l_2)$ has the entries $w_{ij} := b_{ij}^{(k)}$ One can easily verify that our initial matrix Therefore the formula (1) in the introductory section, with l_2 in the capacity of E and F, provides the estimate $\|\mathbf{u}\| \le \|\mathbf{v}\|\|\mathbf{w}\|$. What does it give?

Our first concern is $\|\mathbf{v}\|$. Because of the column quantization of l_2, this number, accordingly with what was mentioned above, is the norm of the operator $\hat{v} : \mathbb{C}^{Nn} \to nl_2$, taking orts in $\mathbb{C}^{Nn}$ to copies of N orts $e_1, \ldots, e_N \in l_2$, belonging to various Hilbert summands of $nl_2 = l_2 \dot{\oplus} \ldots \dot{\oplus} l_2$. Since all these mentioned vectors in nl_2 are pairwise orthogonal, the operator $\hat{v}$ is isometric. Therefore $\|\hat{v}\|$ and hence $\|\mathbf{v}\|$ are equal to 1.

Turn to $\|\mathbf{w}\|$. This is the norm of the operator from $\mathbb{C}^n$ to Nnl_2 that takes an arbitrary n-tuple $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ to the Nn-tuple $(\ldots, \sum_{j=1}^{n} \xi_j b_{ij}^k, \ldots)$ with $k = 1, \ldots, N$, $i = 1, \ldots, n$ of vectors in l_2.

Therefore, by the definition of the norm in a Hilbert sum,

$$\|\mathbf{w}\|^2 = \max \sum_{i=1}^{n} \sum_{k=1}^{N} \left\| \sum_{j=1}^{n} \xi_j b_{ij}^k \right\|^2,$$

where the maximum is taken over all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ with $\sum_{j=1}^{n} |\xi_j|^2 = 1$.

And what about $\|R_n(\mathbf{u})\|$? This is the norm of the operator that takes the same ξ to the n-tuple (this time of vectors in $l_2^2 = l_2 \dot{\otimes} l_2$)

$$(\ldots, x_i, \ldots) \text{ with } 1 \leq i \leq n \quad \text{where} \quad x_i = \sum_{j=1}^{n} \xi_j \left(\sum_{k=1}^{N} [e_k \dot{\otimes} b_{ij}^k] \right) = \sum_{k=1}^{N} \left[e_k \dot{\otimes} \sum_{j=1}^{n} \xi_j b_{ij}^k \right].$$

Using again the structure of the norm in a Hilbert sum, and also the pairwise orthogonality of vectors $e_k \dot{\otimes}(\cdot)$ for $k = 1, \ldots, N$, we obtain

$$\begin{aligned} \|R_n(\mathbf{w})\|^2 &= \max \sum_{i=1}^{n} \sum_{k=1}^{N} \left\| e_k \dot{\otimes} \sum_{j=1}^{n} \xi_j b_{ij}^k \right\|^2 \\ &= \max \sum_{i=1}^{n} \sum_{k=1}^{N} \|e_k\|^2 \left\| \sum_{j=1}^{n} \xi_j b_{ij}^k \right\|^2 = \max \sum_{i=1}^{n} \sum_{k=1}^{N} \left\| \sum_{j=1}^{n} \xi_j b_{ij}^k \right\|^2, \end{aligned}$$

where the maximum is taken over the same ξ as before. It follows that $\|R_n(\mathbf{u})\| = \|\mathbf{w}\|$.

Combining the letter equality with what we already know about the norms of $\mathbf{u}$ and $\mathbf{v}$, we obtain that $\|\mathbf{u}\| \leq \|R_n(\mathbf{u})\|$. Therefore R, being completely contractive, is completely isometric. Together with the obvious observation that the image of R is dense in l_2^2, this gives the desired result. $\triangleright$

Proposition 2.3 *The Banach algebra l_2 with the coordinate-wise multiplication is an $\overset{h}{\otimes}$- and hence an $\overset{o}{\otimes}$-algebra with respect to the column quantization. Moreover, the respective product morphism participates in the commutative diagram*

$$\begin{tikzcd} l_2(\mathbb{N}) \overset{h}{\otimes} l_2(\mathbb{N}) \arrow[rr, "R"] \arrow[dr, "\pi_{\overset{h}{\otimes}}"'] & & l_2(\mathbb{N} \times \mathbb{N}) \arrow[dl, "\nabla"] \\ & l_2(\mathbb{N}) & \end{tikzcd} \tag{8}$$

where ∇ is a completely contractive operator, taking x to $a; a(n) := x(n,n)$. (Compare the operator of the same name in the diagram (7)*.)*

$\triangleleft$ At first we shall show that ∇ is completely contractive. (As a matter of fact, this is a particular case of future Proposition 4.2, but here the proof is most transparent.) Indeed, for any $n \in \mathbb{N}$, we consider the amplification $\nabla_n : M_n(l_2^2) \to M_n(l_2)$ and take an arbitrary $\mathbf{x} = (x_{ij}) \in M_n(l_2^2)$. Being identified with an operator in $\mathcal{B}(\mathbb{C}^n, nl_2^2)$, our $\mathbf{x}$ takes $\xi = (\xi_1, \ldots, \xi_n)$ to the n-tuple $(\ldots, \sum_{j=1}^{n} \xi_j x_{ij}, \ldots)$. Therefore

$$\|\mathbf{x}(\xi)\|^2 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \xi_j x_{ij} \right|^2 = \sum_{i=1}^{n} \sum_{k,m=1}^{\infty} \left| \sum_{j=1}^{n} \xi_j x_{ij,km} \right|^2,$$

where $x_{ij,km}$ is the km-th coordinate of the double sequence x_{ij}. Similarly, in the guise of an operator in $\mathcal{B}(\mathbb{C}^n, nl_2)$, $\nabla_n(\mathbf{x})$ takes the same ξ to $(\ldots, \sum_{j=1}^n \xi_j \nabla(x_{ij}), \ldots)$. Therefore, remembering what the coordinates of $\nabla(x_{ij})$ are, we have

$$\|[\nabla_n(\mathbf{x})](\xi)\|^2 = \sum_{i=1}^{n} \sum_{k=1}^{\infty} \left| \sum_{j=1}^{n} \xi_j x_{ij,kk} \right|^2 .$$

We see that $\|[\nabla_n(\mathbf{x})](\xi)\| \leq \|\mathbf{x}(\xi)\|$ for any ξ and hence $\|\nabla_n(\mathbf{x})\| \leq \|\mathbf{x}\|$. Thus ∇ has the desired property.

Together with the previous proposition, this implies that the composition $\nabla \circ R$ is a completely contractive operator, taking elementary tensors to the product of their factors. But by Theorem 0, the existence of such an operator is equivalent to the multiplicative boundedness of the multiplication in l_2. Thus l_2 is an $\overset{h}{\otimes}$-algebra, and the rest is clear. $\triangleright$

Theorem 2.5 (i) *The Banach algebra l_2 is not $\widehat{\otimes}$-projective, however,*

(ii) *the $\overset{h}{\otimes}$-algebra l_2 is $\overset{h}{\otimes}$-projective.*

$\triangleleft$ (i) Consider the bioperator $\mathcal{S} : l_2 \times l_2 \to l_1$, taking a pair $(\ldots, a_k, \ldots)$, $(\ldots, b_k, \ldots)$ to $(\ldots, a_k b_k, \ldots)$; by virtue of the Cauchy–Bunyakovsky inequality, it is well defined and bounded. Consequently, it gives rise to its associated bounded operator $S : l_2 \widehat{\otimes} l_2 \to l_1$. Observing the action of R on elementary tensors, we see that the composition $\mathbf{in} \circ S : l_2 \widehat{\otimes} l_2 \to l_2$, where $\mathbf{in} : l_1 \to l_2$ is the natural embedding, is exactly the product morphism $\pi_{\widehat{\otimes}}$ for the $\widehat{\otimes}$-*algebra* l_2. It follows that the image of $\pi_{\widehat{\otimes}}$ is l_1, and not all l_2. Hence $\pi_{\widehat{\otimes}}$, not being surjective, can not have a right inverse map. The rest is clear.

(ii) Acting like in Example 2 above, we set $\Delta : l_2 \to l_2^2 : a \mapsto x$, where $x(n,n) := a(n)$ and $x(m,n) := 0$ if $m \neq n$. After identifying $M_n(l_2)$ with $\mathcal{B}(\mathbb{C}^n, nl_2)$ and $M_n(l_2^2)$ with $\mathcal{B}(\mathbb{C}^n, nl_2^2)$, an argument, very close to what was used for ∇ in the previous proposition, shows that Δ is a completely isometric operator. At the same time, it is obviously a morphism of l_2-modules, and it is a right inverse to ∇. Therefore, adding this morphism to the commutative diagram (8), we see that $\rho := i^{-1} \circ \Delta$ is a morphism in l_2-$\overset{h}{\otimes}$-**mod** and a right inverse to $\pi_{\overset{h}{\otimes}}$. The rest is clear. $\triangleright$

Remark Being considered as an $\overset{o}{\otimes}$-algebra, l_2 is also left projective. This follows, for example, from the well known fact that the Banach quantum spaces $l_2 \overset{o}{\otimes} l_2$ and $l_2 \overset{h}{\otimes} l_2$ coincide up to a completely isometric isomorphism (cf. [6, p. 163]). Moreover, l_2 as an $\overset{h}{\otimes}$-, as well as an $\overset{o}{\otimes}$-algebra, has the much stronger property to be biprojective that was mentioned in the previous remark (cf. [1]).

Concluding, we proceed from function algebras to group algebras.

Let G be a locally compact group, and let $L_1(G)$ be the Banach space of those functions (more precisely, of equivalence classes of functions) that are integrable with respect to the left invariant Haar measure on G. It is a classical example of a Banach algebra. The appropriate multiplication is called convolution and is defined by the Haar integral $a{*}b(s) := \int_G a(t)b(t^{-1}s)dt$. At the same time, after the appearance of quantized functional analysis, $L_1(G)$ is usually considered as a quantum Banach space with respect to the so-called maximal quantization.

We recall that a quantization of a Banach space E is called *maximal*, if the norm in $M_n(E)$ ("in the n-th floor") is the supremum of norms, given by all possible quantizations of E. The respective quantum Banach space is also called *maximal*, and it is often denoted by $\max E$. (It is obvious that these notions are well defined for any E.) One can easily show that every bounded operator from $\max E$ into an arbitrary quantum space is automatically completely bounded. Also it is known, that, for every quantum Banach space F, there exists a complete isometric isomorphism between quantum Banach spaces $\max E \overset{o}{\otimes} F$ and $\max(E \widehat{\otimes} F)$, leaving elementary tensors unmoved [4, p. 289]. Combining both facts, we see that any Banach algebra A, endowed with the maximal quantization, is an $\overset{o}{\otimes}$-algebra.

In what follows, speaking of $L_1(G)$ as of a quantum Banach space, we always mean the maximal quantization.[4]

Thus, we see that $L_1(G)$ is not only a Banach (i.e. $\widehat{\otimes}$-) algebra, but it is also an $\overset{o}{\otimes}$-algebra. (However, one could show that it is, generally speaking, not an $\overset{h}{\otimes}$-algebra.)

Theorem 2.6 *The algebra $L_1(G)$ is left projective as an $\widehat{\otimes}$-algebra and as an $\overset{o}{\otimes}$-algebra.*

$\triangleleft$ Denote by $\pi_{\widehat{\otimes}}$ and $\pi_{\overset{o}{\otimes}}$ the respective product morphisms for $L_1(G)$ in **A**-$\widehat{\otimes}$-**mod** and **A**-$\overset{o}{\otimes}$-**mod**. Recall that our algebra is non-degenerate (more of this, by Cohen's factorization theorem, every function is the convolution product of two others). Consequently, by virtue of Theorem 1.3, it is sufficient to show that $\pi_{\widehat{\otimes}}$ (respectfully, $\pi_{\overset{o}{\otimes}}$) has a right inverse morphism in **A**-$\widehat{\otimes}$-**mod** (respectfully, **A**-$\overset{o}{\otimes}$-**mod**).

Consider the Banach space $L_1(G \times G)$ of functions, integrable with respect to the left invariant Haar measure on $G \times G$. Obviously, this is an $L_1(G)$-$\widehat{\otimes}$-module with respect to the outer multiplication

$$a \cdot u(s,t) := \int_G a(r)u(r^{-1}s,t)dr \quad \text{where } a \in L_1(G) \text{ and } u \in L_1(G \times G)$$

[4] The same quantization of $L_1(G)$ could be introduced in an alternative way, as the so-called quantization of a predual space; here we mean, of course, the predual of $L_\infty(G) \subseteq \mathcal{B}(L_2(G))$ with its standard quantization. But we do not use this observation now.

("convolution in the first variable"). This space participates in the diagram

$$\begin{tikzcd}
L_1(G) \mathbin{\widehat{\otimes}} L_1(G) \arrow[rr, "\mathbf{ig}"] \arrow[dr, "\pi_{\overset{o}{\otimes}}"'] & & L_1(G \times G) \arrow[dl, "\Pi"] \\
& L_1(G) &
\end{tikzcd}$$

where $\mathbf{ig}$ is the well known isometric isomorphism of Grothendieck (taking $a \otimes b$ to the function $u(s,t) := a(s)b(t)$), and Π acts via $u \mapsto a(s) := \int_G u(t, t^{-1}s)dt$. It is easy to verify that $\mathbf{ig}$ and Π are morphisms of left $L_1(G)$-modules and that the diagram is commutative.

Now fix an arbitrary compact set in G of Haar measure 1 and denote by χ its characteristic function. The main ingredient of our proof is the map $\varrho : L_1(G) \to L_1(G \times G), a \mapsto u(s,t) := \chi(t^{-1})a(st)$.

Let $\Delta(s)$ for $s \in G$ denote, as usually, the modular function of the group G. Using the well known relations $\int_G a(st)ds = \Delta(t^{-1}) \int_G a(s)ds$ and $\int_G b(t^{-1})\Delta(t^{-1})dt = \int_G b(t)dt$ for $a, b \in L_1(G)$ as well as Fubini's theorem, we have

$$\begin{aligned}
\|\varrho(a)\| &= \int_{G \times G} \chi(t^{-1})|a(st)|\, d(s \times t) \\
&= \int_G \chi(t^{-1})\Delta(t^{-1}) \left(\int_G |a(s)|\, ds \right) dt \\
&= \left(\int_G \chi(t^{-1})\Delta(t^{-1})\, dt \right) \|a\| = \int_G \chi(t)dt\|a\| = \|a\|.
\end{aligned}$$

Thus ϱ is an isometric operator. Further,

$$\begin{aligned}
&\varrho(a * b)(s,t) = \chi(t^{-1})[a * b](st) \\
&\quad = \chi(t^{-1}) \int_G a(r)b(r^{-1}st)dr = \int_G a(r)(\chi(t^{-1})b(r^{-1}st))dr = [a \cdot (\varrho(b)](s,t),
\end{aligned}$$

and hence ϱ is a morphism of left $L_1(G)$-modules. Finally, we have

$$\Pi(\varrho(a)) = \int_G [\varrho(a)](t, t^{-1}s)dt = \int_G \chi(s^{-1}t)a(tt^{-1}s)dt = a(s) \int_G \chi(s^{-1}t)dt = a(s),$$

and hence $\Pi \circ \varrho = \mathbf{1}_{L_1(G)}$. Combining all these observations, we see that ϱ is a morphism in $L_1(G)$-$\widehat{\otimes}$-**mod** and a right inverse of Π. It follows that $\rho_{\widehat{\otimes}} := \mathbf{gr}^{-1} \circ \varrho$ is a morphism in the same category and a right inverse of $\pi_{\widehat{\otimes}}$. This completes the proof in the traditional ("$\widehat{\otimes}$–") context.

Now recall the contractive operator $j_1 : L_1(G) \mathbin{\widehat{\otimes}} L_1(G) \to L_1(G) \overset{o}{\otimes} L_1(G)$, discussed in the introductory section. (Here, of course, we specify both E and F as $L_1(G)$.)[5] Set $\rho_{\overset{o}{\otimes}} :=$

[5]In fact, this concrete operator is an isometric isomorphism, but we do not need it.

$j_1 \circ \rho_{\widehat{\otimes}}$. Since j_1 is obviously a morphism of left $L_1(G)$-modules, the same is true for $\rho_{\overset{o}{\otimes}}$. Further, of course we have $\pi_{\overset{o}{\otimes}} \circ j_1 = \pi_{\widehat{\otimes}}$, and consequently $\pi_{\overset{o}{\otimes}} \circ \rho_{\overset{o}{\otimes}} = \pi_{\overset{o}{\otimes}} \circ j_1 \circ \rho_{\widehat{\otimes}} = \mathbf{1}_{L_1(G)}$. Finally, because of the choice of the maximal quantization of $L_1(G)$, the operator $\rho_{\overset{o}{\otimes}}$, being bounded, is automatically completely bounded. It follows that $\rho_{\overset{o}{\otimes}}$ is a morphism in $L_1(G)$-$\widehat{\otimes}$-**mod** and a right inverse of $\pi_{\overset{o}{\otimes}}$. This completes the proof in the "$\overset{o}{\otimes}-$" context. $\triangleright$

Dales and Polyakov [5], in the framework of traditional homology, have proved or disproved the projectivity (as well as the injectivity and the flatness) of several other important Banach modules over $L_1(G)$.

To conclude the section, we would like to say several words about another type of group algebras. We mean the so-called Fourier algebras of locally compact groups, introduced by P. Eymard in 1964. Gradually it was realized that these algebras play in harmonic analysis a very important role, comparable with the role of L_1-algebras. More of this, the both classes of algebras are rather intimately connected. Namely, the dual spaces of the L_1-algebra and of the Fourier algebra of any locally compact group, being equipped with a certain additional structure, turn out to be in some natural relation of a duality. This duality, discovered by G. I. Kac and L. I. Vainerman, and, independently, by M. Enock and J.-M. Schwartz, and defined in the general framework of the so-called Kac algebras, can be considered as a "right" generalization of the classical Pontryagin duality to non-Abelian groups. It is presented in the monograph [8].

Let G be a locally compact group, and let $L_2(G)$ be the Hilbert space of functions, square integrable with respect to the left invariant Haar measure on G. Denote by $A(G)$ the set of functions on G of the form $\varphi = \xi * \check{\eta}$, where $\xi, \eta \in L_2(G)$ and $\check{\eta}(t) := \eta(t^{-1})$ for $t \in G$. (It is easy to see that $A(G)$ is a dense subset in $C_0(G)$.) The fundamental and non-trivial fact is that $A(G)$ is a Banach algebra with respect to the pointwise multiplication and the norm

$$\|\varphi\| := \inf\{\|\xi\|\|\eta\| : \varphi = \xi * \check{\eta}; \xi, \eta \in L_2(G).$$

The Banach algebra $A(G)$ is called the *Fourier algebra of* G. It is always non-degenerate, and it is unital $\iff G$ is compact.

Is such an algebra left projective?

The answer is, of course, positive, if G is abelian. Indeed, let $\hat{G}$ be the Pontryagin dual locally compact group to G. It is well known that the Fourier transform $F : L_1(\hat{G}) \to C_0(G)$ has $A(G)$ as its image and, being corestricted on $A(G)$, provides an isometric isomorphism between the respective Banach algebras. Therefore the projectivity of the $\widetilde{\otimes}$-algebra $A(G)$ for an Abelian G can be considered as the particular case of Theorem 2.6.

But this result cannot be extended to non-abelian groups. The matter is that there are groups G such that the canonical morphism $\pi : A(G) \widehat{\otimes} A(G) \to A(G)$ not only fails to be a retraction in $A(G)$-$\widehat{\otimes}$-**mod**, but even to be surjective: This was shown by H. Steiniger (apparently unpublished) for the case of the discrete group $\mathbb{F}_2$, the free group with two generators. Since, as it was already mentioned, every Fourier algebra is non-degenerate, Theorem 1.3 implies that $A(\mathbb{F}_2)$ *is not a left projective* $\widehat{\otimes}$*-algebra.*

The situation, however, changes, if if we consider Fourier algebras in the framework of quantized functional analysis. Let $VN(G)$ be the von Neumann algebra of G (that is, the least von Neumann algebra on $L_2(G)$ that contains all left translation operators.) Then, as it is well known, $VN(G)$ is (up to an isometric isomorphism) the dual Banach space of $A(G)$. (Note that the respective duality is well defined, for $a \in VN(G)$ and $\varphi \in A(G); \varphi = \xi * \check{\eta}, \xi, \eta \in L_2(G)$, by $(a, \varphi) = \langle a(\eta), \bar{\xi} \rangle$.) Further, since $VN(G)$, is an operator norm closed subspace in $\mathcal{B}(L_2(G))$, it is automatically a quantum Banach space with respect to the standard quantization, discussed in Section 0). Now we recall that if a Banach space is a predual space of the "first floor space" of some quantum Banach space, then it can be endowed with a special quantization "of an operator predual space" [6, p.317]. This is exactly what we do with $A(G)$. As it was shown by Effros and Ruan, the quantum Banach space $A(G) \overset{o}{\otimes} A(G)$ is completely isometrically isomorphic to $A(G \times G)$, and this easily implies that $A(G)$ is an $\overset{o}{\otimes}$-algebra. (However, generally speaking, it is not an $\overset{h}{\otimes}$-algebra.)

The following theorem will be only formulated here. It is a partial case of a general result of Z.-J. Ruan and G. Xu [25], concerning Kac algebras. Aristov [1] and, independently, Wood [30] gave its direct proof.

Theorem 2.7 *If G is a discrete group, then the $\overset{o}{\otimes}$-algebra $A(G)$ is left projective.*

(In fact, these authors have proved that, for a discrete G, $A(G)$ possesses a much stronger property, the so-called biprojectivity.)

3 Spatially projective operator algebras on Banach spaces

Let E be a Banach space, so far (but not for a long time) arbitrary. Let $\mathcal{B}(E)$ be the Banach algebra of all bounded operators on E, equipped, as usually, with the composition as the multiplication, and with the operator norm, denoted by $\| \cdot \|_\infty$. In this section we widen the meaning of the term "operator algebra": now an *operator algebra on E* is any subalgebra in $\mathcal{B}(E)$ that is a Banach algebra with respect to a certain norm $\| \cdot \| \geq \| \cdot \|_\infty$.

If A is such an algebra, then, apart from its left ideals, one important example of an A-module immediately comes to our mind. Namely, E itself is evidently a left Banach A-module with the outer multiplication $a \cdot x := a(x)$ for $a \in A$ and $x \in E$. (in other words, the outer multiplication is the action of an operator on a vector). This particular A-module is called *spatial* (or sometimes *natural*).

Definition 3.1 An operator algebra A on a Banach space E is called *spatially* $\widehat{\otimes}$*-projective* (or, when there is no danger of misunderstanding, just *spatially projective*), if the spatial left A-$\widehat{\otimes}$-module E is projective.

Note that in this section we consider only the traditional version of projectivity. Partially it is because we cannot suggest a reasonable quantization of $\mathcal{B}(E)$ for a non-Hilbert space E.

We proceed to discuss one of typical questions of the homological theory of operator algebras: *Which operator algebras are spatially projective?*

Let us begin with a rather important sufficient condition for spatial projectivity. For $x \in E$ and $f \in E^*$, we denote by $x \bigcirc f$ the operator on E, taking y to $[f(y)]x$. If $x, f \neq 0$, then $x \bigcirc f$ is obviously an operator of rank 1 (in fact, $\mathrm{Im}(x \bigcirc f) = \mathrm{span}\{x\}$), and $\|x \bigcirc f\|_\infty = \|x\|\|f\|$. Evidently, for any $a \in \mathcal{B}(E)$, $x, y \in E$, and $f \in E^ast$ we have

$$a(x \bigcirc f) = a(x) \bigcirc f \quad \text{and, in particular,} \quad (x \bigcirc f)(y \bigcirc f) = [f(y)]x \bigcirc f.$$

It is known (and easy to prove) that every bounded operator of rank 1 has the form $x \bigcirc f$ for some $x \in E$ and $f \in E^*$.

Fix, for a moment, some non-zero $f \in E^*$ and set $C_f := \{x \bigcirc f; x \in E\}$.

Definition 3.2 A subset in $\mathcal{B}(E)$ of the form C_f with $f \neq 0$ is called a *column of rank 1 operators*.

Theorem 3.1 *Let an operator algebra A on a Banach space E contain a column of rank 1 operators, say, C_f. Then it is spatially projective.*

$\triangleleft$ **Lemma** *The set C_f is a closed left principal ideal in A, generated by an idempotent element.*

$\triangleleft$ It follows from the first relation in (1) that C_f is a left ideal in $\mathcal{B}(E)$ and hence in A. Take $y \in E$ such that $f(y) = 1$ Then, for any $x \in E$, it follows from the second relation in (1) that $p := y \bigcirc f$ is a left ideal idempotent element in A, and C_f is generated by p. Finally, if a sequence $a_n \in C_f$ converges to a in A, then $a_n = a_n p$ implies $a = ap$. Consequently, C_f is closed. $\triangleright$

The end of the proof Let A_+ be the unitization of A as a Banach algebra. Consider the natural embedding $\mathbf{in} : C_f \to A_+$; of course, it is a morphism in $\mathbf{A}$-$\widehat{\otimes}$-$\mathbf{mod}$. Further, according to the Lemma, C_f is generated by some idempotent element, say p. Consider the map $\tau : A_+ \to C_f : a \mapsto ap$; obviously, it is a morphism in $\mathbf{A}$-$\widehat{\otimes}$-$\mathbf{mod}$, such that $\tau \circ (\mathbf{in}) = \mathbf{1}_{C_f}$. We see that C_f is a retract of A_+ in $\mathbf{A}$-$\widehat{\otimes}$-$\mathbf{mod}$. Since the latter module is a free module (of rank 1), it follows from Theorem 1.2'(ii) (with $\mathbb{C}$ as E) that the former module is projective.

Now consider the map $i : C_f \to E : x \bigcirc f \mapsto x$; obviously it is a bijection. Further, the relation (1) implies that i is a morphism of left A-modules, and the estimation $\|x \bigcirc f\| \geq \|x \bigcirc f\|_\infty = \|x\|\|f\|$ implies that i is a bounded operator. But C_f, being closed in A, is a Banach space. Therefore, by virtue of Banach's theorem, we have that the inverse operator of i is also bounded, and hence i is an isomorphism in $\mathbf{A}$-$\widehat{\otimes}$-$\mathbf{mod}$. But we already know that the left A-$\widehat{\otimes}$-module C_f is projective; hence, the same is true for E. $\triangleright$

$\triangleright$

Is a spatially projective operator algebra bound to contain a column of rank 1 operators? Of course, not: it is sufficient to observe that the algebra of scalar operators $\{\lambda \mathbf{1}_E\}$ is certainly spatially projective. However, the question becomes more interesting, if we shall restrict ourselves to the consideration of the so-called indecomposable algebras. Recall that an operator algebra A on a Banach space E is called *indecomposable*, if there is no decomposition of E into the direct sum of two non-trivial closed subspaces, invariant with respect to A. It is clear that the operator algebra, possessing a column of rank 1 operators, is always indecomposable. Could it be that within the class of indecomposable algebras those that are spatially projective are exactly those that possess a column of rank 1 operators ?

It turns out that for some important classes of operator algebras the answer is "yes". Not surprisingly, the majority of results of this kind concerns the most investigated case of operator algebras on Hilbert spaces. Somewhat later we shall see that both discussed properties are equivalent for indecomposable operator C^*-algebras. Now we shall formulate the similar result for another class of operator algebras, this time essentially non-selfadjoint.

Let A be an operator algebra on a Hilbert space H. Denote by $\mathrm{Lat}(A)$ the set of invariant subspaces of this algebra. Recall that A is called *reflexive*, if any $a \in \mathcal{B}(H)$ such that every space in $\mathrm{Lat}(A)$ is invariant with respect to a, belongs to A. (We immediately see that a reflexive algebra is always weak-operator closed and hence uniformly closed in $\mathcal{B}(H)$). A reflexive algebra is called *CSL-algebra*, if all projections on spaces in $\mathrm{Lat}(A)$ commute. ("CSL" is an abbreviation for "commutative subspace lattice"). The most popular CSL-algebras are the so-called *nest algebras*, defined as reflexive algebras A with $\mathrm{Lat}(A)$ linearly ordered under inclusion. These algebras can be considered as reasonable infinite-dimensional generalizations of the algebra of upper triangular matrix. Certainly, nest algebras are indecomposable.

Theorem 3.2 (Yu. O. Golovin) *Let A be an indecomposable CSL-algebra on a Hilbert space H. Then the following conditions are equivalent:*

(i) *A is spatially projective;*

(ii) *the closure of the algebraic sum of all spaces in* $\mathrm{Lat}(A)$*, other than* H*, does not coincide with* H*;*

(iii) A *contains a column of rank 1 operators.*

The proof is given in [9].

Note that for a nest algebra the second condition means that H, as a biggest element of a linearly ordered set $\mathrm{Lat}(A)$, has an immediate predecessor.

Example 3.1 Take the concrete Hilbert space $L_2(0,\infty)$ and set, for any $s > 0$, $H_s := \{f \in L_2(0,\infty) : f(t) = 0 \text{ if } 0 < t < s\}$. Now consider two nest algebras: A_1 with $\mathrm{Lat}\, A_1 := \{H_n; n = 1, 2, \dots\}$ and A_2 with $\mathrm{Lat}\, A_2 := \{H_s; s > 0\}$. Then, by virtue of Golovin's result, the first algebra is spatially projective (you are invited to display its columns of rank 1 operators) whereas the second one is not.

Nevertheless, indecomposable operator algebras with no column of rank 1 operators do exist. Here is apparently the simplest example.

Exercise 3.1 Show that the operator algebra on $\mathbb{C}^2$, consisting of operators that have, in the natural basis, matrices of the form

$$\begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}$$

is spatially projective and indecomposable, but it does not contain any column of rank 1 operators.

Note that the indicated algebra is neither reflexive nor semi-simple; so it is rather "bad". The following result, this time concerning a "very good" algebra, is much more interesting:

Theorem 3.3 (S. B. Tabaldyev) *There exists a uniformly closed operator algebra on a Hilbert space with the following properties:*

(i) *it is indecomposable, reflexive and semi-simple;*

(ii) *it is commutative and hence has no column of rank 1 operators;*

(iii) *and nevertheless it is spatially projective.*

The concrete algebra, suggested by Tabaldyev, was called by him a Sobolev algebra. It acts on the Sobolev space $W^1[0,1]$ of functions $f \in L_2[0,1]$, whose generalized derivatives f' are also regular generalized functions from $L_2[0,1]$. As it is well known, $W^1[0,1]$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle := \int_0^1 f(t)\overline{g(t)}dt + \int_0^1 f'(t)\overline{g'(t)}dt.$$

The Sobolev algebra consists of all operators m_f with $f \in W^1[0,1]$, acting by the pointwise multiplication, i.e. $m_f : g \mapsto fg$ for $f \in W^1[0,1]$. See the details in [29, pp. 207-209].

However, if we replace the condition of indecomposability by much stronger condition of irreducibility, examples of that kind are impossible:

Proposition 3.1 (Tabaldyev) *Let A be a spatially projective operator algebra on a Banach space. If, in addition, A is irreducible, then it contains a column of rank 1 operators.*

The proof see [29, p. 204].

In a well known class of operator algebras that is intermediate between indecomposable and irreducible algebras, the situation is not clear:

Problem Problem. Let a *topologically* irreducible operator algebra on a Banacg space be spatially projective. Does it imply that it contains a column of rank 1 operators?

If we know, in addition, that our algebra A on E is semi-simple, and also that either A or E has the approximation property, then it is irreducible and, as a corollary, the answer is "yes" ([29, p. 205]).

4 Projectivity in Hilbert modules

From now on, we again consider the algebras and modules as object of quantized as well as of classical functional analysis, and accordingly we use freely the notation "$\widetilde{\otimes}$" of the unpersonified tensor product. Besides, the term "operator algebra" again means only a uniformly closed subalgebra of $\mathcal{B}(H)$.

In this section we concentrate on another outstanding class of $\widetilde{\otimes}$-modules, the so-called Hilbert modules over C^*-algebras. We shall present a criterion for the projectivity of these modules and, as a particular case, we shall describe all spatially projective operator C^*-algebras. It will turn out that both results depend on our choice of homological theory (that is, on the choice of "$\widetilde{\otimes}$"). We shall see that the "quantized" approach gives us a larger class of projective modules than the "classical" approach. However, if we leave the class of C^*-algebras and turn to non-selfadjoint operator algebras, we encouter quite different phenomena. That will be shown at the very end of the section.

Here is the principal object of our study in this section:

Definition 4.1 Let A be a C^*-algebra. A left Banach A-module H is called a *Hilbert A-module* if it has an underlying Hilbert space and, besides, the identity $\langle a \cdot x, y\rangle = \langle x, a^* \cdot y\rangle$ holds.

Remark We must warn the reader that the terminology concerning these modules is still not fixed in the literature. In fact, there are not less than five different objects bearing in various papers the name of Hilbert modules (cf. [16, p. 79]).

It is evident that the notion of a Hilbert module over a C^*-algebra is equivalent to that of a representation of a C^*-algebra on a Hilbert space. (We mean, of course, an involutive representation.) Namely, if A and H are respective algebra and module, then the map $T : A \to \mathcal{B}(H), a \mapsto T(a)$ with $T(a)x := a \cdot x$ for $x \in H$ is obviously an involutive representation. It will be called *the representation, associated with the module* H. Conversely, if we are given a representation of A, say T, then H immediately becomes a Hilbert A-module with the outer multiplication $a \cdot x := [T(a)](x)$ for $a \in A$ and $x \in H$.

Needless to say, the most natural examples of Hilbert modules are provided by spatial modules over operator C^*-algebras.

The previous definition, in the form as it is given, concerns a $\widehat{\otimes}$-module over a $\widehat{\otimes}$-algebra. But we shall show that, as a matter of fact, it provides automatically a $\overset{h}{\otimes}$-module over an $\overset{h}{\otimes}$-algebra and hence a $\overset{o}{\otimes}$-module over an $\overset{o}{\otimes}$-algebra with respect to some reasonable quantizations of A and H. These are the well-known standard and, respectively, column quantization. Recall what they are.

As to a given (abstract) C^*-algebra A, we always take its quantization provided by its arbitrarily chosen faithful representation in a Hilbert space H. In other words, as a quantization we take an arbitrary injective $*$-homomorphism of A into $\mathcal{B}(H)$; recall that the latter is automatically isometric. But observe: after this the algebra $M_n(A)$ for $n = 1, 2, \ldots$ itself becomes a C^*-algebra (being a C^*-subalgebra of $\mathcal{B}(nH)$), and we recall that $*$-isomorphisms of C^*-algebras are automatically isometric. Therefore the resulting quantum Banach space does not depend on a particular choice of a faithful representation. The indicated quantization of a C^*-algebra will be called *standard*.

Note that *every *-homomorphism between C^*-algebras, quantized in the indicated way, is automatically completely contractive*. (It is because its amplifications are also *-homomorphisms between C^*-algebras, and therefore they are contractive.)

Being equipped with the standard quantization, a C^*-algebra becomes an $\overset{h}{\otimes}$- and hence an $\overset{o}{\otimes}$-algebra. (Indeed, for any $n = 1, 2, \ldots$, the space $M_n(A)$ is itself a C^*-algebra, and the respective amplification of the multiplication in A is itself a multiplication in this C^*-algebra. Hence, the latter bioperator is contractive.)

Now we proceed to a quantization of H. It is well known that Hilbert spaces can be quantized by a great multitude of different ways , and many of these ways are of a great use in studying various problems of functional analysis. For our present purposes, however, we need only one concrete quantization. As we shall see, it is a generalization of the concrete quantization of the spaces l_2 and l_2^2, cosidered in Section 1.

In what follows we denote by $x \bigcirc y$ with $x, y \in H$ the rank-one operator on H, taking z to $\langle z, y\rangle x$. (Compare the notation $x \bigcirc f$ in the previous section.) Fix an arbitrary unit vector $e \in H$ and consider the map $H \to \mathcal{B}(H) : x \mapsto x \bigcirc e$; it is, of course, an isometric operator and thus a quantization of H. This particular quantization is called the *column* *quantization*, and the resulting quantum space is called the *column Hilbertian space*, or just *column space*. It is easy to see that the matrix-norm of this quantum space does not depend on a particular choice of $e \in H$. More of this, for any n the Banach space $M_n(H)$ can be identified with the space $\mathcal{B}(\mathbb{C}^n, nH)$, if we identify a $n \times n$ matrix with entries in H with the operator from $\mathbb{C}^n$ to nH, depicted by this matrix. (This operator acts on a vector of $\mathbb{C}^n$, arranged as a column of complex numbers, by multiplying the given matrix by this column.)

Proposition 4.1 *Any Hilbert module H over a C^*-algebra A is an A-$\overset{h}{\otimes}$-module, and hence A-$\overset{o}{\otimes}$-module, with respect to the standard quantization of A and the column quantization of H. Moreover, the outer multiplication $\dot{m} : A \times H \to H$ is a multiplicatively contractive bioperator.*

$\triangleleft$ Fix a natural n and consider the respective multiplicative amplification of $\dot{m}$, that is the bioperator $\dot{m}^{(n)} : M_n(A) \times M_n(H) \to M_n(H)$, taking a pair $\mathbf{a} = (a_{ij}) \in M_n(A)$, $\mathbf{x} = (x_{ij}) \in M_n(H)$ to the matrix with entries $y_{ij} := \sum_{k=1}^n \dot{m}(a_{ik}, x_{kj}) \in H$ for $1 \leq i, j \leq n$. We must show that this bioperator is contractive.

Let $T : A \to \mathcal{B}(H)$ be the representation, associated with our module. Then we obviously have $y_{ij} = \sum_{k=1}^n [T(a_{ik})](x_{kj})$. Recall that H is a column space, and therefore we can identify $M_n(H)$ with $\mathcal{B}(\mathbb{C}^n, nH)$. Let T_n be the amplification of T. Then, using the previous equality, it is easy to observe that $\dot{m}^{(n)}(\mathbf{a}, \mathbf{x})$, now an operator in $\mathcal{B}(\mathbb{C}^n, nH)$, turns out to be the operator composition $[T_n(\mathbf{a})] \circ \mathbf{x}$. But T, being a $*$-homomorphism of C^*-algebras, is completely contractive (see above). Thus $\|T_n(\mathbf{a})\| \leq \|\mathbf{a}\|$, and hence $\|\dot{m}^{(n)}(\mathbf{a}, \mathbf{x})\| \leq \|\mathbf{a}\|\|\mathbf{x}\|$. And this is just what we need. $\triangleright$

From now on, speaking about a Hilbert module H as about a $\widetilde{\otimes}$-module over an $\widetilde{\otimes}$-algebra A, we always mean, in the both two "quantum" cases, the standard quantization of A and the column quantization of H.

We begin to prepare the theorem that will describe projective Hilbert modules. The first well known observation is of independent interest.

Proposition 4.2 *Let $\varphi : H \to K$ be a bounded operator between two column spaces. Then it is automatically completely bounded, and $\|\varphi\|_{cb} = \|\varphi\|$.*

$\triangleleft$ Fix a natural n and consider the operator $\varphi \dot{\otimes} \mathbf{1} : nH = H \dot{\otimes} \mathbb{C}^n \to K \dot{\otimes} \mathbb{C}^n = nK$. Identifying $M_n(H)$ with $\mathcal{B}(\mathbb{C}^n, nH)$ and $M_n(K)$ with $\mathcal{B}(\mathbb{C}^n, nK)$, we easily see that the n-th amplification $\varphi_n : M_n(H) \to M_n(K)$ of φ takes $a \in \mathcal{B}(\mathbb{C}^n, nH)$ to the operator composition $(\varphi \dot{\otimes} \mathbf{1}) \circ a \in \mathcal{B}(\mathbb{C}^n, nK)$. Therefore $\|\varphi_n(a)\| \leq \|\varphi \dot{\otimes} \mathbf{1}\|\|a\| = \|\varphi\|\|a\|$ holds. The rest is clear. $\triangleright$

The following class of module morphisms will serve us as one of our principal tools. Therefore it deserves a special name.

Definition 4.2 Let X be a left $\widetilde{\otimes}$-module over a (so far arbitrary) $\widetilde{\otimes}$-algebra A. Then any $\widetilde{\otimes}$-bounded morphism from X to the basic algebra A is called a *module character of* X, or just a *character of* X, if there is no danger of confusion.

We say that a left A-$\widetilde{\otimes}$-module X has a *sufficient set of characters*, if, for any $x \in X$, $x \neq 0$ there exists a character χ of X such that $\chi(x) \neq 0$.

It is well known in abstract algebra that any projective left module over an algebra has a sufficient set of characters (now we mean, of course, the algebraic prototype of the latter notion). We shall need a functional-analytic version of this result. At first we shall prove a preparatory assertion that will be used right now and also, for other purposes, later.

In what follows, for any vector $e \in H$ we denote by $\check{e}$ the functional $y \mapsto \langle y, e \rangle$ on H. If E is an arbitrary $\widetilde{\otimes}$-space, such a functional gives rise to the operator $\mathbf{1}\widetilde{\otimes}\check{e} : E\widetilde{\otimes}H \to E\widetilde{\otimes}\mathbb{C}$. Identifying $E \widetilde{\otimes} \mathbb{C}$ with E, we can consider $\mathbf{1} \widetilde{\otimes} \check{e}$ as a morphism from $E \widetilde{\otimes} H$ to E, well defined by taking $x \otimes y$ for $x \in E$ and $y \in H$ to $\langle y, e \rangle x$.

Proposition 4.3 *Let H and E be as above, and u be a non-zero vector in $E \widetilde{\otimes} H$. Then there exists $e \in H$, $e \neq 0$ such that the morphism $\mathbf{1} \widetilde{\otimes} \check{e} : E \widetilde{\otimes} H \to A$ takes u to a non-zero vector in H.*

$\triangleleft$ We shall show that the desired e can be found among vectors of an arbitrary orthonormal basis in H. Fix such a basis, and denote it by e_ν for $\nu \in \Lambda$.

Denote by $N(\Lambda)$ the family of all finite subsets of Λ, directed by the inclusion relation, and, for any $\lambda \in N(\Lambda)$, denote by $p_\lambda : H \to H$ the projection onto the finite-dimensional Hilbert space $K_\lambda := span\{e_\nu; \nu \in \lambda\}$. It follows from Proposition 2 that $\|p_\lambda\|_{cb} = 1$ for all $\lambda \in N(\Lambda)$. Consequently, for all λ we have $\|\mathbf{1} \widetilde{\otimes} p_\lambda\|_{cb} = 1$ for the operator $\mathbf{1} \widetilde{\otimes} p_\lambda : E \widetilde{\otimes} H \to E \widetilde{\otimes} H$. The latter equality easily implies the estimate

$$\|v - (\mathbf{1} \widetilde{\otimes} p_\lambda)(v)\| \leq 2\|v - w\| + \|w - (\mathbf{1} \widetilde{\otimes} p_\lambda)(w)\|$$

for all $v, w \in A \widetilde{\otimes} H$ and $\lambda \in N(\Lambda)$.

We claim that, for any $v \in E \widetilde{\otimes} H$, the net $(\mathbf{1} \widetilde{\otimes} p_\lambda)(v)$ converges to v. Indeed, since every $y \in H$ is a limit of $p_\lambda(y)$, it is true if v is an elementary tensor. Hence it is true for all sums of elementary tensors and, by virtue of the previous inequality, for their cluster points, that is for all $v \in E \widetilde{\otimes} H$.

Since our given u is not zero, it follows that $(\mathbf{1} \widetilde{\otimes} p_\lambda)(u) \neq 0$ at least for one λ. Hence, by additivity, there exists $\nu \in \Lambda$ such that the operator $\mathbf{1} \widetilde{\otimes} q_\nu$, where q_ν is the rank-one projection $e_\nu \bigcirc e_\nu$, does not take u to zero. But for every elementary tensor $v \in E \widetilde{\otimes} H$, say $v := x \otimes y$, we have

$$(\mathbf{1} \widetilde{\otimes} q_\nu)(v) = x \otimes \langle y, e_\nu \rangle e_\nu = (\mathbf{1} \widetilde{\otimes} \check{e_\nu})(x \otimes y) \otimes e_\nu = (\mathbf{1} \widetilde{\otimes} \check{e_\nu})(v) \otimes e_\nu.$$

Now again the habitual transfer from elementary tensors to arbitrary elements of $E \widetilde{\otimes} H$ works, and we have $(\mathbf{1} \widetilde{\otimes} q_\nu)(u) = (\mathbf{1} \widetilde{\otimes} \check{e_\nu})(u) \otimes e_\nu$. The rest is clear. $\triangleright$

Remark As a matter of fact, if we shall restrict ourselves to the "$\widehat{\otimes}$-case", the previous proposition can be considerably strengthened. Namely, let A be a Banach algebra and X be a left projective Banach A-module. Take the underlying Banach spaces of A and X, and suppose that at least one of them has the approximation property. Then, as it was observed by Yu. V. Selivanov, X has a sufficient set of characters. Details see, e.g., [10, Proposition 4.4].

Proposition 4.4 *Let H be a left $\widetilde{\otimes}$-module over C^*-algebra A. Suppose that H is non-degenerate and projective. Then it has a sufficient set of characters.*

$\triangleleft$ Fix a non-zero $x \in X$. By virtue of Theorem 3, there exists a morphism $\rho : H \to A \widetilde{\otimes} H$, right inverse to the respective outer product morphism and therefore injective. In particular, $u := \rho(x) \neq 0$.

Apply the previous proposition to the case A as E and the present u. We get an operator $\mathbf{1} \widetilde{\otimes} \check{e} : A \widetilde{\otimes} H \to A$ such that $\mathbf{1} \widetilde{\otimes} \check{e}(u) \neq 0$. Observing its action on elementary tensors, we see that it is a morphism in $\mathbf{A}$-$\widehat{\otimes}$-$\mathbf{mod}$. It remains to set $\chi := (\mathbf{1} \widetilde{\otimes} \check{e}) \circ \rho$. $\triangleright$

Later it will turn out that in the framework of "quantum" homology projective Hilbert modules are exactly those with a sufficient set of characters. At the same time, in the "classical" context the modules with sufficient set of characters are not bound to be projective.

Now we introduce an important notion that will participate in the formulation of the main result.

Definition 4.3 Let A be a C^*-algebra. A projection (that is, self-adjoint idempotent element) $p \in A$, $p \neq 0$ is called *elementary*, if pap is a multiple of p for any $a \in A$.

Let p be an elementary projection in a C^*-algebra A. We introduce the

$$I_p := \{ap : a \in A\};$$

in other words, I_p is the principal left ideal in A, generated by p. Recall several well-known facts of the structure theory of C^*-algebras:

Proposition 4.5 (i) *I_p is a minimal left ideal in A (in other words, an irreducible submodule of the A-module A). Conversely, any minimal left ideal in A is of the form I_p for some elementary projection $p \in A$. Moreover, the map $p \mapsto I_p$ is a bijection between the set of elementary projections in A and the set of minimal left ideals in A;*

(ii) *the ideal I_p is always closed, and its norm, induced from A, is the norm of a Hilbert space. Moreover, it is generated by the inner product, well defined by the equation $\langle ap, bp\rangle p = pb^*ap$ for $a, b \in A$.*

For the proof see, e.g., [24, Ch. IV, Section 10]).

In fact, we can say more:

Proposition 4.6 *The ideal I_p, as a submodule of A, is a Hilbert A-module. Moreover, its quantization as of the subspace of A, and its column quantization provide the same matrix-norm.*

$\triangleleft$ Let $a, b, c, d \in A$, and $x := cp, y := dp \in I_p$. Then

$$\langle ax, y\rangle p = \langle acp, dp\rangle p = pd^*acp = \langle cp, a^*dp\rangle p = \langle x, a^*y\rangle p,$$

and hence we obtain the first assertion.

Further, take a matrix $\mathbf{x} = (x_{ij}) \in M_n(I_p)$. Denote by $\|\mathbf{x}\|_1$ its norm with respect to the standard quantization of A, and by $\|\mathbf{x}\|_2$ its norm with respect to the column quantization.

Since $\|\cdot\|_1$ is a C^*-algebra norm, and the adjoint matrix $\mathbf{x}^*$ has the entries $(x^*)_{ij} := (x_{ji})^*$, the number $\|\mathbf{x}\|_2^2$ is the norm of the matrix $\mathbf{y} := \mathbf{x}^*\mathbf{x}$ with the entries

$$y_{ij} = \sum_{k=1}^{n}(x^*)_{ik}x_{kj} = \sum_{k=1}^{n}(x_{ki})^*x_{kj} = \sum_{k=1}^{n}p(x_{ki})^*x_{kj}p = \sum_{k=1}^{n}\langle x_{kj}, x_{ki}\rangle p.$$

It obviously follows that $\|\mathbf{x}\|_1^2$ is the norm of the scalar $n \times n$ matrix α with the entries $\alpha_{ij} := \sum_{k=1}^{n}\langle x_{kj}, x_{ki}\rangle$ for $1 \le i, j \le n$.

On the other hand, $\|\mathbf{x}\|_2$ is the norm of the respective operator in $\mathcal{B}(\mathbb{C}^n, nH)$, and hence $\|\mathbf{x}\|_2^2$ is the norm of the operator composition $\mathbf{f} \circ \mathbf{x} \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^n)$, where $\mathbf{f} \in \mathcal{B}(nH, \mathbb{C}^n)$ is the adjoint operator of $\mathbf{x}$. Obviously this latter operator has the matrix $(f_{ij} \in H^*)$, where $f_{ij} : H \to \mathbb{C}$ takes y to $\langle y, x_{ji}\rangle$. It remains to observe that $\mathbf{f} \circ \mathbf{x}$ is depicted by the same matrix α as before. $\triangleright$

In what follows, Hilbert A-modules of the form I_p, where p is an elementary projection in A, will be called *elementary Hilbert A-modules*.

Now we are almost ready to obtain the simplest particular case of a future general theorem and, doing this, to describe the "elementary bricks" for the building of arbitrary projective Hilbert modules. This initial result will concern *irreducible Hilbert modules*, that is, non-zero Hilbert modules without proper submodules (in pure algebraic sense) save $\{0\}$. [6]

[6]Note that, by virtue of Kadison's representation theorem, a Hilbert module over a C^*-algebra without proper *closed* submodules automatically has no proper submodules at all.

Theorem 4.1 *Let H be an irreducible Hilbert modules over a C^*-algebra A, considered as a $\widetilde{\otimes}$-module over an $\widetilde{\otimes}$-algebra. The following assertions are equivalent:*

(i) *H is $\widetilde{\otimes}$-projective;*

(ii) *H has a sufficient set of characters (in* $\mathbf{A}$-$\widetilde{\otimes}$-$\mathbf{mod}$*);*

(iii) *H has at least one non-zero character (in* $\mathbf{A}$-$\widetilde{\otimes}$-$\mathbf{mod}$*);*

(iv) *H is algebraically isomorphic to an elementary Hilbert A-module;*

(v) *H is $\widetilde{\otimes}$-isometrically $\widetilde{\otimes}$-isomorphic to an elementary Hilbert A-$\widetilde{\otimes}$-module.*

In particular, (because of the presence of the pure algebraic condition (iii)) the class of projective irreducible Hilbert A-$\widetilde{\otimes}$-modules does not depend on the particular choice of "$\widetilde{\otimes}$", i.e. it is the same in all three theories.

(As to the concluding assertion of the theorem, we shall see later that outside the class of irreducible modules the situation is different.)

$\triangleleft$ (i) $\Longrightarrow$ (ii). This is provided by Proposition 4.

(ii) $\Longrightarrow$ (iii). This is clear.

(iii) $\Longrightarrow$ (iv). Let χ be a non-zero character of H. Since H is irreducible, χ is an injective map and hence its corestriction $\chi^0 : H \to \mathrm{Im}(\chi)$ is an algebraic isomorphism of A-modules. It follows that the sub-A-module $\mathrm{Im}(\chi)$ in A is also irreducible, and hence it is a minimal left ideal in A. It remains to use Proposition 4(i).

(iv) $\Longrightarrow$ (v). Let $i : H \to I_p$ be an algebraic isomorphism of A-modules. Take $x \in H$ such that $i(x) = p$; since i is injective, and $i(p \cdot x) = p[i(x)] = p$, we have $p \cdot x = x$. Now take arbitrary $y, z \in H$. Since H is irreducible, there are $a, b \in A$ such that $y = a \cdot x = ap \cdot x$ and $z = b \cdot x = bp \cdot x$ and hence $i(y) = ap$ and $i(z) = bp$. Therefore, using the form of the inner product in I_p (see Proposition 4(ii)) and the definition of a Hilbert module, we have

$$\begin{aligned}\|x\|^2\langle i(y), i(z)\rangle &= \|x\|^2\langle ap, bp\rangle = \langle\langle ap, bp\rangle x, x\rangle \\ &= \langle\langle ap, bp\rangle p \cdot x, x\rangle = \langle(pb^*ap) \cdot x, x\rangle = \langle ap \cdot x, bp \cdot x\rangle = \langle y, z\rangle.\end{aligned}$$

Thus the operator $\|x\|i$ is an isometric morphism from H onto I_p, and this completes our assertion in the "$\widehat{\otimes}$"-case. Since in both "quantum" cases the spaces H and (by virtue of Proposition 5) I_p have the column quantization, the rest follows from Proposition 2.

(v) $\Longrightarrow$ (i). Since p is a right identity for I_p, the latter is $\widetilde{\otimes}$-projective by Proposition 2.1. $\triangleright$

The most transparent particular case of this theorem is apparently the following assertion. From now on, the term "spatially $\widetilde{\otimes}$-projective operator C^*-algebra" means, of course, that the spatial module over this algebra is $\widetilde{\otimes}$-projective.

Theorem 4.2 *Let A be an irreducible operator C^*-algebra on a Hilbert space H. Then A is spatially $\widetilde{\otimes}$-projective $\Longleftrightarrow$ it contains all compact operators (i.e. $A \supseteq \mathcal{K}(H)$).*

$\triangleleft$ "$\Longleftarrow$". Since A, together with $\mathcal{K}(H)$, contains a lot of columns of rank 1 operator, H is $\widehat{\otimes}$-projective by virtue of Theorem 3.1. Hence the concluding assertion of the previous theorem guarantees that H, being an irreducible module, is also $\overset{h}{\otimes}$- and $\overset{o}{\otimes}$-projective.

"$\Longrightarrow$". It follows from the previous theorem that A contains an elementary projection, say p. Take arbitrary $x, y \in \mathrm{Im}(p)$ with $x \neq 0$. Since A is irreducible, x is its cyclic vector. Hence $y = a(x)$ holds for some $a \in A$. We remember that pap is a multiple of p; therefore $y = [pap](x)$ is a multiple of $x = p(x)$. It follows that p is an operator of rank 1. But the following is well known (and easy to show; do it!): if an irreducible C^*-algebra contains at least one operator of rank 1, then it contains all compact operators. The rest is clear. $\triangleright$

Recall that an indecomposable operator C^*-algebra is automatically irreducible. We therefore redeem a promise, given in the previous section. Namely, as a direct corollary of the previous theorem, we obtain that every spatially projective indecomposable operator C^*-algebra is bound to have columns of rank 1 operators.

Counterexample Consider the well-known fermion (or CAR-) algebra. It is irreducible, but contains no compact operator. Hence, it is not spatially $\widetilde{\otimes}$-projective, whatever concrete "$\widetilde{\otimes}$" we choose. The same is true for general Glimm algebras.

In fact the modules, participating in Theorem 2, admit a complete classification. We have to restrict ourselves to a mere formulation of the relevant result. In what follows, the word "isomorphism" means whatever you can imagine: from a pure algebraic isomorphism (through a topological isomorphism of Banach modules) to a completely isometric isomorphism of quantized modules.

We recall that projections p and q in a C^*-algebra A are called *equivalent* or, more precisely, *equivalent in the sense of Murray and von Neumann* if there exists $v \in A$ (called a *partial isometry*) such that $v^*v = p$ and $vv^* = q$ and consequently, as an easy calculation shows, $v = vp = qv$ and $v^* = pv^* = vq^*$.

The following assertion (Proposition 21 in [13]) is rather easy, and we leave its proof to the listener/reader.

Proposition 4.7 *Let p and q be elementary projections in A. Then they are equivalent $\Longleftrightarrow$ $paq \neq 0$ for some $a \in A$ $\Longleftrightarrow$ A-modules I_p and I_q are isomorphic.*

This proposition, being combined with the previous theorem, immediately implies:

Theorem 4.3 *The assignment $p \mapsto I_p$ induces a bijection between the set of equivalence classes of elementary projections in A and the set of isomorphism classes of irreducible projective A-$\widetilde{\otimes}$-modules.* $\triangleleft\triangleright$

Exercise 4.1 Let A be an operator C^*-algebra, containing at least one operator of rank 1. Show that all irreducible projective A-$\widetilde{\otimes}$-modules coincide, up to an isomorphism, with the spatial A-module.

We proceed from irreducible to general Hilbert modules.

Unfortunately, the lack of space/time prevents us from giving the thorough description and classification of projective objects within this class of modules. Nevertheless, we shall try to illuminate some crucial points with sufficient degree of completeness.

To begin with, we shall provide some valuable information about the behavior of characters of our modules. As a matter of fact, we have a full knowledge of their structure (see [21]). However, for our present aims we need only part of the relevant general result.

Theorem 4.4 *Let H be a Hilbert module over a C^*-algebra A, and let $\chi : H \to A$ be its non-zero character. Then the image of χ contains an elementary projection.*

Regretfully, we have no possibility to present here the complete proof of this theorem, given in [16, p. 20–22]. However, we shall point out its principal stages. It may well happen that our listener/reader will be able to restore, embarking from these "lemmas", all missing details.

First of all, relying heavily on the "Hilbert" and "C^*-" stuff, we get the rather technical

Lemma 1 *Let $x \in H$ and let $a := \chi(x)$. Suppose that there are selfadjoint normed elements $b_1, \dots, b_n \in A$ such that $b_k b_l = b_l b_k = 0$ for $k, l = 1, \dots, n$ and, for some $\theta > 0$, we have $\|b_k a\| \geq \theta$ for $k = 1, \dots, n$. Then the number n of these elements does not exceed $(\theta)^{-2}\|\chi\|^2\|x\|^2$.*

From this lemma, using the apparatus of the continuous functional calculus in C^*-algebras, one can deduce

Lemma 2 *Let a be a non-zero positive element in the image of χ, and $\Omega \subset \mathbb{R}^+$ be its spectrum. Then this spectrum consists of isolated points, save, perhaps, 0.*

Thanks to this lemma, we can apply "δ-functions" to elements in the image of χ. With their help, we get

Lemma 3 *Let a and Ω be as above. Then the image of χ contains a projection p such that $pa = ap = pap$, and the latter element is a non-zero multiple of p.*

From this lemma, in its turn, follows

Lemma 4 *If a projection in $\mathrm{Im}(\chi)$ is not elementary, then it strictly majorizes another projection in $\mathrm{Im}(\chi)$.*

On the other hand, the same Lemma 1, being a source of the formulated lemmas, also implies

Lemma 5 *Let $p = p_0 > p_1 > \ldots > p_n$ be a family of strictly decreasing non-zero projections in A, and $p = \chi(x)$ for some $x \in H$. Then $n \leq \|\chi\|^2\|x\|^2$. As a corollary, every projection in the image of χ majorizes a projection, which is minimal in A.*

Now one can obtain the formulated theorem, combining three last lemmas.

As the second step to the main results, we use the previous theorem to obtain

Proposition 4.8 *Let H be a Hilbert module over a C^*-algebra. Suppose H has a non-zero character. Then it has a closed irreducible submodule, isomorphic to some elementary Hilbert module.*

$\triangleleft$ Let χ be a mentioned character. Then, by virtue of the previous theorem, there exist $x \in H$ and an elementary projection $p = \chi(x)$. Replacing x by $p \cdot x$, we can assume that $p \cdot x = x$. Take a submodule $H_x := \{a \cdot x; a \in A\}$ in H. Our aim is to show that this is just what we need.

Let a sequence $y_n = a_n \cdot x \in H_x$ converge to $y \in H$. Since the map χ is continuous, $\chi(y_n) = a_n \cdot \chi(x) = a_n p \in I_p$ converge to $\chi(y)$. Since I_p is closed, $\chi(y) = ap$ for some $a \in A$. But then

$$a \cdot x = ap \cdot x = \lim_{n\to\infty}(a_n p \cdot x) = \lim_{n\to\infty}(a_n p \cdot x) = \lim_{n\to\infty}(a_n \cdot x) = \lim_{n\to\infty} y_n = y.$$

Therefore $y \in H_x$, and hence H_x is closed.

Further, χ evidently maps H_x onto I_p. Consider the respective birestriction $\chi_0 : H_x \to I_p$; of course, it is a morphism of A-modules. Take $y \in H_x; y = a \cdot x$; then $y = ap \cdot x$. Therefore, if $y \neq 0$, then $ap \neq 0$ and hence $\chi_0(y) = \chi(a \cdot x) = a\chi(x) = ap \neq 0$. We see that χ_0 is an injective morphism of H_x onto I_p. Consequently, it is an isomorphism. The rest is clear. $\triangleright$

Theorem 4.5 *Let H be a Hilbert module over a C^*-algebra A such that the set of its characters is sufficient. Then H decomposes into a Hilbert sum of its closed irreducible submodules. Moreover, every of these submodules is isomorphic to an elementary Hilbert module.*

$\triangleleft$ If K is a closed submodule in H, the "direct difference" $H \ominus K$ is obviously also a submodule in H. This enable us to deduce the desired assertion from the previous theorem with the help of a ritual dance around Zorn's Lemma. The respective partially ordered set is the set of all families of pairwise orthogonal submodules of H, isomorphic to some elementary Hilbert module. We left the details to the listener/reader. $\triangleright$

Formally this theorem belongs to the representation (= module) theory of C^*-algebras. But its impact on the homological theory is obvious. We remember (Proposition 4) that a Hilbert module, which is projective in the sense of at least of one of our three homological theories, automatically satisfies the conditions of the previous theorem and thus has a Wedderburn-type decomposition of the indicated form. Now we are at the threshold of the theorem, showing that in both "quantum" homological theories the converse is also true. As a corollary, we shall get a complete description of $\widetilde{\otimes}$-projective Hilbert modules for $\widetilde{\otimes} = \overset{h}{\otimes}$ and $\widetilde{\otimes} = \overset{o}{\otimes}$.

We want to rearrange our decomposition, grouping isomorphic summands. As it could be expected because of Theorem 3, the set of equivalence classes of elementary projections in A plays an important role. We denote it by $\mathcal{P}(A)$, or just $\mathcal{P}$. It will be convenient to identify $\mathcal{P}$ with some fixed (arbitrarily chosen) maximal family of mutually non-equivalent elementary projections in A and respectively write, say, $p \in \mathcal{P}$. If $\beta : \mathcal{P} \to \overline{\mathbb{N}}$ is an arbitrary cardinality-valued function, we denote by $\beta(p)I_p$ the Hilbert sum of $\beta(p)$ copies of the Hilbert space I_p and introduce, as one of our main objects, the quantum column Hilbertian A-module $\dot{\oplus}\{\beta(p)I_p : p \in \mathcal{P}\}$, denoted by I^β. (if $\mathcal{P}$ is empty, we put $I^\beta := 0$).

Theorem 4.6 *The following properties of a non-degenerate Hilbert module H over a C^*-algebra A are equivalent:*

(i) *H is projective as a $\overset{o}{\otimes}$-module over a $\overset{o}{\otimes}$-algebra;*

(ii) *H is projective as a $\overset{h}{\otimes}$-module over a $\overset{h}{\otimes}$-algebra;*

(iii) *H has a sufficient set of characters;*

(iv) *H is completely isometrically isomorphic to the Hilbert sum of a family of elementary A-modules, that is (by Theorem 3) to I^β for some $\beta : \mathcal{P} \to \overline{\mathbb{N}}$.*

$\triangleleft$ (i) $\Longrightarrow$ (ii). This follows from Proposition 1.7.

(ii) $\Longrightarrow$ (iii). This is Proposition 3.

(iii) $\Longrightarrow$ (iv). This is Theorem 5.

(iv) $\Longrightarrow$ (i). **And now it is time to work in earnest...**

Again we intend to use Theorem 1.3. Accordingly, our aim is to produce a completely bounded morphism of A-modules $\rho : I^\beta \to A \overset{o}{\otimes} I^\beta$, right inverse to the outer product morphism $\pi := \pi_{I^\beta}$. We shall provide such a morphism, and the latter, in addition, will happen to be completely contractive.[7]

For any $p \in \mathcal{P}$, we denote by $\overline{\beta(p)}$ the segment of the transfinite line from 1 to $\beta(p)$. If $m \in \overline{\beta(p)}$, we denote by I_p^m the "m-th replica" (or copy) of I_p in a Hilbert sum $\beta(p)I_p$.

[7]And even better than this; see the future Exercise 2.

In what follows, for $x \in I_p(\subseteq A)$ the notation x^m means that we consider the respective replica of of x in I_p^m.

Denote by I_0^β the linear span or, what is now the same, the direct linear sum of all $I_p^m : p \in \mathcal{P}, m \in \overline{\beta(p)}$. Consider the linear operator $\rho_0 : I_0^\beta \to A \overset{o}{\otimes} I^\beta$, well defined by taking a replica $x^m \in I_p^m$ of $x \in I_p$ to

$$\rho_0(x^m) := x \otimes p^m.$$

Observe that, for any $x^m \in I_p^m$ and $a \in A$, we have $\rho_0(a \cdot x^m) = \rho_0((a \cdot x)^m) = ax \otimes p^m = a \cdot (x \otimes p^m) = a \cdot \rho_0(x^m)$, and $\pi(\rho_0(x^m)) = x \cdot p^m = (xp)^m = x^m$. By linearity, this implies that the operator ρ_0 has the properties

$$\rho_0(a \cdot y) = a \cdot \rho_0(y) \tag{9}$$

and

$$\pi(\rho_0(y)) = y \tag{10}$$

for all $a \in A$ and $y \in I_0^\beta$.

Fix $n \in \mathbb{N}$ and an arbitrary matrix $\mathbf{y} = (y_{ij}) \in M_n(I_0^\beta)$. To begin with, we want to show that, for the matrix $\mathbf{u} := \rho_{0n}(\mathbf{y}) \in M_n(A \overset{o}{\otimes} I^\beta)$, where ρ_{0n} is the amplification of ρ_0 (i.e. for $\mathbf{u}$ with the entries $u_{ij} := \rho_0(y_{ij}) \in A \overset{o}{\otimes} I^\beta$) we have $\|\mathbf{u} \leq \|\mathbf{y}\|$.[8]

Since I_0^β is the direct sum of the spaces I_p^m, we we can express the matrix entries y_{ij} in the form

$$y_{ij} = \sum_{l=1}^{L} \sum_{k=1}^{K} x_{lk,ij}^k.$$

In this expression $x_{lk,ij}^k \in I_{p_l}^{m_k}$ is the "m_k-th replica of a certain element $x_{lk,ij} \in I_{p_l}$, where $p_l \in \mathcal{P}$ for $l = 1, \ldots, L$ are certain elementary projections, $m_k \in \overline{\beta(p_l)}; k = 1, \ldots, K$ are ordinals, and from now on we write, for brevity, the superscript k instead of m_k.

Take, for every $l = 1, \ldots, L$, the span of all $x_{lk,ij}$ in I_{p_l}, and then some orthonormal basis, say $e_{l1}, \ldots, e_{lM_l}$ of that span. Then we have, for all relevant l, k, i, j, the expansion

$$x_{lk,ij} = \sum_{m=1}^{M_l} \lambda_{lkm,ij} e_{lm}, \quad \text{and consequently} \quad x_{lk,ij}^k = \sum_{m=1}^{M_l} \lambda_{lkm,ij} e_{lm}^k$$

for some $\lambda_{lkm,ij} \in \mathbb{C}$.

The following important introduced vectors will be used a little bit later.

Lemma 1 (i) *For all $l = 1, \ldots, L, m, m' = 1, \ldots, M_l$ we have $e_{lm}^* e_{lm'} = p_l$, if $m = m'$, and $e_{lm}^* e_{lm'} = 0$, if $m \neq m'$;*

[8] The advanced reader will realize that our argument is rather near to the proof of the well known fact that the $\overset{o}{\otimes}$-product of two column Hilbertians is again a column Hilbertian. Moreover, we could, after some preparation, use this fact, showing that the image of ρ is actually contained in such a $\overset{o}{\otimes}$-product. But it will not make our life much easier. We prefer the direct proof, which seems to us more instructive.

(ii) *for all* $l, l' = 1, \dots, L, l \neq l'$ *and all* $m = 1, \dots, M_l, m' = 1, \dots, M^{m'}$ *we have* $e^*_{l'm'} e_{lm} = 0$.

$\triangleleft$ (i). Since vectors e_{lm} for all m belong to I_{p_l}, we have $e_{lm} = e_{lm} p_l$. Therefore, using the form of an inner product in I_{p_l} (see Proposition 4(ii)), we have

$$e^*_{lm} e_{lm'} = p_l e^*_{lm} e_{lm'} p_l = \langle e_{lm}, e_{lm'} \rangle p_l.$$

The rest is clear.

(ii). Suppose that, on the contrary, $e^*_{l'm'} e_{lm} \neq 0$. Let us denote, for brevity, this element by a. Since $a = p_{l'} e^*_{l'm'} e_{lm} p_l$, we have $a^* a = p_l a^* p^*_{l'} a p_l$ and $a a^* = p_{l'} a p^*_l a^* p_{l'}$. Therefore $a^* a$ is a multiple of p_l, whereas $a a^*$ is a multiple of $p_{l'}$. Set $b := a / \|a\|$. The C^*-identity for the norm in A immediately implies $b^* b = p_k$ and $b b^* = p_l$. But this means that the projections p_l and p_k, despite they represent different elements of $\mathcal{P}$, are equivalent. We came to a contradiction. $\triangleright$

The continuation of the proof Remembering that I^β has the column quantization of I^β, we identify the matrix $\mathbf{y}$ with the respective operator in $\mathcal{B}(\mathbb{C}^n, (I^\beta)^n)$, also denoted by $\mathbf{y}$. This operator takes $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ to

$$\mathbf{y}(\xi) = \left(\sum_{j=1}^n \xi_j y_{1j}, \dots, \sum_{j=1}^n \xi_j y_{nj} \right).$$

The i-th vector in this n-tuple is

$$\begin{aligned} \sum_{j=1}^n \xi_j y_{ij} &= \sum_{j=1}^n \xi_j \left(\sum_{l=1}^L \sum_{k=1}^K x^k_{lk,ij} \right) \\ &= \sum_{j=1}^n \xi_j \left(\sum_{l=1}^L \sum_{k=1}^K \left(\sum_{m=1}^{M_l} \lambda_{lkm,ij} e^k_{lm} \right) \right) = \sum_{l=1}^L \sum_{k=1}^K \sum_{m=1}^{M_l} \left(\sum_{j=1}^n \xi_j \lambda_{lkm,ij} \right) e^k_{lm}. \end{aligned}$$

Since the system e^k_{lm} for all possible l, k, m is orthonormal in I^β, we have

$$\|\mathbf{y}\|^2 = \max \sum_{i=1}^n \left\| \sum_{j=1}^n \xi_j y_{ij} \right\|^2 = \max \sum_{i=1}^n \sum_{l=1}^L \sum_{k=1}^K \sum_{m=1}^{M_l} \left| \sum_{j=1}^n \xi_j \lambda_{lkm,ij}) \right|^2$$

where the maximum is taken over all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ such that $\sum_{j=1}^n |\xi_j|^2 = 1$.

Keeping this in our memory, we turn to the matrix $\mathbf{u} := \rho_{0n}(\mathbf{y})$. Its entries have the form

$$u_{ij} = \sum_{l=1}^L \sum_{k=1}^K x_{lk,ij} \otimes p^k_l.$$

(Recall that by p_l^k we denote the replica of p_l in $I_l^{m_k}$.) Therefore

$$u_{ij} = \sum_{l=1}^{L}\sum_{k=1}^{K}\left[\left(\sum_{m=1}^{M_l}\lambda_{lkm,ij}e_{lm}\right)\otimes p_l^k\right] = \sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{m=1}^{M_l}[\lambda_{lkm,ij}(e_{lm}\otimes p_l^k)].$$

To present it in slightly more convenient form, we set, for all l, m, k, i, j, participating in the calculations above, and for all $s = 1, \dots, L$, $\mu_{slkm,ij} := \lambda_{lkm,ij}$, if $s = l$, and $\mu_{slkm,ij} := 0$ otherwise. Then obviously we have

$$u_{ij} = \sum_{s,l=1}^{L}\sum_{k=1}^{K}\sum_{m=1}^{M_l}\mu_{slkm,ij}[e_{lm}\otimes p_s^k]. \tag{11}$$

Observe that this formula deals with $N := M_1 + \ldots + M_L$ elements $e_{lm} \in A$ and KL vectors $p_s^k \in I^\beta$. For brevity, we renumerate arbitrarily the double indexes "$(\cdot)_{lm}$" by single numbers $t = 1, \dots, N$, and the double indexes "$(\cdot)_s^k$" by single numbers $r = 1, \dots, KL$. *From now on we shall write e_t, q_r and $\mu_{tr,ij}$ instead of respective elements e_{lm}, vectors p_s^k and complex numbers $\mu_{slkm,ij}$.* The formula (11) transforms into

$$u_{ij} = \sum_{t=1}^{N}\sum_{r=1}^{KL}\mu_{tr,ij}[e_t\otimes q_r].$$

After this, we introduce the following three matrices:

1^0. A certain matrix $\mathbf{v} \in M_{n,Nn}(A)$. It has the form of the "block-row"

$$bfv_1, \dots \mathbf{v}_t, \dots, \mathbf{v}_N),$$

where, for any $t = 1, \dots, N$, $\mathbf{v}_t = (v_{t,ij}) \in M_n(A)$ is the diagonal matrix with entries e_t in the main diagonal and 0 in other places.

2^0. A certain matrix, more precisely, a row $\mathbf{w} \in M_{1,KL}(I^\beta)$. It has the form

$$(q_1, \dots, q_r, \dots, q_{KL}).$$

3^0. A certain scalar ("usual") matrix $\gamma \in M_{NKLn,n}(\mathbb{C})$. It has the form of the "block-column"

$$\begin{pmatrix} \vdots \\ \gamma_{tr} \\ \vdots \end{pmatrix},$$

where the respective blocks are $n \times n$ matrices $\gamma_{tr} := (\mu_{tr,ij}); t = 1, \dots, N, r = 1, \dots KL$.

Now let us have a look at the tensor product $\mathbf{v}\otimes\mathbf{w}$. According to what was said above, we see a matrix in $M_{n\times 1,Nn\times KL}(A\otimes I^\beta)$ that obviously has the form of the "block-row" $(\dots, \mathbf{z}_{tr}, \dots)$ with $1 \le t \le N, 1 \le s \le KL$, where $\mathbf{z}_{tr} \in M_n(A\otimes I^\beta)$ is the diagonal matrix with entries $e_t \otimes q_r$ in the main diagonal and 0 in other places. (We can arrange these matrices from left to right in the lexicographical order).

We came to the crucial observation. From the structure of our matrices it easily follows that the $n \times n$ matrix $(v \otimes w)\gamma$ has, in the "ij-th" place, for $1 \leq i, j \leq n$, the entries $\sum_{t=1}^{N} \sum_{r=1}^{KL} \mu_{tr,ij}[e_t \otimes q_r]$. But by (6) these entries are exactly u_{ij}. We got the equality

$$\rho_0(\mathbf{y}) = \mathbf{u} = (\mathbf{v} \otimes \mathbf{w})\gamma.$$

Consequently, we can apply the formula (2) in Section 0, where we set, of course, A as E, I^β as F, our concrete matrices of the same notation as $\mathbf{v}$, $\mathbf{w}$ and γ, and the identity matrix in $M_n(\mathbb{C})$ as α. This gives the estimate

$$\|\rho_{0n}(\mathbf{y})\| \leq \|\mathbf{v}\|\|\mathbf{w}\|\|\gamma\|. \tag{11}$$

But what can be said about the norms of our three matrices?

Lemma 2 $\|\mathbf{v}\| = \|\mathbf{w}\| = 1$, *whereas* $\|\gamma\| = \|\mathbf{y}\|$.

$\triangleleft$ Consider at first $\mathbf{v} \in M_{n,Nn}(A)$. The norm of this rectangular matrix is, by definition, the norm of the square matrix in $M_{Nn}(A)$ with $\mathbf{v}$ as its upper block-row and with 0 in other places; we preserve for this square matrix the same notation $\mathbf{v}$. But $M_{Nn}(A)$ is a C^*-algebra; therefore $\|\mathbf{v}\|^2 = \|\mathbf{v}^*\mathbf{v}\|$ holds. Further, $\mathbf{v}^*\mathbf{v}$ can be represented as the block-matrix of the size $N \times N$ with blocks from $M_n(A)$. In this "big" matrix the block with the subscript "tr" for $1 \leq t, r \leq N$ is obviously the $n \times n$ matrix with $e_t^* e_r$ in the main diagonal and 0 in other places. Now we recall that by e_t we began, after an arbitrary renumeration, to denote the vectors e_{lm} for $l = 1, \ldots, L, m = 1, \ldots, M_l$, and these vectors were the subject of Lemma 1. It is evident that in the new notation of our vectors Lemma 1 exactly asserts that $e_t^* e_r = p_t$, *if* $t = r$, *and* $e_t^* e_r = 0$ *otherwise*. Consequently, our "big" matrix $\mathbf{v}^*\mathbf{v}$ has no non-zero entries, save in the main diagonal, and those are non-zero projections. This implies, of course that $\|\mathbf{v}^*\mathbf{v}\| = 1$, and hence $\|\mathbf{v}\| = 1$.

Now we turn to the row $\mathbf{w} = (q_1, \ldots, q_{KL}) \in M_{1,KL}(I^\beta)$. Because of the column quantization of I^β, its norm is the norm of the operator in $\mathcal{B}(\mathbb{C}^{KL}, I^\beta)$, depicted by this row. This operator takes the t-th ort (= vector of the natural basis) in $\mathbb{C}^{KL}$ to q_t. Recall that by q_t we began, after an arbitrary renumeration, to denote the vectors p_l^k for $l = 1, \ldots, L, k = 1, \ldots, K$, and that these vectors, because of their choice, form an orthonormal system in I^β. This means, of course, that our operator is isometric. Consequently, $\|\mathbf{w}\| = 1$.

Finally, take the scalar matrix γ. Its norm is the norm of the respective operator in $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^{NKLn})$. This means that

$$\|\gamma\|^2 = \max \sum_{i=1}^{n} \sum_{t=1}^{N} \sum_{r=1}^{KL} \left| \sum_{j=1}^{n} \xi_j \mu_{tr,ij} \right|^2,$$

where the maximum is taken over all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ such that $\sum_{j=1}^n |\xi_j|^2 = 1$. Recall that by $\mu_{tr,ij}$ we began, after an arbitrary renumeration, to denote the complex numbers $\mu_{slkm,ij}$ for $s, l = 1, \dots, L$, $k = 1, \dots, K$, $m = 1, \dots, M_l$, and the latter, in their turn, were taken equal to $\lambda_{lkm,ij}$, if $s = l$, and to 0 otherwise. Therefore, we have

$$\|\gamma\|^2 = \max \sum_{i=1}^n \sum_{s,l=1}^L \sum_{k=1}^K \sum_{m=1}^{M_l} \left| \sum_{j=1}^n \xi_j \mu_{slkm,ij} \right|^2 = \max \sum_{i=1}^n \sum_{l=1}^L \sum_{k=1}^K \sum_{m=1}^{M_l} \left| \sum_{j=1}^n \xi_j \lambda_{lkm,ij} \right|^2,$$

where the maximum is taken over the same ξ as before. We see that exactly $\|\gamma\| = \|\mathbf{y}\|$. $\triangleright$

The end of the proof This lemma, together with the estimate (11), gives $\|\rho_{0n}(\mathbf{y})\| \leq \|\mathbf{y}\|$ for any $n \in \mathbb{N}$ and $\mathbf{y} \in M_n(I_0^\beta)$. Thus the operator $\rho_{0n} : M_n(I_0^\beta) \to M_n(A \overset{o}{\otimes} I^\beta)$ is contractive. In particular, taking $n = 1$, we see that our initial operator $\rho_0 : I_0^\beta \to A \overset{o}{\otimes} I^\beta$ is contractive. Since I_0^β is a dense subspace in I^β, ρ_0 has a unique extension to a contractive operator $\rho : I^\beta \to A \overset{o}{\otimes} I^\beta$. Clearly, for any n, the respective amplification $\rho_n : M_n(I^\beta) \to M_n(A \overset{o}{\otimes} I^\beta)$ is just the extension by continuity of ρ_{0n} and therefore it is also contractive. We see that ρ is *completely* contractive.

It remains to observe that the equality (9), combined with the continuity of ρ, implies that ρ is a morphism of A-modules whereas the equality (10) similarly implies that ρ is a right inverse of the canonical projection π. Now Theorem 1.3 closes the matter. $\triangleright$

Exercise 4.2 Show that, as a matter of fact, the just constructed morphism ρ is completely isometric.

Thus we have just seen that a "quantum projective" Hilbert module over a C^*-algebra is exactly what one can build from the "elementary bricks", described in Theorem 1, by taking arbitrary Hilbert sums.

Proceeding from the "quantum" to the "classical" context, we discover that the situation becomes more complicated.

The reason of the appearance of new difficulties is as follows. Now, trying to prove that a module of the type I^β is $\widehat{\otimes}$-projective, we should like to construct a morphism from such a module to $A \otimes I^\beta$, where the latter is equipped with the $\widehat{\otimes}$-norm. But this norm is considerably bigger than $\overset{o}{\otimes}$-norm, with which we have just worked. Therefore an operator from I^β to $A \otimes I^\beta$, that was bounded with respect to the $\overset{o}{\otimes}$-norm in the latter space, can fail to be bounded after the replacing this norm by the $\widehat{\otimes}$-norm. More of this, one can observe that the concrete morphism ρ_0, constructed in the previous proof, is, generally speaking, no more bounded.

Nevertheless, the "classically projective" Hilbert modules also can be completely described, and this can be done with the help of some modification of the morphism ρ_0, discussed above. The existing proof of the relevant result, as a whole, is more difficult than that of Theorem 6 (and this is despite we have "only one floor" to work with). We shall present only its formulation.

Definition 4.4 A cardinality-valued function $\beta : \mathcal{P} \to \overline{\mathbb{N}}$ is *essentially finite*, if the cardinality $\beta(p)$ is finite for all $p \in \mathcal{P}$ such that $\dim I_p = \infty$.

Theorem 4.7 *Let H be a non-degenerate Hilbert module over a C^*-algebra A. Then its following properties are equivalent:*

(i) *H is $\widehat{\otimes}$-projective;*

(ii) *there is an essentially finite cardinality-valued function $\beta : \mathcal{E} \to \overline{\mathbb{N}}$ such that H is topologically isomorphic to the Banach A-module I^β;*

(iii) *the same, with "isometrically" instead of "topologically".*

The complete proof is presented in [13].

Counterexample Let A be an operator C^*-algebra, acting on a Hilbert space H and containing at least one operator of rank 1, and hence all of $\mathcal{K}(H)$. Take any projection p of rank 1. Then it is easy to see that $\dim I_p = \infty$. (In fact, the A-module I_p is isomorphic to the spatial module H; cf. Exercise 1.) Therefore Theorems 6 and 7 imply that the A-module $\aleph_0 I_p$, the Hilbert sum of a countable family of copies of I_p, is $\overset{h}{\otimes}$- and $\overset{o}{\otimes}$-projective, but it is not $\widehat{\otimes}$-projective.

(In fact, the most difficult part in the proof of Theorem 7 is just to demonstrate that such a module is not $\widehat{\otimes}$-projective.)

Recall that, within the class of irreducible modules, $\widetilde{\otimes}$-projective Hilbert modules admit the complete classification (Theorem 3). This classification can be extended to general $\widetilde{\otimes}$-projective Hilbert modules. For this aim, apart from the mentioned theorem, we use

Proposition 4.9 *Let $\beta, \beta' : \mathcal{P} \to \overline{\mathbb{N}}$ be two arbitrary cardinality-valued functions. If the Hilbert modules H^β and $H\beta'$ are topologically isomorphic, then $\beta = \beta'$.*

For the proof, see [13, Theorem 4].

Theorems 6 and 7, combined with this proposition, give the theorem, formulated below. There, in the "classical" part (i) the word "isomorphism" means either topological or, according to your wish, isometric isomorphism of Banach modules. In the "quantum" part (ii) of the theorem the same word means whatever you choose between topological isomorphism of Banach modules and completely isometric isomorphism of quantum Banach modules.

Theorem 4.8 *Let A be a C^*-algebra, $\mathcal{P}$ be the set of equivalence classes of its elementary projections. The the assignment $\beta \mapsto H^\beta$ generates a bijection*

(i) *between the class of all essentially finite cardinal-valued functions on $\mathcal{P}$ and the class of isomorphism classes of non-degenerate $\widehat{\otimes}$-projective Hilbert modules over A;*

(ii) *between the class of arbitrary cardinal-valued functions on $\mathcal{P}$ and the class of isomorphism classes of non-degenerate $\overset{h}{\otimes}$-projective (or, equivalently, $\overset{o}{\otimes}$-projective) Hilbert modules over A.*

Following the line of our most transparent illustrations, opened by Theorem 2 and Exercise 1, you are invited to do

Exercise 4.3 Let A be as in the counter-example above. Taking Theorem 8 for granted, show that any $\widetilde{\otimes}$-projective Hilbert A-module has, up to an isomorphism, the form mH, where m is a uniquely determined cardinality. This cardinality must be finite in the "classical" case and can be arbitrary in the "quantum" case.

Remark Observe, in the context of this exercise, that the A-modules $\aleph_0 H$ and $\aleph H$ are not topologically isomorphic, but they are certainly algebraically isomorphic. Thus we see that the variety of types of isomorphisms, considered in the previous theorem, cannot include (contrary to what was said in Theorem 3) the pure algebraic isomorphism of modules.

As you have noticed, all the time our modules were assumed to be non-degenerate. However, we did not loose generality. Degenerate modules do not create new problems:

Exercise 4.4 Let H be an arbitrary Hilbert module over a C^*-algebra A, H_e be its non-degenerate submodule. Show that

(i) if A is unital, then H is $\widetilde{\otimes}$-projective $\iff$ H_e is $\widetilde{\otimes}$-projective

(ii) if A is not unital and H is $\widetilde{\otimes}$-projective, then H is non-degenerate (that is $H = H_e$).

5 Wedderburn algebras and spatially projective C^*-algebras

From now on, we concentrate on the outstanding particular case of Hilbert modules. This is the case of spatial modules over operator C^*-algebras. Theorems 6 and 7, being applied to these modules, lead to the complete description of spatially projective C^*-algebras in both of contexts, the classical one and the quantum one. In fact, this description, the future Theorem 3, is equivalent to the combination of Theorems 6 an 7: each of them can be deduced from another one (see [16] [13]). The argument has a somewhat technical character. We shall omit it, restricting ourselves with the following hint about the whole business.

Exercise 5.1 Let A be an operator C^*-algebra, and H be the spatial A-module. Further, let p be an elementary projection in A, and x be a vector in the image of p. Show that the set $H_{p,x} := \{a(x); a \in A\}$ is an irreducible closed submodule of H, isomorphic to I_p. Then show that, if A has a sufficient set of characters, the converse is true: every irreducible submodule of H has the form $H_{p,x}$ for some p and x as above.

Instead of going to details of the proofs, we shall say more about the connections of the spatial projectivity of operator algebras with some old questions of the structure theory of operator algebras.

As we shall see, the conditions of the spatial projectivity of an operator algebra will turn out to be equivalent (or closely related) to the existence of what could be reasonably called the Wedderburn structure of this algebra, on the lines of the classical Wedderburn structure theorems for semisimple finite-dimensional operator algebras. We mean the celebrated result, established in the beginning of 20th century. In the modern language it can be formulated as follows.

Wedderburn Theorem *Let A be a non-degenerate algebra of operators acting on a finite-dimensional linear space H. Suppose that A is semi-simple. Then there are decompositions $H = \bigoplus\{H'_m; m = 1, \dots, n\}$ and, for any m, $H'_m = H_m \otimes K_m$, such that A consists of all operators* $\mathbf{a}$ *with the following properties:*

(i) *H'_m for $m = 1, \dots, n$ are invariant subspaces of* $\mathbf{a}$*;*

(ii) *the restriction of* $\mathbf{a}$ *on every H'_m with $m = 1, \dots, n$ has the form $\mathbf{b} \otimes \mathbf{1}$ where $\mathbf{b}$ is an operator acting on H_m, and $\mathbf{1}$ is the identity operator on K_m.*

In the more traditional language of matrices the described structure obviously means the following. Operators in A are depicted, with respect to some basis in H, by diagonal block matrices, and each of these "big" blocks is, in its turn, a scalar block matrix. In other words, an operator in A has the block matrix of the form

$$\begin{pmatrix}
a & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots \\
0 & a & \dots & 0 & 0 & \dots & 0 & 0 & \dots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \dots & \dots \\
0 & 0 & \dots & a & 0 & \dots & 0 & 0 & \dots \\
0 & 0 & \dots & 0 & b & \dots & 0 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \dots \\
0 & 0 & \dots & 0 & 0 & \dots & b & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & c & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

(Here the sizes of the "small" blocs $a, b, c, \ldots$ are respectively $\dim H_1, \dim H_2, \ldots$, that is the linear dimensions of left tensor factors. On the other hand, the numbers of identical "small" blocks inside the "big" blocs are respectively $\dim K_1, \dim K_2, \ldots$, the dimensions of of right tensor factors.)

With the establishment of functional analysis some people, and notably von Neumann began to be interested in possible functional-analytic (i.e. infinite-dimensional) generalization of this theorem. We recall that this interest was one of the main impulses that led von Neumann to discover what is now called "von Neumann algebras"(cf, e.g., [27]). The most desirable structure of algebras that could be thought as a "right" infinite-dimensional analogue of the classical Wedderburn structure, could be described in the following terms.

Let H be an arbitrary Hilbert space. Suppose that H is represented as a Hilbert sum $H = \dot{\bigoplus}\{H'_\nu : \nu \in \Lambda\}$, and every H'_ν, in its turn, is represented as a Hilbert tensor product $H'_\nu = H_\nu \dot{\otimes} K_\nu$. In this situation we shall say, for brevity, that a *double composition* (H_ν, K_ν) for $\nu \in \Lambda$ of H is given. Further, we shall say that the indicated double decomposition is *essentially finite*[9], if, for any ν, at least one of Hilbert dimensions of the spaces H_ν and K_ν is finite.

Let (H_ν, K_ν) for $\nu \in \Lambda$ be a double decomposition of a Hilbert space H, and, for some $\nu \in \Lambda$, A_ν, A_ν be an operator algebra on H_ν. We shall denote by $A_\nu \dot{\otimes} \mathbf{1}$ the algebra of all operators, acting as $a \dot{\otimes} \mathbf{1}$ with $a \in A_\nu$ on $H_\nu \dot{\otimes} K_\nu$ and as zero on $H \ominus (H_\nu \dot{\otimes} K_\nu)$. Further, the symbol $\bigoplus_\infty$ will denote the direct ($= l_\infty$-) sum of operator algebras.

Definition 5.1 An operator algebra A on H is called a *Wedderburn (operator) algebra*, if it has the form $\bigoplus_\infty\{\mathcal{B}(H_\nu) \dot{\bigotimes} \mathbf{1}; \nu \in \Lambda\}$ for some double decomposition (H_ν, K_ν) for $\nu \in \Lambda$ of H. (In other words, A consists of all bounded operators, acting as $a \dot{\otimes} \mathbf{1}$ with $a \in \mathcal{B}(H_\nu)$ on $H_\nu \dot{\otimes} K_\nu$.)

Sometimes we shall use the expression "the Wedderburn algebra, associated with such-and-such double decomposition"; its meaning is clear.

In matrix language this means that a Wedderburn algebra A consists of all operators depicted, exactly as in the picture above, by diagonal block-matrices such that every "big" block is a scalar block-matrix. However, now it is done with respect to some *orthonormal* basis in H, and, of course, we must speak about bounded operators. Apart from this, now the number of "big" blocks, the sizes of "small" blocks $a, b, c, \ldots$, and the numbers of identical "small" blocks, situated in the main diagonals of the "big" blocks, are arbitrary cardinalities.

The condition of the essential finiteness of a given double composition evidently means for the respective Wedderburn algebra the following thing. Any of "big" blocks of matrices

[9]This new "essential finiteness" is intimately connected with the property, of the same name, of cardinal-valued functions in the previous section. One can guess about it, comparing Theorem 4.8(i) with the future Theorem 3.

of operators, belonging to these algebras, must have at least one of the following two properties: either this block, as a block-matrix, has a finite size, or the "small" blocks, belonging to that "big" one, are matrices of a finite size. (The number of "big" blocks is, of course, not restricted and can be any cardinality.)

Obviously, any Wedderburn algebra is a von Neumann algebra with a discrete (= isomorphic to $l_\infty(\cdot)$) center. It is known that at the beginning of 30s von Neumann supposed that the converse is also true. But in 1935, jointly with Murray, he discovered that even a von Neumann algebra with the scalar center, that is a factor, is not bound to be a Wedderburn algebra. Now, in retrospective, we know that the respective counter-examples, the celebrated "continuous" factors, represent one of major discoveries of 20th century mathematics.

What kind of additional conditions, imposed on a von Neumann algebra, distinguish Wedderburn algebras? A venerable algebraic tradition suggests to look for such conditions in the realm of homology. And indeed, one can prove:

Theorem 5.1 (i) *The class of all Wedderburn algebras coincides with the class of spatially $\overset{h}{\otimes}$-projective as well as with the class of spatially $\overset{o}{\otimes}$-projective von Neumann algebras*

(ii) *the class of Wedderburn algebras, associated with essentially finite double decompositions, coincides with the class of spatially $\widehat{\otimes}$-projective von Neumann algebras.*

The original proof of the part (ii) of this theorem see [12]. (This was the earliest result concerning homological properties of Hilbert modules.)

We see, in particular, that the class of quantum spatially projective von Neumann algebras is wider, than class of traditionally spatially projective von Neumann algebras. To display the simplest illustration, consider an infinite-dimensional Hilbert space, say H_0. Let A be what is called the algebra $\mathcal{B}(H_0)$ in the standard form. This means that we set $H := H_0 \dot{\otimes} H_0$ and take, in the capacity of A, the algebra $\mathcal{B}(H_0)\dot{\bigotimes}\mathbf{1}$ (that is, the Wedderburn algebra, associated with the double decomposition, consisting of only one pair (H_0, H_0)). Since the condition of the essential finiteness is obviously not fulfilled, our algebra is spatially projective in the quantum, but not in the traditional homology theory.

But what happens in the larger class of arbitrary (generally speaking, not weak-operator closed) operator C^*-algebras? To formulate the relevant criterion, let us introduce a new term.

Definition 5.2 We say that an operator algebra A on H (so far arbitrary) is *coherent with a double composition* (H_ν, K_ν) *for* $\nu \in \Lambda$ of that space, if, for any $\nu \in \Lambda$, there exists an operator algebra A_ν on $H_n u$ with the following properties:

(i) A is a subalgebra of $\bigoplus_\infty \{A_\nu \dot{\bigotimes} \mathbf{1}; \nu \in \Lambda\}$

(ii) for any $\nu \in \Lambda$, the algebra A_ν contains at least one column of rank 1 operators.

Again, taking a proper orthonormal basis in H, we see that operators in A are depicted by matrices, consisting of "big" and "small" blocks as above. (The difference is that now A is not bound to contain all such operators.) As to the meaning of the additional condition (ii), we can choose our orthonormal basis in such a way that the following happens. Taking matrices of operators in A and looking at any of their "small" blocks, one can find in this block all possible matrices with zero entries outside the first column.

Fix a double decomposition (H_ν, K_ν) for $\nu \in \Lambda$ of H. Evidently, among all operator algebras that are coherent with this decomposition, there is the largest one, and this is the respective Wedderburn algebra. Besides, it is equally clear that among these algebras there is no smallest.

However, such a smallest algebra exists, if we shall restrict ourselves with the consideration of C^*- (i.e. self-adjoint) operator algebras. Indeed, if A is a C^*-algebra, the same is true for all A_ν with $\nu \in \Lambda$. Since any A_ν contains a column of rank 1 operator, then, being self-adjoint, it contains all operators of finite rank and hence, being uniformly closed, it contains all compact operators. Thus $A \supseteq \mathcal{K}(H_\nu)$. But A is also uniformly closed, this time in $\mathcal{B}(H)$. Therefore A contains the algebra $\bigoplus_0 \left\{\mathcal{K}(H_\nu)\dot{\bigotimes}\mathbf{1}; \nu \in \Lambda\right\}$, where $\bigoplus_0$ is the symbol for the restricted ($= c_0$-) sum of operator algebras. Consequently, $\bigoplus_0 \left\{\mathcal{K}(H_\nu)\dot{\bigotimes}\mathbf{1}; \nu \in \Lambda\right\}$ is the smallest C^*-algebra, coherent to the given double decomposition of our Hilbert space.[10]

Now we are able, at last, to formulate the promised "spatial" version of Theorems 4.6 and 4.7.

Theorem 5.2 *Let A be an operator C^*-algebra, acting on a Hilbert space H. Then:*

(i) *A is spatially $\overset{h}{\otimes}$-projective $\Longleftrightarrow$ A is spatially $\overset{o}{\otimes}$-projective $\Longleftrightarrow$ A is coherent to some double decomposition of H.*

(ii) *A is spatially $\widehat{\otimes}$-projective $\Longleftrightarrow$ A is coherent to some essentially finite double decomposition of H.*

In other words, the spatial projectivity of A in any of two quantum homological theories means exactly that, for some double decomposition (H_ν, K_ν) for $\nu \in \Lambda$ of H we have the inclusions

$$\bigoplus_0 \left\{\mathcal{K}(H_\nu)\dot{\bigotimes}\mathbf{1}; \nu \in \Lambda\right\} \subseteq A \subseteq \bigoplus_\infty \left\{\mathcal{B}(H_\nu)\dot{\bigotimes}\mathbf{1}; \nu \in \Lambda\right\}.$$

[10] If the algebras $\bigoplus_\infty \left\{A_\nu)\dot{\bigotimes}\mathbf{1}; \nu \in \Lambda\right\}$ deserve to be called "Wedderburn" among von Neumann algebras, the algebras $\bigoplus_0 \left\{\mathcal{K}(H_\nu)\dot{\bigotimes}\mathbf{1}; \nu \in \Lambda\right\}$ could reasonably claim the same honorary title among *general* operator C^*-algebras.

At the same time, the spatial projectivity of A in the traditional homological theory means exactly that the same inclusions hold with respect to some *essentially finite* double decomposition of the given Hilbert space.

What can happen, if we pass to arbitrary, not necessarily self-adjoint, operator algebras? In the framework of the quantum homological theory, the "$\Longleftarrow$" part of the previous criterion still holds:

Proposition 5.1 [16] *An arbitrary operator algebra on H, coherent to some double decomposition of H, is spatially $\overset{h}{\otimes}$- and $\overset{o}{\otimes}$-projective.*

However, in the traditional theory the respective sufficient condition of the spatial projectivity is no longer valid. In fact, there is no guarantee, even if we assume that all tensor factors, participating in a given double decomposition, have finite dimensions. Consider, for example, the Hilbert space

$$(\mathbb{C}\dot{\otimes}\mathbb{C})\dot{\oplus}(\mathbb{C}^2\dot{\otimes}\mathbb{C}^2)\dot{\oplus}(\mathbb{C}^3\dot{\otimes}\mathbb{C}^3)\dot{\oplus}\cdots$$

As it was shown in [16, pp. 22–24], *there exists an operator algebra, coherent to the relevant double decomposition, that is not spatially $\widetilde{\otimes}$-projective.*

The complete description of spatially $\widehat{\otimes}$-projective operator algebras, coherent to double decompositions of Hilbert spaces, was obtained by M. E. Polyakov [20].

Thus we see that a Hilbert module over an operator algebra can be at the same time projective in quantum homological theories and non-projective in the classical homological theory ("after forgetting about the quantization"). But the phenomena of the opposite sense also happen: there are Banach modules that are projective in the traditional context, but they fail to have this property after some quantization of these modules and their basic algebras. We shall conclude our notes with the example of that kind, due to O. Yu. Aristov (still unpublished). It relies heavily on a following construction of an $\overset{h}{\otimes}$-algebra, given by D. Blecher and C. le Merdy [3].

Fix an infinite-dimensional Hilbert space H and consider the set $\mathcal{N}(H)$ of nuclear (= trace-class) operators on H. We recall that $\mathcal{N}(H)$ is a Banach algebra with respect to the so-called nuclear norm, and that, as a Banach space, it is isometrically isomorphic to the space $H \mathbin{\widehat{\otimes}} \overline{H}$, where $\overline{H}$ denotes the complex conjugate space of H. The respective isomorphism (going back to von Neumann) is well defined by taking every rank 1 operator $x \bigcirc y$ with $x, y \in H$ to $x \otimes y$. Blecher and Merdy discovered that there exists a quantization of $\mathcal{N}(H)$ such that this algebra becomes an $\overset{h}{\otimes}$-algebra. In the light of Blecher's theorem, mentioned in our introductory section, this assertion is equivalent to the following (rather surprising!) statement: there exists a topological isomorphism of the Banach algebra $\mathcal{N}(H)$ onto some *uniformly closed* operator algebra on some Hilbert space. (And this is although the original nuclear norm in $\mathcal{N}(H)$ is by no means equivalent to the operator norm.)

Here is their construction. Consider H, and also $\overline{H}$ as a quantized spaces, but this time not with respect to the column quantization, as we are accustomed to, but with the maximal quantization (discussed in Section 2). Take the Haagerup tensor product $\max H \overset{h}{\otimes} \max \overline{H}$. It turns out that the Banach space in the first floor of this quantized Banach space coincides, up to an isometric isomorphism, with $\mathcal{N}(H)$.

Let us explain, why it is so. To begin with, we have the contracting operator $j_2 \circ j_1 : H \widehat{\otimes} \overline{H} \to \max H \overset{h}{\otimes} \max \overline{H}$ (cf. diagram (3) in Section 0). Further, we consider the quantum Banach space $H_r \overset{h}{\otimes} \overline{H}_c$. Here and later $\overline{H}_c$ denotes the Hilbert space $\overline{H}$, equipped by our habitual column quantization space whereas H_r denotes the so-called row Hilbertian space. (We recall that the latter is defined as H, quantized by the isometric operator $H \to \mathcal{B}(\overline{H} : x \mapsto e \bigcirc x$, where e is an arbitrary fixed vector in $\overline{H}$). It is well known (and it not difficult to prove) that there is a contractive operator from $H_r \overset{h}{\otimes} \overline{H}_c$ to $\mathcal{N}(H)$, taking an elementary tensor $x \otimes y$ to the rank 1 operator $x \bigcirc y$ (cf. [6]). Combining all this information, we obtain the chain of contractive operators

$$\mathcal{N}(H) \longrightarrow H \widehat{\otimes} H \longrightarrow \max H \overset{h}{\otimes} \max \overline{H} \longrightarrow H_r \overset{h}{\otimes} \overline{H}_c \longrightarrow \mathcal{N}(H)$$

where the third arrow depicts the operator, identical on elementary tensors, and other arrows depict operators discussed above. Since the composition of all four operators is obviously the identity operator on $\mathcal{N}(H)$, the composition of two first operators is the desired isometric isomorphism. Note that this map, as well as its classical prototype that participated in its construction, identify rank 1 operators with elementary tensors.

Thus we can equip the space $\mathcal{N}(H)$ with the quantization, induced from the quantum Banach space $\max H \overset{h}{\otimes} \max \overline{H}$. What is essential is that the resulting quantum Banach space is an $\overset{h}{\otimes}$-algebra, in other words, the multiplication in $\mathcal{N}(H)$ is a multiplicatively bounded bioperator. By Theorem 0, this, in its turn, exactly means the existence of the respective product operator, that is of a completely bounded linear operator $\pi : \mathcal{N}(H) \overset{h}{\otimes} \mathcal{N}(H) \to \mathcal{N}(H)$, uniquely determined by taking elementary tensors $a \otimes b$ for $a, b \in \mathcal{N}(H)$ to ab.

To produce such an operator, we consider at first the inner product bilinear functional $\mathcal{F} : \max \overline{H} \times H_c \to \mathbb{C}, (y, x) \mapsto \langle x, y \rangle$. Show that it is multiplicatively contractive. Assign to every $x \in H$ the operator $\hat{x} : \mathbb{C} \to H, \lambda \mapsto \lambda x$, and to every $y \in \overline{H}$ the operator $\tilde{y} : H \to \mathbb{C}, z \mapsto \langle z, y \rangle$. Then, identifying $\mathbb{C}$ with $\mathcal{B}(\mathbb{C})$, we see that $\mathcal{F}(y, x)$ is not other thing than the composition $\tilde{y} \circ \hat{x}$. Now fix $n \in \mathbb{N}$, and consider matrices $\mathbf{y} = (y_{ij}) \in M_n(\max \overline{H})$ and $\mathbf{x} = (x_{ij}) \in M_n(H_c)$. Then, for the n-th multiplicative amplification $\mathcal{F}_n$ of $\mathcal{F}$, we see that $\mathcal{F}_n(\mathbf{y}, \mathbf{x})$ is the matrix $\mathbf{z} \in M_n(\mathbb{C}^n)$ with the entries

$z_{ij} = \sum_{k=1}^{n} \langle x_{ik}, y_{kj} \rangle$ for $1 \le i,j \le n$. Observe that this is exactly the matrix of the operator composition $\tilde{\mathbf{y}} \circ \hat{\mathbf{x}}$, where $\hat{\mathbf{x}} : \mathbb{C}^n \to \mathcal{B}(nH)$ is the operator, depicted by the matrix with the entries $\hat{x}_{ij}$, and $\tilde{\mathbf{y}} : \mathcal{B}(nH) \to \mathbb{C}^n$ is the operator, depicted by the matrix with the entries $\tilde{y}_{ij}$. Consequently $\|\mathcal{F}_n(\mathbf{y}, \mathbf{x})\| \le \|\tilde{\mathbf{y}}\| \|\hat{\mathbf{x}}\|$. But $\|\hat{\mathbf{x}}\|$, by the definition of the column Hilbertian space, is the norm of $\mathbf{x}$ whereas $\|\tilde{\mathbf{y}}\|$ is the norm of $\mathbf{y}$ in the n-th floor of the recently discussed row Hilbertian space $\overline{H}$. Needless to say, we have $\|\tilde{\mathbf{y}}\| \le \|\mathbf{y}\|_{\max}$, where the symbol $\|\cdot\|_{\max}$ denotes the norm in the n-th floor of the maximal quantization of the respective space. Therefore $\|\mathcal{F}_n(\mathbf{y}, \mathbf{x})\| \le \|\mathbf{y}\|_{\max} \|\mathbf{x}\|$.

We see that $\mathcal{F}$ is indeed multiplicatively contractive, and consequently it gives rise to the associated completely bounded linear functional $F : \max \overline{H} \overset{h}{\otimes} H_c \to \mathbb{C}$. We shall need this functional a little bit later. Right now we shall use the composition of F with the contractive operator $J : \max \overline{H} \overset{h}{\otimes} \max H \to \max \overline{H} \overset{h}{\otimes} H_c$, identical on elementary tensors. Denote, for brevity, $F \circ J$ by G. Of course, $G : \max \overline{H} \overset{h}{\otimes} \max H \to \mathbb{C}$ is also a completely contractive functional, that acts on elementary tensors exactly as F does.

Now consider the chain of operators

$$\begin{aligned} &(\max H \overset{h}{\otimes} \max \overline{H}) \overset{h}{\otimes} (\max H \overset{h}{\otimes} \max \overline{H}) \\ &\quad \longrightarrow \max H \overset{h}{\otimes} (\max \overline{H} \overset{h}{\otimes} \max H) \overset{h}{\otimes} \max \overline{H} \\ &\quad \xrightarrow{\mathbf{1} \overset{h}{\otimes} G \overset{h}{\otimes} \mathbf{1}} \max H \overset{h}{\otimes} \mathbb{C} \overset{h}{\otimes} \max \overline{H} \longrightarrow \max H \overset{h}{\otimes} \max \overline{H}. \end{aligned}$$

Here the first arrow depicts the completely isometric isomorphism, provided by the associativity of the Haagerup tensor product, and the last one depicts the identification, determined by $x \otimes \lambda \otimes y \mapsto \lambda(x \otimes y)$. Their composition, after the indicated identification $\max H \overset{h}{\otimes} \max \overline{H}$ with $\mathcal{N}(H)$, provides a completely contractive operator $\pi : \mathcal{N}(H) \overset{h}{\otimes} \mathcal{N}(H) \to \mathcal{N}(H)$, uniquely determined by taking elementary tensors $(x_1 \bigcirc y_1) \otimes (x_2 \bigcirc y_2)$ to $\langle x_2, y_1 \rangle x_1 \bigcirc y_2$, that is to $(x_1 \bigcirc y_1)(x_2 \bigcirc y_2)$. Since the span of operators of rank 1 is dense in $\mathcal{N}(H)$, we see that π is indeed the desired product operator.

Thus, the algebra $\mathcal{N}(H)$ is made an $\overset{h}{\otimes}$-algebra by the recipe of Blecher/le Merdy. Now Aristov enters. At first he observes that the spatial $\mathcal{N}(H)$-module H, equipped this time with our customary column quantization, is a $\mathcal{N}(H)$-$\overset{h}{\otimes}$-module. The argument is as follows. By Theorem 0, it is sufficient to show that the outer product operator $\pi_H : \mathcal{N}(H) \overset{h}{\otimes} H_c \to H_c, a \otimes x \mapsto a(x)$ with $a \in \mathcal{N}(H)$ and $x \in H$ is well defined and completely bounded. Because of the identification of $\mathcal{N}(H)$ with $\max H \overset{h}{\otimes} \max \overline{H}$, this is the same thing as to produce a completely bounded operator from $(\max H \overset{h}{\otimes} \max \overline{H}) \overset{h}{\otimes} H_c$ to H_c, taking elementary tensors of the form $(x \otimes y) \otimes z$ for $x, y, z \in H$ to $(x \bigcirc y)(z)$, that is to $\langle z, y\rangle x$. But look at the composition of the operators in the chain

$$(\max H \overset{h}{\otimes} \max \overline{H}) \overset{h}{\otimes} H_c$$
$$\longrightarrow \max H \overset{h}{\otimes} (\max \overline{H} \overset{h}{\otimes} H_c) \xrightarrow{\mathbf{1}\overset{h}{\otimes}F} \max H \overset{h}{\otimes} \mathbb{C} \longrightarrow \max H \xrightarrow{\mathbf{1}} H_c,$$

where the first operator is provided by the associativity of our tensor product, the third one is the standard identification, and the functional F is the same as before. Obviously, this composition is exactly the desired operator.

Note that the spatial $\mathcal{N}(H)$-module H is certainly projective as a Banach module, i.e. $\widehat{\otimes}$-projective; this follows directly from Theorem 3.1. And nevertheless:

Proposition 5.2 (Aristov) *Consider $\mathcal{N}(H)$ as an $\overset{h}{\otimes}$-algebra with respect to the quantization, induced from $\max H \overset{h}{\otimes} \max \overline{H}$. Then the $\mathcal{N}(H)$-$\overset{h}{\otimes}$-module H_c is not $\overset{h}{\otimes}$-projective (and hence, by Proposition 1.7, it is not $\overset{o}{\otimes}$-projective as well).*

$\triangleleft$ Assume, on the contrary, that H is $\overset{h}{\otimes}$-projective. Then, by Theorem 1.3, there exists a morphism $\rho : H_c \to \mathcal{N}(H) \overset{h}{\otimes} H_c$ in $\mathcal{N}(H)$-$\overset{h}{\otimes}$-**mod**, right inverse to the outer product morphism for H and therefore injective.

Fix an arbitrary non-zero $x \in H$; then $u := \rho(x)$ is also not zero. Consequently, by Proposition 4.3 (with $\mathcal{N}(H)$ as E) there exists $e_1 \in H_c$ such that the completely bounded operator $\mathbf{1} \overset{h}{\otimes} \check{e}_1 : \mathcal{N}(H) \overset{h}{\otimes} H_c \to \mathcal{N}(H), b \otimes y \mapsto \langle y, e_1\rangle b$ where $b \in \mathcal{N}(H)$ and $u \in H$ sends u to a non-zero operator, say a. Observing the action of $\mathbf{1} \overset{h}{\otimes} \check{e}_1$ on elementary tensors, we see that it is a morphism of $\mathcal{N}(H)$-modules.

Now choose any $e_2 \in H$ with $a(e_2) \neq 0$ and take the functional $\check{e}_2 : \overline{H} \to \mathbb{C}, y \mapsto \langle e_2, y\rangle$. Consider the bounded operators $\tau : \mathcal{N}(H) \to \max H : b \mapsto b(e_2)$ and $\mathbf{1} \overset{h}{\otimes} \check{e}_2 : \max H \overset{h}{\otimes} \max \overline{H} \to \max H, z \otimes y \mapsto \langle e_2, y\rangle z$. (Here, of course, we identify $\max H$ with $\max H \overset{h}{\otimes} \mathbb{C}$.) Taking into account the action of τ on operators of rank 1 and the action of $\mathbf{1} \widetilde{\otimes} \check{e}_2$ on elementary tensors, we see that, after the identification of $\mathcal{N}(H)$ with $\max H \overset{h}{\otimes} \max \overline{H}$, these two operators coincide. Therefore, since $\mathbf{1} \overset{h}{\otimes} \check{e}_2$ is completely bounded, the same is true for τ. Note that τ is obviously a morphism of $\mathcal{N}(H)$-modules.

Finally, consider the composition $\varphi := \tau \circ (\mathbf{1} \overset{h}{\otimes} \check{e}_1) \circ \rho : H_c \to \max H$. All its factors are completely bounded operators as well as morphisms of $\mathcal{N}(H)$-modules; therefore φ has both of these properties. Further, the spatial $\mathcal{N}(H)$-module H is evidently irreducible and hence φ is, in addition, an endomorphism of an irreducible module over a Banach algebra. Consequently, by the respective version of Shur's Lemma (see, e.g., [11, Theorem VI.2.48]), φ is a multiple of $\mathbf{1}$. But the factors of φ were chosen just to provide $\varphi(x) \neq 0$. It follows that the identity operator $\mathbf{1}$, considered with the domain H_c and the range $\max H$ is, in its turn, a multiple of φ, and hence it is also completely bounded. Consequently, the column quantization of H is equivalent to the maximal quantization, and this is a knowingly false conclusion. $\triangleright$

Note that, despite, as it was mentioned, the $\overset{h}{\otimes}$-algebra $\mathcal{N}(H)$ is completely isomorphic to a (uniformly closed) operator algebra, the $\mathcal{N}(H)$-module H_c is not a *spatial* module over the latter algebra. It would be interesting to find an example of a spatially $\widehat{\otimes}$-projective, but not spatially $\overset{h}{\otimes}$-projective operator algebra. (Needless to say, such an algebra, if it exists, can not be selfadjoint.)

Epilogue. Homological dimensions

Throughout these lecture notes, the main question, concerning a given module, was whether it is projective or not. If our module is projective, we rejoice, but what if not? It turns out that non-projective modules do not form an amorphic mass of "homoilogically hopeless" objects: they have an hierarchy. There is a number (or ∞) that measures how much our module is "homologically worse" than projective. This is the so-called homological dimension; we intend to explain very briefly what it is and present several related results and problems.

The initial definitions are of a general categorical character. Let $\mathcal{K}$ be an additive category, so far arbitrary, and X be its object. We recall that a *complex over* X is a sequence in $\mathcal{K}$ of the form

$$0 \longleftarrow X \overset{d_{-1}}{\longleftarrow} P_0 \overset{d_0}{\longleftarrow} P_1 \overset{d_1}{\longleftarrow} P_2 \overset{d_2}{\longleftarrow} \cdots \qquad (\mathcal{P})$$

such that $d_{n-1}d_n = 0$ for all $n = 0, 1, 2, \ldots$. We say that a complex $(\mathcal{P})$ over X *splits* if there exist morphisms $s_{-1} : X \to P_0$ and $s_n : P_n \to P_{n+1}$ for $n = 0, 1, 2, \ldots$ in $\mathcal{K}$ such that $d_{-1} \circ s_{-1} = \mathbf{1}$ and $d_n \circ s_n + d_{n+1} \circ s_{n+1} = \mathbf{1}$ for $n = 0, 1, 2, \ldots$.

(Splittable or, as they also say, contractible complexes have, in a sense, the best structure. Their instructive equivalent definitions see, e.g. [10] or [11].)

The following notion that has a preparatory character in our presentation, is of a considerable independent value. It has important applications in the computation of principal homological characteristics of $\widetilde{\otimes}$-algebras, in particular of their cohomology groups (see *idem* and also [15], [1]).

Definition E.1 Let $(\mathcal{K}, \Box : \mathcal{K} \to \mathcal{L})$ be a pre-relative category (cf. Section 1). A complex $(\mathcal{P})$ over X is called a *resolution* of X if the complex

$$0 \longleftarrow X \overset{\Box(d_{-1})}{\longleftarrow} P_0 \overset{\Box(d_0)}{\longleftarrow} P_1 \overset{\Box(d_1)}{\longleftarrow} P_2 \overset{\Box(d_2)}{\longleftarrow} \ldots \qquad (\Box\mathcal{P})$$

in $\mathcal{L}$ splits. A resolution of X is called *projective* if all P_n for $n = 0, 1, 2, \ldots$ are projective (relative to $\Box$).

Remark The construction of a projective resolution of X is, to speak informally, a way to "express this object by means of projective objects". In the categories of Banach and quantum modules, considered in our notes, (and the same is true in a much wider class of pre-relative categories) this is equivalent to the following procedure. At first we try to represent X as a quotient module of some projective module so that the respective quotient map is an admissible morphism. Then we represent the kernel of the latter morphism, in its turn, as a quotient module of another projective module, again with an admissible quotient morphism; after this we do the same with the kernel of the latter and so on. It is not difficult to perceive that the concept of a projective resolution that we are talking about provides a convenient "synchronous" way of writing down such a process.

It is obvious that in our principal categories of Banach and quantum modules resolutions are always exact sequences in the habitual sense of linear algebra. Further, *in categories of Banach modules resolutions are those and only those complexes that are exact and, in addition, all kernels (or, equivalently, images) of all participating morphisms have, as subspaces in respective P_n, Banach complements* (prove this!). However, such an assertion is no more true for categories of quantum modules.

Exercise E.1 Let $(\mathcal{K}, \square : \mathcal{K} \to \mathcal{L})$ be a (not just pre-relative but) relative category. Show that every object in $\mathcal{K}$ possesses at least one projective resolution.

Let $\mathcal{P}$ (see above) be a resolution of an object in a pre-relative category. We say that it has the *length* n, if $P_n \neq 0$, and $P_k = 0$ for all $k > n$. If there is no such an n, we say that our resolution has the length ∞.

Definition E.2 For a given X, the length of its shortest projective resolution is called the *projective homological dimension of* X and is denoted by $\mathrm{dh}_{\mathcal{K}} X$. (If all projective resolutions of X have the infinite length, we accordingly set $\mathrm{dh}_{\mathcal{K}} X := \infty$.)

(The root of the concept of projective homological dimension is Hilbert's famous syzygy theorem, in which, speaking in modern terms, it was shown that every (abstract) module over the algebra of polynomials in n variables has a projective and even a free resolution of length at most n.)

As a matter of fact, "homological dimension" is a generic name: one can (and sometimes must) to consider the so-called injective homological dimension, the flat (or week) homological dimension etc. (cf. [15]). But since we restrict ourselves by the projective homological dimension, we shall omit the adjective "projective".

In particular, we have the definition of the homological dimension of a left $\widetilde{\otimes}$-module over an $\widetilde{\otimes}$-algebra. With a fixed "$\widetilde{\otimes}$" we shall write, for the case $\mathcal{K} = \mathbf{A}$-$\widetilde{\otimes}$-$\mathbf{mod}$, $\mathrm{dh}_A X$ instead of $\mathrm{dh}_{\mathcal{K}} X$.

It is easy to observe that, in more traditional language, $\mathrm{dh}_A X$ is the least n for which X can be represented as $P_0/(P_1 \ldots /(P_{n-1}/P_n)\ldots)$ where all modules are projective and all respective quotient maps are admissible. In particular, $\mathrm{dh}_A X = 0$ *means exactly that* X *is projective*.

As it is usually the case with key notions, that of homological dimension admits important alternate approaches (see, e.g. [10, Theorem III.5.4]). We shall formulate only one assertion of that kind. It helps considerably to compute $\mathrm{dh}_A X$ in concrete situations.

Proposition E.1 *Let* $(\mathcal{K}, \Box : \mathcal{K} \to \mathcal{L})$ *be a pre-relative category, and* $n \in \mathbb{N}$*. Then the following properties of* $X \in \mathcal{K}$ *are equivalent:*

(i) $\mathrm{dh}_A X \leq n$*;*

(ii) *if we have a resolution* $\mathcal{P}$ *of* X *(see above) of the length* n *such that* $P_0, \ldots, P_{n-1}$ *are projective, then the last non-zero object* P_n *is also projective.*

From this one can easily deduce:

Corollary E.1 *Let* A *be an* $\widetilde{\otimes}$*-algebra,* $X \in$ **A**-$\widetilde{\otimes}$-**mod**. *Then* $\mathrm{dh}_A X \leq n \iff$ *in an arbitrary projective resolution* $\mathcal{P}$ *of* X *the* A-$\widetilde{\otimes}$*-module* $\mathrm{Im}(d_{n-1})$ *(coinciding with* $\mathrm{Ker}(d_{n-2})$*) is* $\widetilde{\otimes}$*-projective.*

(In other words, the respective "shortened" resolution

$$0 \longleftarrow X \longleftarrow P_0 \; \overleftarrow{P}_1 \longleftarrow \cdots \longleftarrow P_{n-1} \xleftarrow{\mathbf{in}} \mathrm{Im}(d_{n-1}) \longleftarrow 0 \longleftarrow \cdots$$

remains projective.)

Exercise E.2 Let X be a unital $\widetilde{\otimes}$-module over a unital $\widetilde{\otimes}$-algebra A. Then the homological dimension of X as (i) of an object of **UA**-$\widetilde{\otimes}$-**mod**, and (ii) as an of an object in **A**-$\widetilde{\otimes}$-**mod** is the same quantity.

Example E.1 Let Ω be a metrizable compact topological space, $A := C(\Omega)$, $t \in \Omega$. Consider the A-$\widetilde{\otimes}$-module $\mathbb{C}_t$, defined as the complex plane with the outer multiplication $a \cdot z := a(t)z$. It is not difficult to show, with the help of Theorem 2.2, that this module has a projective resolution

$$0 \longleftarrow \mathbb{C}_t \xleftarrow{d_{-1}} A \xleftarrow{d_0} A \xleftarrow{d_1} I_t \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots,$$

where $I_t := \{a \in A : a(t) = 0\}$, d_1 is a natural embedding, and $d_0 : a \mapsto a(t)$. Besides, it is easy to verify that our module $\mathbb{C}_t$ is $\widetilde{\otimes}$-projective $\iff$ t is an isolated point in Ω. Thus we have that $\mathrm{dh}_A \mathbb{C}_t = 0$, if t is an isolated point in Ω and $\mathrm{dh}_A \mathbb{C}_t = 1$ otherwise.

Example E.2 Let Ω be a paracompact (say, metrizable) topological space, and A be the Banach algebra $C_0(\Omega)$. Consider the Banach space $C_b(\Omega)$, consisting of all continuous bounded functions on Ω and equipped by the uniform norm. Obviously, it is a Banach A-module with respect to the point-wise multiplication. Then we have

$$\mathrm{dh}_A C_b(\Omega) = 2.$$

In particular, $\mathrm{dh}_{c_0} l_\infty = 2$. The proof is based on some manipulations with the projective tensor norm, and it is omitted here; see, e.g. [10, pp. 211-212].

Example E.3 Let A be $\mathbb{C}$ with zero multiplication. Consider the A-$\widetilde{\otimes}$-module A and its projective resolution

$$0 \longleftarrow A \xleftarrow{\pi} A_+ \xleftarrow{d} A_+ \xleftarrow{d} A_+ \longleftarrow \cdots$$

where $\pi, d : \lambda e + z \mapsto \lambda z$. With the help of Proposition 1, it is easy to show that $\mathrm{dh}_A A = \infty$.

Now, starting from modules, we introduce one of principal numerical characteristics of the algebras themselves.

Definition E.3 Let A be an $\widetilde{\otimes}$-algebra. The number (or ∞) $\sup\{\mathrm{dh}_A X : X \in$ **A**-$\widetilde{\otimes}$-**mod**$\}$ is called the *left global homological dimension of* A, or just *global dimension of* A. It is denoted by $\mathrm{dg}A$.

It easily follows from Corollary 1.3 that $\mathrm{dg}A = \mathrm{dg}A_+$. In other words, the global dimension does not change after the unitization of a given algebra.

A typical problem in topological homology is *to compute global dimensions of "popular" $\widetilde{\otimes}$-algebras, serving in this or that branch of analysis*. Beginning with the first results of 1972, a considerable number of various results was accumulated. However, there remains a lot of open problems, sometimes rather old and challenging.

To begin the relevant brief discussion, we shall indicate an important class of $\widetilde{\otimes}$-algebras, the global dimension of which is obliged to be ≤ 2. (As it turned out, the number 2 plays a very conspicuous role in topological homology, being the lower value of $\mathrm{dg}A$ for "non-trivial", in a sense, classes of A. This somehow resembles the role of 1 in the pure algebraic homology.)

For this purpose, we shall the first and the last time in our presentation consider some other modules than the left ones. These are two-sided modules, or, for short, bimodules.

Let A be an $\widetilde{\otimes}$-algebra. As you can guess, A-$\widetilde{\otimes}$-*bimodule* is a $\widetilde{\otimes}$-space X, endowed by a structure of an A-bimodule in the algebraic sense such that both of the bioperators of outer multiplications are $\widetilde{\otimes}$-bounded. With the fixed A, all A-$\widetilde{\otimes}$-bimodules constitute the category, denoted by $\mathbf{A}$-$\mathbf{A}$-$\widetilde{\otimes}$-**mod**. Morphisms in that category are, by definition, maps that are morphisms of bimodules in the algebraic sense and at the same time $\widetilde{\otimes}$-bounded operators. The category $\mathbf{A}$-$\mathbf{A}$-$\widetilde{\otimes}$-**mod** is considered as pre-relative with respect to the obviously defined forgetful functor to the category $\widetilde{\otimes}$-**Ban**. Thus we can speak about its projective objects, accordingly called *projective* A-$\widetilde{\otimes}$-*bimodules*. (As a matter of fact, the category $\mathbf{A}$-$\mathbf{A}$-$\widetilde{\otimes}$-**mod** is — just as $\mathbf{A}$-$\widetilde{\otimes}$-**mod** — relative, but we do not need this now).

The first and extremely important example is the basic algebra A itself; it is an A-$\widetilde{\otimes}$-bimodule with its inner multiplication in the role of the outer multiplications. Obviously, the product operator $\pi : A \widetilde{\otimes} A \to A$ is the morphism in $\mathbf{A}$-$\mathbf{A}$-$\widetilde{\otimes}$-**mod**.

Definition E.4 An $\widetilde{\otimes}$-algebra A is called *biprojective* if the A-$\widetilde{\otimes}$-bimodule A is projective.

The following assertion, taken by some recent authors as the initial definition of biprojectivity [26], provides a standard method to check the property.

Proposition E.2 *An $\widetilde{\otimes}$-algebra A biprojective $\iff$ the product morphism $\pi : A \widetilde{\otimes} A \to A$ has a right inverse in* $\mathbf{A}$-$\mathbf{A}$-$\widetilde{\otimes}$-**mod**.

For the proof see, e.g. [10, Proposition IV.5.6] or [11, Theorem VII.1.69].

Which $\widetilde{\otimes}$-algebras we know and respect are biprojective and which are not? We shall mention some of related facts, accumulated up to the present day (see [15] and/or [1] for references):

- (cf. Theorem 2.1(ii)). *Every biprojective commutative $\widetilde{\otimes}$-algebra must have discrete Gel'fand spectrum*
- *A C^*-algebra is $\widehat{\otimes}$-biprojective $\iff$ it is $\overset{o}{\otimes}$-biprojective $\iff$ it decomposes into the c_0-sum of a family of full matrix algebras.* At the same time, *a C^*-algebra is $\overset{h}{\otimes}$-biprojective $\iff$ it decomposes into the c_0-sum of a family of algebras $\mathcal{K}(H)$ (for various H). In particular, the algebra $\mathcal{K}(H); \dim H = \infty$ is not prt- and $\overset{o}{\otimes}$-biprojective, but it is $\overset{h}{\otimes}$-biprojective.* (Pay attention to the fact that in the question of biprojectivity the two different types of quantum algebras behave differently!)
- (This is a particular case of the previous assertion; compare it with Theorem 2.1). *The $\widetilde{\otimes}$-algebra $C_0(\Omega)$, where Ω is a locally compact topological space, is biprojective $\iff \Omega$ is discrete.*
- (cf. Theorem 2.6). *The $\widetilde{\otimes}$-algebra $L_1(G)$, where G is a locally compact group, is biprojective $\iff G$ is compact.*
- (cf. Theorem 2.7). *The $\overset{o}{\otimes}$-algebra $A(G)$, where G is a locally compact group, is biprojective $\iff G$ is discrete.*

- *The $\widehat{\otimes}$-algebra $\mathcal{N}(E)$, where E is a Banach space, is biprojective $\Longleftrightarrow$ E has the approximation property of Grothendieck.*

From all this we can learn, in particular, the following lesson: for an $\widetilde{\otimes}$-algebra, to be biprojective is much more difficult than to be left projective.

Biprojective algebras have a considerable independent value; in particular, they have an interesting structure theory [28]. However, in the present context we need them because of their following application that served, by the way, as the initial stimulus of their appearance.

Theorem E.1 *Let A be a biprojective $\widetilde{\otimes}$-algebra. Then* $\mathrm{dg}A(=\mathrm{dg}A_+) \leq 2$.

The proof is based on a consideration of a special resolution of length 2, the so-called entwining resolution of a given module; see *idem*.

Trying to estimate the global dimension from below, we encounter a phenomenon that has no analogue for abstract algebras and at the same time for non-normed topological algebras, even metrizable Arens–Michael algebra. To begin with the abstract algebra $\mathbb{C}[t]$ and the topological algebra $\mathcal{O}(\mathbb{C})$, both classes contain a multitude of various semi-simple infinite-dimensional commutative algebras of the global dimension 2. However:

Theorem E.2 (Global Dimension Theorem) *Let A be a commutative $\widehat{\otimes}$- (i.e. Banach) algebra with an infinite Gel'fand spectrum, in particular, an infinite-dimensional function algebra. Then* $\mathrm{dg}A \geq 2$.

The existing proof of this theorem is, probably, longer, than the proof of any other theorem in topological homology. It combines various considerations, concerning the theory of Banach algebras, homological algebra, geometry of Banach spaces and topology. Its detailed exposition is contained in an article of S. Pott [22].

As a direct corollary of the formulated theorem, the global dimension of a function Banach algebra is either 0, when it is isomorphic to $\mathbb{C}^n$, or, otherwise, it is 2 or more. Consequently, the global dimensions of Banach function algebras have at least one "forbidden" value: 1. This provokes a natural vaguely formulated question: how much widespread is this phenomenon among "algebras for analysis"? Certainly, it does not cover all Banach algebras: for example, the algebra of 2×2 matrices with zeroes in the second row has just 1 as its global dimension. However, this algebra and other known examples of similar kind by no means belong to well-beloved classes of functional analysts: they are neither semi-simple nor commutative. In this connection, the following conjecture seems to be reasonably stated:

Conjecture 1 *Let A be a semi-simple infinite-dimensional Banach algebra (not necessarily commutative). Is it true that* $\mathrm{dg}A \geq 2$*? (and if so, 1 turns out to be a forbidden value for global dimensions in this larger class of algebras as well).*

If we know, in addition, that A is biprojective and has, as a Banach space, the approximation property, the answer is "yes" (Selivanov). The same is true for a wide class of C^*-algebras that includes all non-unital algebras and all separable postliminal, or GCR-algebras (Aristov). However, the important case of unital simple C^*-algebras, like, say, the fermion algebra, remains to be obscure.

(References for the mentioned results and for what will be said below, see, e.g. [15].)

At the present moment, we know exact values of the global dimension for quite a few $\widetilde{\otimes}$-algebras. Sometimes we get them as a direct corollary of Theorems 1 and 2, sometimes an additional argument is needed. Here is the relevant incomplete list:

- *the Banach (=$\widehat{\otimes}$-) algebras* $l_1, C_0(\Omega)$ *for infinite discrete* Ω, $L_1(G)$ *for infinite compact* $G, C^*(G)$ *for the same* G, $\mathcal{N}(H)$ *with* $\dim H = \infty$ *— all of them have* $\mathrm{dg}A = 2$.
- *for all the aforementioned Banach algebras* A *we have* $\mathrm{dg}\,\underbrace{A \widehat{\otimes} \ldots \widehat{\otimes} A}_{n\ factors} = 2n$;
- *the Banach algebras* l_p *with* $1 < p < \infty$, $L_1(G)$ *for amenable non-compact* G, $\mathcal{N}(E)$ *for* E *without the approximation property,* $C_(\Omega)$ *where* Ω *is a "huge" segment of the transfinite line, have* $\mathrm{dg}A = \infty$.

...and nobody knows a semi-simple Banach algebra A with $\mathrm{dg}A = 3$.

As to quantum algebras, we still know very little. It is easy to show that $\mathrm{dg}l_2 = 2$ for l_2 in the capacity of an $\overset{h}{\otimes}$-, as well as of $\overset{o}{\otimes}$-algebra. Most probably (although it is still not done accurately), the equality $\mathrm{dh}_{\mathcal{K}(H)}\mathcal{B}(H) = 2$ that is known many years for the Banach algebra $\mathcal{K}(H)$ with $\dim H = \infty$, is also valid for the both "quantum guises" of the same algebra. This, by virtue of Theorem 1, would immediately imply $\mathrm{dh}_{\mathcal{K}(H)}\mathcal{B}(H) = 2$ for the $\overset{h}{\otimes}$-case. However, for the $\widehat{\otimes}$- and $\overset{o}{\otimes}$-cases it gives only the estimate $\mathrm{dh}_{\mathcal{K}(H)}\mathcal{B}(H) \geq 2$, and the exact value of the global dimension of $\mathcal{K}(H)$ in these two cases remains to be unknown.

Until recently, all known examples of A with $\mathrm{dg}A = 2$ were constructed from biprojective algebras by means of unitizations and direct sums. Tabaldyev has found an algebra of somewhat another kind: he proved that $\mathrm{dg}C(\Omega) = 2$, *if* Ω *is a countable compact space with* $\Omega^{(n)} = \emptyset$ *for some* $n \in \mathbb{N}$.

Shamefully, for many years *we have not known, what are the global dimensions of such immensely popular Banach algebras as* $C[0,1]$, l_∞, *the Volterra algebra* $L_1[0,1]$, $\mathcal{B}(H)$ *and (as it was already discussed)* $\mathcal{K}(H)$.

And, of course, it would be very interesting to know, *whether the global dimension theorem is valid, if we shall replace in its formulation Banach algebras by quantum algebras.*

We conclude our notes with the formulation of of problem that is also, at least in the context of Banach algebras, rather old.

Problem Let A be an $\widetilde{\otimes}$-algebra such that all left A-$\widetilde{\otimes}$-modules are projective (in other words, $\mathrm{dg}A = 0$). Does it imply that A is classically simple (that is, the direct sum of a finite family of full matrix algebras?

(The converse is well known to be true, and it is not difficult to prove.)

The affirmative answer is obtained under some additional condition on A, for example, if it is a commutative algebra, C^*-algebra or a semi-simple algebra with the approximation property. We want emphasize, however, in the zoo of non-normed topological algebras, the animals with $\mathrm{dg}A = 0$ abound: already such an algebra as $\mathbb{C}^M$, where M is a set of an arbitrary cardinality, has this property.

Remark As it is mentioned in [17, Remark 6.15], R. Smith has obtained a result that seems to be very near to answer the problem under discussion in the class of $\overset{h}{\otimes}$-algebra. The thing is that there is another important numerical characteristic of $\widetilde{\otimes}$-algebras, the so-called *bidimension* or *cohomological dimension* $\mathrm{db}A$. This is defined as the homological dimension of the unitization A_+ as of an A-$\widetilde{\otimes}$-bimodule (and also, equivalently, in terms of cohomology groups of A; cf. [15]). We have always $\mathrm{db}A \geq \mathrm{dg}A$ (and, by contrast with pure algebra, we do not know whether there exist examples of the strict inequality). What happens, if we replace, in the problem that was just discussed, $\mathrm{dg}A$ by $\mathrm{db}A$? The resulting problem, in the context of Banach algebras, is also well known. What Smith has proved (in equivalent terms) is as follows: *any $\overset{h}{\otimes}$-algebra A of the cohomological dimension 0 is classically semi-simple*.

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