Markov diagrams for some non-Markovian systems

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Background

- Hofbauer (1979) used Markov diagrams to determine maximal measures of piecewise monotonic increasing transformations on the interval.
- In 1997 Buzzi extended Hofbauer's construction to arbitrary smooth interval maps, and to any subshift in 2010.

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Objective:

1. Describe the construction of the Buzzi Markov diagrams of Sturmian systems.
2. Discuss some properties of the constructed diagrams.

Notation

Let $\mathcal{A}$ be a finite *alphabet*. The *full $\mathcal{A}$-shift* is the collection of all bi-infinite sequences of symbols from $\mathcal{A}$. If $\mathcal{A}$ has n elements

$$\Sigma(\mathcal{A}) = \Sigma_n = \mathcal{A}^{\mathbb{Z}} = \{x = (x_i)_{i \in \mathbb{Z}} : x_i \in \mathcal{A} \text{ for all } i \in \mathbb{Z}\}.$$

The *one-sided full $\mathcal{A}$-shift* is the collection of all infinite sequences of symbols from $\mathcal{A}$ and is denoted

$$\Sigma(\mathcal{A})^+ = \Sigma_n^+ = \mathcal{A}^{\mathbb{N}} = \{x = (x_i)_{i \in \mathbb{N}} : x_i \in \mathcal{A} \text{ for all } i \in \mathbb{N}\}.$$

The *shift transformation* is $\sigma : \Sigma(\mathcal{A}) \to \Sigma(\mathcal{A})$ and $\Sigma^+(\mathcal{A}) \to \Sigma^+(\mathcal{A})$ defined by

$$(\sigma x)_i = x_{i+1} \quad \text{for all } i.$$

The pair (Σ_n, σ) is called the *n-shift dynamical system*.

A *subshift* is a pair (X, σ) (or (X^+, σ)), where $X \subset \Sigma_n$ (or $X^+ \subset \Sigma_n^+$) is a nonempty, closed, shift-invariant set.

Let X be a subset of a full shift, and let $\mathcal{L}_n(X)$ denote the set of all n-blocks that occur in points in X. The *language of X* is the collection

$$\mathcal{L}(X) = \bigcup_{n=0}^{\infty} \mathcal{L}_n(X).$$

Definitions

Let $\mathcal{A}$ be a finite alphabet with $X^+ \subset \mathcal{A}^{\mathbb{N}}$ a one-sided subshift.

Natural extension

The *natural extension* of X^+ is

$$\tilde{X} = \{x \in \mathcal{A}^{\mathbb{Z}} : \text{ for all } p \in \mathbb{Z} \;\; x_p x_{p+1} \ldots \in X^+\}.$$

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- Let $a^{(n)} = a_0^{(n)} a_1^{(n)} a_2^{(n)} a_3^{(n)} ...$ be points in X^+.
- Define $b^{(n)} = 0^\infty . a^{(n)}$.
- Set $x_n(a^{(n)}) = \sigma^n b^{(n)}$.
- $(x_n(a^{(n)}))$ is a sequence of two-sided sequences.

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Proposition

Let X^+ and $(x_n(a^{(n)}))$ be as described. Then $\tilde{X}$ is the set of limit points of all $(x_n(a^{(n)}))$, $a^{(n)} \in X^+$ for all $n \geq 0$.

Corollary

$\mathcal{L}(X^+) = \mathcal{L}(\tilde{X})$ if and only if for every block B in $\mathcal{L}(X^+)$ and for all $n \geq 0$ there exists $a^{(n)} \in X^+$ such that B appears in $a^{(n)}$ starting at position n. In particular, if X^+ is minimal, then $\mathcal{L}(X^+) = \mathcal{L}(\tilde{X})$.

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Follower set

The *follower set* of a block $a_{-n}a_{-n+1}...a_0$ is

$$\{b_0b_1... \in X^+ : \text{ there exists } b \in \tilde{X} \;\; b_{-n}...b_0 = a_{-n}...a_0\},$$

denoted $\text{fol}(a_{-n}a_{-n+1}...a_0)$

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Remark

- This defines a "block-to-ray" follower set.

Significant block

A *significant block* of $\tilde{X}$ is $a_{-n}a_{-n+1}...a_0$ such that

$$\mathrm{fol}(a_{-n}a_{-n+1}...a_0) \subsetneq \mathrm{fol}(a_{-n+1}a_{-n+2}...a_0).$$

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Significant form

The *significant form* of $a_{-n}a_{-n+1}...a_0$ is

$$\text{sig}(a_{-n}...a_0) = a_{-k}...a_0$$

where $k \leq n$ is maximum such that $a_{-k}...a_0$ is significant.

The Buzzi Markov diagram

Buzzi Markov diagram (1997)

The *Buzzi Markov diagram* $\mathcal{D}$ of a subshift X is the oriented graph whose vertices $V_{\mathcal{D}}$ are the significant blocks of $\tilde{X}$ and whose arrows are defined by

$$a_{-n}...a_0 \to b_{-m}...b_0 \iff b_{-m}...b_0 = \mathrm{sig}(a_{-n}...a_0 b_0)$$

and $a_{-n}...a_0 b_0$ is in the language of $\tilde{X}$.

Sturmian sequence

Complexity function

Let u be a sequence or bisequence. The *complexity function* of u, denoted p_u, maps n to the number of blocks of length n that appear in u.

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Sturmian

A sequence u is called *Sturmian* if it satisfies the following equivalent conditions:

1. u has complexity $p_u(n) = n + 1$ (Coven and Hedlund, 1973).
2. u is an irrational rotational sequence (Hedlund and Morse, 1940).
3. u is balanced and aperiodic.

Sturmian systems

Sturmian system

Let u be a Sturmian sequence. Let X_u^+ be the closure of $\{\sigma^n(u) : n \in \mathbb{N}\}$. Then (X_u^+, σ) is the dynamical system associated with the Sturmian sequence u.

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Remark

- Sturmian systems are minimal, so $\mathcal{L}(X_u^+) = \mathcal{L}(\tilde{X}_u)$.

Properties of Sturmian systems

Left special block

Let u be a Sturmian sequence u. The unique block of length n that can be extended to the left in two different ways is called a *left special block*, denoted $L_n(u)$.

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Right special block

The unique block of length n that can be extended to the right in two different ways is called a *right special block*, and is denoted $R_n(u)$. The block $R_n(u)$ is precisely the reverse of $L_n(u)$

Buzzi Markov diagram of a Sturmian system

Theorem

Let X_u^+ be a one-sided Sturmian system, with $l = l_1 l_2 l_3 \ldots$ the left special sequence of X_u^+. The Buzzi Markov Diagram of X_u^+ is the directed graph with vertices 0,1, $0L_n$, and $1L_n$, $n \geq 1$, and whose arrows are defined by

1. $0 \to 1$, $0 \to 00$, and $1 \to 10$ if $l_1 = 0$, and
   $1 \to 0$, $1 \to 11$, and $0 \to 01$ if $l_1 = 1$,
2. $0L_n \to 0L_{n+1}$, $1L_n \to 1L_{n+1}$,
3. If xL_n and wL_m, $n \geq m$, are consecutive right special blocks
   - $xL_n \to wL_{m+1}$ if $x \neq w$
   - $xL_n \to \mathrm{sig}(wL_m y)$, $y \neq l_{m+1}$, if $x = w$.

Buzzi Markov diagram of a Sturmian system

Example

The *Fibonacci sequence*, $f = 010010100100101001\ldots$, is the fixed point of the *Fibonacci substitution*

$$\begin{aligned}\phi : 0 &\mapsto 01 \\ 1 &\mapsto 0.\end{aligned}$$

- The Fibonacci sequence is Sturmian.
- The left special sequence of X_f^+ is f.

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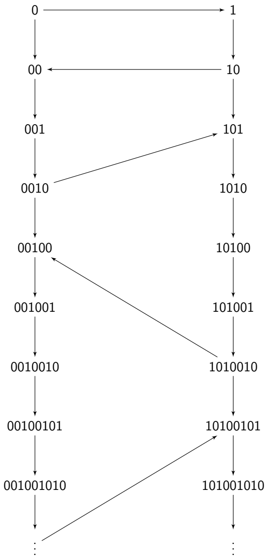
Significant blocks

0, 1, 00, **10**, 001, 101, **0010**, 1010, 00100, 10100, 001001, 101001, ...

The Buzzi Markov diagram of X_f^+



Paths on the Buzzi Markov diagram

Markov shift

Given a Buzzi Markov diagram $\mathcal{D}$ of a subshift X the corresponding *Markov shift* is

$$\hat{X} = \{\alpha \in V_{\mathcal{D}}^{\mathbb{Z}} : \text{ for all } p \in \mathbb{Z} \;\; \alpha_p \to \alpha_{p+1} \text{ on } \mathcal{D}\}.$$

Natural Projection

Let $\hat{\pi}$ denote the natural continuous projection defined by

$$\hat{\pi} : \alpha \in \hat{X} \mapsto a \in \tilde{X}$$

with a_n the last symbol of the block α_n for all $n \in \mathbb{Z}$.

Eventually Markov

A sequence $a \in \tilde{X}$ is *eventually Markov* at time $p \in \mathbb{Z}$ if there exists $N = N(x, p)$ such that for all $n \geq N$

$$\mathrm{fol}(a_{p-n} \ldots a_p) = \mathrm{fol}(a_{p-N} \ldots a_p).$$

The *eventually Markov part* $\tilde{X}_M \subset \tilde{X}$ is the set of $a \in \tilde{X}$ which are eventually Markov at all times $p \in \mathbb{Z}$.

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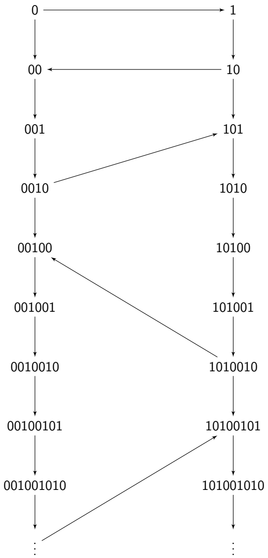
Theorem (Hofbauer, 1979; Buzzi, 2010)

The natural projection $\hat{\pi}$ from $\hat{X}$ to the subshift $\tilde{X}$ defined by

$$\hat{\pi} : \alpha \in \hat{X} \mapsto a \in \tilde{X}$$

with a_n the last symbol of the block α_n for all $n \in \mathbb{Z}$, is well defined and is a Borel isomorphism from $\hat{X}$ to $\tilde{X}_M$.

Buzzi Markov diagram of X_f^+



Observe: $\hat{X}_f$ is the empty set!

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Proposition

Let $\tilde{X}$ be the natural extension of a one-sided subshift X^+. If there exists a point $x \in \tilde{X}$ that is eventually Markov at any time $p \in \mathbb{Z}$, then there exists a periodic point in X^+.

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Corollary

If X^+ is an infinite minimal subshift, then the eventually Markov part of $\tilde{X}$ is empty.

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Corollary

If X^+ is an infinite minimal subshift, then the eventually Markov part of $\tilde{X}$ is empty.

Consequence

- If X^+ is infinite minimal then the isomorphism $\hat{\pi} : \hat{X} \to \tilde{X}_M$ is a map from the empty set to the empty set.

Paths starting with a block of length one

One-sided Markov shift

Given a Markov diagram $\mathcal{D}$ of a subshift X the corresponding *one-sided Markov shift* is

$$\hat{X}^+ = \{\alpha \in V_{\mathcal{D}}^{\mathbb{N}} : \text{ for all } p \in \mathbb{N} \;\; \alpha_p \to \alpha_{p+1} \text{ on } \mathcal{D} \text{ and } |\alpha_0| = 1\}.$$

Projection

Let $\hat{\pi}^+$ denote the continuous projection defined by

$$\hat{\pi}^+ : \alpha \in \hat{X}^+ \mapsto a \in X^+$$

with a_n the last symbol of the block α_n for all $n \in \mathbb{N}$.

Another isomorphism

Theorem

Let X^+ be a one-sided subshift such that $\mathcal{L}(X^+) = \mathcal{L}(\tilde{X})$. Then the map

$$\hat{\pi}^+ : \hat{X}^+ \to X^+$$

is a bi-continuous isomorphism.

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Remarks

- If $\mathcal{L}(X^+) = \mathcal{L}(\tilde{X})$, then given a block $B \in \mathcal{L}(X^+)$ there exists a finite path on $\mathcal{D}$ starting with a block of length one that projects to B.
- All paths leading into the same vertex have the same "futures."

Questions

- What else can these diagrams can tell us about the structures of such systems?
  - Can invariant measures be represented?
  - Can we detect unique ergodicity or minimality?
- Are the vertex labelings on a Buzzi Markov diagram unique up to a permutation of symbols?
- How does the Buzzi Markov diagram of a β-shift relate to the Buzzi Markov diagram of one of its factors.
  - Given a factor of a β-shift that is not a β-shift, can we construct its Buzzi Markov diagram and use it to find its unique measure of maximal entropy?

Thank you!