

PIMS TALK, VANCOUVER - 2013

- joint work with Tom
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All measures will be shift-invariant probability measures.

A Markov random field (MRF) is a probability measure on $A^{\mathbb{Z}^d}$ such that for all $A, B \subset \mathbb{Z}^d$ finite satisfying $\partial A \subset B \subset A^c$, $a \in A^A$, $b \in A^B$

$$\mu([a]_A | [b]_B) = \mu([a]_A | [b]_{\partial A})$$

The collection of \leftarrow \rightarrow outer boundary. these objects is called a specification. (need not correspond to a measure).

Question: Given $\text{supp}(\mu)$ can there exist a finite description of the specification for any MRF with support μ ?

A nearest neighbour interaction is $V: A^{\square, \square} \rightarrow \mathbb{R}$ a function

A Gibbs state is an MRF with μ such that
$$\mu([x]_A | [c]_{\partial A}) = \sum_{c \in \square, \square} V(x|c)$$
 for all

topological support \leftarrow $\mathbb{Z}^d | \partial A$
 $x \in \text{supp}(\mu)$

Hammersley - Clifford Theorem: If $\text{supp}(\mu)$ has a safe symbol then μ is an MRF $\iff \mu$ is Gibbs with some nearest neighbour interaction

(say) This does not hold without
 safe symbol. There constructions
 by Mussowis in \mathbb{Z}^d finite graphs.
 μ is a probability shift-invariant measure
 Note: $\Rightarrow \text{supp}(\mu)$ is a shift

Note: μ is a MRF $\Rightarrow \text{supp}(\mu)$ is a shift space

(say) Further is true. It is a topological Markov field.

X.n.n. SFT.

$$\Delta_x = \{(x, y) \mid x \text{ and } y \text{ differ at finitely many sites}\}$$

(say) let us reparametrise the specifications

Markov Cocycles X-TMR
 $c: \Delta_x \rightarrow \mathbb{R}$

such that

- $c(x, y) = c(x, z) + c(z, y)$
- $c(x, y)$ is a function of $x|_{F \cup \partial F}, y|_{F \cup \partial F}$ where $F = \{i \mid x(i) \neq y(i)\}$.
- shift invariant

Gibbs Cocycle: Markov Cocycle + there exists (Schmidt & Peterseim)

interaction V such that

$$c(x, y) = \sum_{C \in \mathbb{D}} V(x|_C) - V(y|_C)$$



(3)

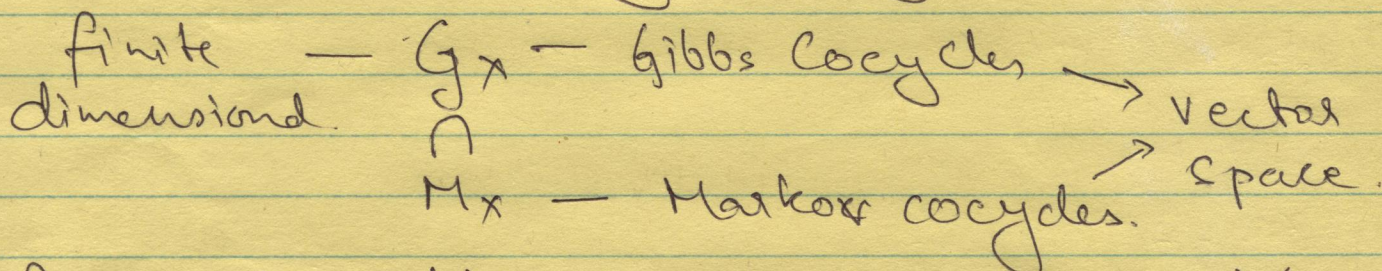
1-1 correspondence between Markov Cocycles and specifications. If μ is an MRF

$$c(x,y) = \log \frac{\mu([x]_F | [x]_{\partial F})}{\mu([y]_F | [y]_{\partial F})} \quad \text{where}$$

x, y differ at F is

a Markov Cocycle.

Say (Random Nika dym Cocycles)



finite description corresponds to M_X

Stronger version of Hammersley Clifford
 X has a safe symbol. Then $M_X = G_X$.

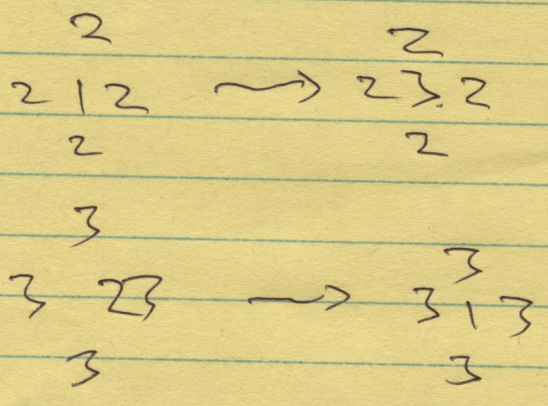
Pivot property X has pivot property

if for all $(x,y) \in \Delta_X$ there exists

$x = x_1, x_2, x_3, \dots, x_n = y$ such that
 x_i, x_{i+1} differ at a single site

Note: Then $c(x,y) = c(x, \sum_{i=1}^{n-1} c(x_i, x_{i+1}))$
 $- M_X$ finite dim

Main Examples X - 3 coloured checkerboard



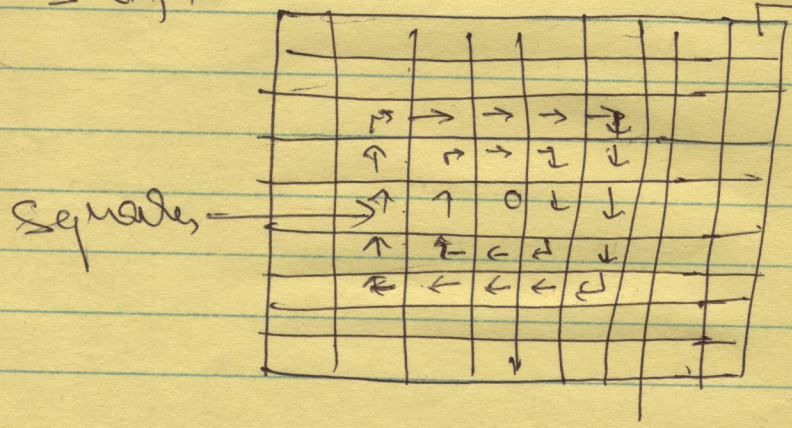
$\dim(M_X) = 3$

$\dim(G_X) = 2$

But every Markov random field is Gibbs.

Is the pivot property important?

* Close cousin of checkerboard island shift



- X: Allowed blocks
- all $2 \times 1, 1 \times 2$ blocks.
 - topologically transitive
 - Square of various sizes
 - and two distinct colours
 - $\dim(M_X) = \infty$