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Self-Interacting Walk and Functional Integration

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a series of lectures by
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Abstract

These lectures are directed at analysts who are interested in learning some of the standard tools of theoretical physics, including functional integrals, the Feynman expansion, supersymmetry and the Renormalization Group. The lectures are centered on the problem of determining the asymptotics of the end-to-end distance of a self-avoiding walk on a D-dimensional simple cubic lattice as the number of steps grows. When $D = 4$, the end-to-end distance has been conjectured to grow as Const. $n^{1/2} \log^{1/8} n$, where n is the number of steps. We include a theorem, obtained in joint work with John Imbrie, that validates the $D = 4$ conjecture in the simplified setting known as the "Hierarchical Lattice".

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1 Self Avoiding Walk and Differential Forms

Self-avoiding walk can be defined in more than one way. A single self-avoiding walk is usually a sequence of n *distinct* nearest neighbor sites beginning with the origin in an infinite D-dimensional lattice such as $\mathbb{Z}^D$, but there are different ways to put a probability measure on the space of all such self-avoiding walks. To begin with, in these lectures, one should have in mind the case where n is fixed and each self-avoiding walk has equal probability. What is the typical end-to-end distance of such walks when n is very large?

This question is a canonical member of a family of problems that are simple to state but difficult to answer. One has a chain of objects in space with some forces or constraints and one wants to learn about the typical configurations. Our colleagues in biology and physical chemistry, who are trying to predict the conformation of proteins, are contending with the big cousins in this family, but as mathematicians we have the right and even an obligation to consider simplest cases first. However, in low dimensions, such as two and three, this is

already a considerable challenge in need of truly new ideas. In higher dimensions it becomes easier because the constraint of no self-intersection has less effect. In these lectures we concentrate on four dimensions, make further simplifications, which include using a rather artificial lattice, and finally reach a tractable problem. We shall use the term "self repelling" to signal a simplification in which walks with self-intersections have reduced probability, as opposed to zero probability.

Our basic strategy has come from the work of theoretical physicists, especially Luttinger, who emphasized an isomorphism between self-interacting walks and some very special (supersymmetric) classes of perturbations of Gaussian random fields. The analysis of perturbations of Gaussian processes is one of the achievements of the Glimm-Jaffe school of Constructive Quantum Field Theory. By this isomorphism those methods become relevant for this family of problems. Our goal in these notes is to describe this program and to oppose a commonly held opinion that it is impossible to do analysis with the functional integrals that appear in theoretical physics.

We begin with a review of continuous time simple random walk on a lattice such as $\mathbb{Z}^D$. Any such walk defines a collection of local times τ_x — the times spent at lattice sites x. Provided the walk is confined to a finite subset of the lattice there is an isomorphism relating these local times to finite dimensional integrals involving differential forms $\phi\bar{\phi} + d\phi d\bar{\phi}$. In principle self-repelling walk problems can be solved by approximately evaluating such integrals, provided the approximation is uniform in the dimension of the integral.

The need for uniformity arises because the restriction of the walk to a finite lattice will affect the statistics of the walk unless the size of the finite lattice is large compared with the typical size of the walk. The dimension of the integral is (twice) the number of sites in the finite lattice.

Laplace discovered an approximate method for evaluating integrals which has a wonderful stability as the number of dimensions of the integral is increased. This is because it applies to integrals which are approximately Gaussian and Gaussian integration has a natural extension to spaces of infinite dimension. There is an asymptotic expansion to all orders whose leading term is the Laplace approximation. The famous Feynman graphs are an algorithm that generates the formulas for the coefficients in this expansion.

Such an expansion is not uniformly asymptotic for the integrals that arise from self-avoiding walk, but if it is applied in a different way guided by the *Renormalization Group* then it is expected to be uniformly asymptotic. The second lecture describes this idea in a general way without reference to the self-repelling walk problem.

In the third lecture we return to the self-repelling walk problem but it is greatly simplified by changing the topology of the lattice to the *hierarchical* lattice. The fourth lecture is a brief introduction to the corresponding analysis on the simple cubic lattice.

1.1 Self-Repelling Walk

Fix an integer $T \geq 0$. A *self-avoiding walk* with T steps is a sequence

$$X = X_0, X_1, \ldots, X_T$$

of *distinct* nearest neighbor sites on the simple cubic lattice $\mathbb{Z}^D$, starting with $X_0 = 0$. We define an expectation $\langle \cdot \rangle_T$ using normalized counting measure on the space of all X. The

mean squared end-to-end distance is $\langle X_T^2 \rangle$. In five or more dimensions, Hara and Slade [HS92] have proved that

$$\langle X_T^2 \rangle \sim \text{ const } T. \tag{1.1}$$

Here $f(T) \sim g(T)$ means that $f(T)/g(T) \to 1$ as $T \to \infty$. The linear growth in T would also hold without the self-avoiding constraint. The effect of the self-avoiding constraint is only to change the constant. However for four dimensions it is conjectured in [BLGZJ73] that there are logarithmic corrections

$$\langle X_T^2 \rangle \sim \text{ const } T \ln^{1/4} T. \tag{1.2}$$

The lower dimension makes it more likely that the walk will find its past and be affected by it and this is reflected in a logarithmic increase in the size of the walk. To emphasize how little this problem is understood we mention that there is not yet a general proof that $\langle X_T^2 \rangle$ exceeds the corresponding expectation for simple random walk (without the self-avoiding constraint).

We will be concentrating on four dimensions, but the reader may be interested to learn that in two dimensions there is a conjecture that the exponent is exactly $8/5$; in three there is no precise conjecture but numerical experiments suggest that there is an exponent and it is approximately 1.23. The book [MS93] is a good reference on self-avoiding walk.

The conjecture (1.2) is based on arguments that predict the *same* logarithmic correction for walks that are allowed to intersect themselves but are punished by a factor $\exp(-\lambda) < 1$ for every pair of times s, t such that $X_s = X_t$[1]. This new problem with parameter λ is called the *Domb-Joyce* model. The Domb Joyce model is easier to analyze because one can assume λ is small. We will pass to this new problem but with one further change, namely walks will be parametrized by a continuous time instead of discrete time. This makes it possible to use the isomorphism between local times and Gaussian processes which will be described later.

Finite State Markov Process. Consider a finite subset of lattice points labeled

$$\Lambda = \{0, 1, \ldots, N\}.$$

Let a, b, i, j be any points in Λ. X_t is the position in Λ at time $t \in [0, T]$. a is the starting point for the walk, $X_0 = a$. But, now t is a continuous time. The position X_t is a random variable whose law is determined as follows: given that $X_t = j$ at time t, the probability that it is at a different position k an infinitesimally short time dt later is $-A_{jk}dt$, where A is a matrix which is non-positive off-diagonal and positive on the diagonal. By working with a general matrix A we avoid making any assumption on the topology of the lattice. We include the outcome that there is no change in position by writing

$$P\{X_{t+\,dt} = k | X_t = j\} = \delta_{jk} - A_{jk}\, dt, \tag{1.3}$$

where δ_{jk} is the Kronecker delta. From this differential law one finds that the probability of transition from a to b in time T is given by the exponential of A:

$$P_a\{X_T = b\} = (e^{-TA})_{a,b}. \tag{1.4}$$

[1] As $\lambda \to 0$ the time of onset of the log correction will become larger and larger in such a way as to restore the linear growth law in the limit.

The probabilities add to one if $\sum_k A_{jk} = 0$, but we will assume instead that $\delta := \sum_k A_{jk} \geq 0$. Then there is some missing probability which is interpreted as the process X_t exiting the state space Λ to go to an additional state called the *graveyard* where it remains forever. Thus in time dt there is a probability of δdt of passing to the graveyard. This is called *killing on first exit*. A is called the generator of the process X_t.

We define the *local time* spent at j by

$$\tau_j = \int_0^T \delta_{jX_s}\, ds.$$

Examples of functions of τ that will interest us are

$$\sum_j V(j)\tau_j = \int_0^T V(X_s)\, ds, \tag{1.5}$$

$$I(\tau) = \sum_j \tau_j \tau_j = \iint_0^T \delta_{X_s X_t}\, ds\, dt.$$

The function $I = I(\tau)$ measures the time the process spends intersecting itself up to time T and so the factor $\exp(-\lambda I(\tau))$ is the continuous time analogue of "punishing the random walk by $\exp(-\lambda)$" for each self intersection, as mentioned above in the context of the Domb-Joyce model.

Continuous Time Self-Repelling Walk. Let $\mathbb{E}_a^{[0,T]}$ be the expectation for the Markov process we have defined. We will be studying

$$\mathbb{E}_a^{[0,T]}\left(e^{-\lambda I}\right) \text{ and } \mathbb{E}_a^{[0,T]}\left(e^{-\lambda I} X_T^2\right).$$

The ratio

$$\langle X_T^2 \rangle \equiv \frac{\mathbb{E}_a^{[0,T]}\left(e^{-\lambda I} X_T^2\right)}{\mathbb{E}_a^{[0,T]}\left(e^{-\lambda I}\right)}$$

as $T \to \infty$ is the expected squared end-to-end distance of a *Continuous Time Self-Repelling Walk*. The calculation of this ratio is our ultimate goal. We will assume that λ is a small but positive number, so that the self-repulsion is weak.

One reason why this problem is hard is that the numerator and denominator of $\langle X_T^2 \rangle$ are exponential in T. This very rapidly changing exponential must be canceled very exactly before one can hope to show that the ratio is linear with logarithmic corrections.

The self-avoiding walk should be the result of taking a limit $\lambda \to \infty$, but no one has studied this limit. Taking λ large has the side effect of killing the walk while it rests at a lattice site, which will alter the relation between the number of jumps and T as $\lambda \to \infty$.

1.2 Differential Forms and Self-Normalizing integrals

In this subsection we prepare for a reformulation of the problem which will involve differential forms. Let $\phi = u + iv$ so that

$$d\phi = du + i\, dv, \qquad d\bar{\phi} = du - i\, dv.$$

We multiply forms by the wedge product (suppressed in notation). Forms anticommute:

$$du\,dv = -dv\,du$$

and $du\,du = 0$, so

$$d\phi\,d\bar{\phi} = (du + i\,dv)(du - i\,dv) = -2i\;du\,dv. \tag{1.6}$$

If F is a smooth function and ω a differential form on $\mathbb{C}^n$, we define the form $F(\omega)$ by the Taylor expansion around the degree 0 part of ω. Note that the expansion always terminates, since the degree of a form cannot exceed $2n$.

For example, if $A > 0$, and $\omega_A = \phi A\bar{\phi} + d\phi A d\bar{\phi}$, we have

$$\exp\{-\omega_A\} = \exp\{-\phi\,A\,\bar{\phi} - d\phi\,A\,d\bar{\phi}\} = \exp\{-\phi\,A\,\bar{\phi}\}(1 - d\phi\,A\,d\bar{\phi}).$$

When integrating forms we adopt the convention that only forms of the same degree as the dimension of the integration contribute. Then we have

$$\begin{aligned}\int \exp\{-\omega_A\} &= -\int \exp\{-\phi A\bar{\phi}\}\,d\phi\,A\,d\bar{\phi} \\ &= 2i\int \exp\{-A(u^2+v^2)\}A\,du\,dv \\ &= 2\pi i.\end{aligned} \tag{1.7}$$

The factor $2\pi i$ is put out of sight by setting $\tilde{A} = A/(2\pi i)$. Let $S_A = \phi\,A\,\bar{\phi} + d\phi\,\tilde{A}\,d\bar{\phi}$. Then the integral of the form $\exp(-S_A)$ equals one.

By similar arguments,

$$\int e^{-S_A}\phi\bar{\phi} = \frac{1}{A}$$

This generalizes to higher dimensions. Let

$$\phi = (\phi_1, \ldots, \phi_N) \qquad A = (A_{jk})$$

and

$$\phi A\bar{\phi} = \sum_{jk} \phi_j A_{jk}\bar{\phi}_k.$$

Assume that A has positive real part: $\mathrm{Re}\,\phi A\bar{\phi} > 0$ for $\phi \neq 0$. Then:

$$\begin{aligned}\int_{\mathbb{C}^N} e^{-S_A} &= 1 \\ \int_{\mathbb{C}^N} e^{-S_A}\phi_a\bar{\phi}_b &= C_{ab},\end{aligned} \tag{1.8}$$

where $C = A^{-1}$. We prove the first of these claims at the end of this section and explain the connection with supersymmetry.

1.3 Feynman-Kac Formula and the τ Isomorphism

Assume that A is the generator for the Markov process introduced in (1.3). Equation (1.4) can be paraphrased as

$$(e^{-TA})_{a,b} = \mathbb{E}_a^{[0,T]}\left(\delta_{bX_T}\right).$$

Suppose that V is a diagonal matrix with entries $V(i)$ on the diagonal and we want to represent $\exp(-T[A+V])$ in a similar way. This is achieved by the Feynman-Kac formula, which says

$$(e^{-T[A+V]})_{a,b} = \mathbb{E}_a^{[0,T]}\left(F(\tau)\delta_{bX_T}\right) \tag{1.9}$$

where

$$F(t_1,\ldots,t_N) = e^{-\sum V(j)t_j}.$$

A proof for Brownian motion is given in [Sim79, p49] and the same proof also applies to Markov processes with finite state spaces. In comparing the statement of the Feynman-Kac formula in the literature with ours, note that F can be rewritten in terms of an integral over time using (1.5).

If $\mathrm{Re}\,\phi A\bar{\phi} > 0$ for $\phi \neq 0$ and $V(i)$ is imaginary, we can integrate both sides from 0 to ∞ with respect to T to get

$$(A+V)^{-1}_{a,b} = \int_0^\infty dT\, \mathbb{E}_a^{[0,T]}\left(F(\tau)\delta_{bX_T}\right), \tag{1.10}$$

and this brings us to the main result of this section,

Theorem 1. *(τ isomorphism) Let $F = F(t_1,\ldots,t_N)$ be any smooth function with compact support. Then*

$$\int_0^\infty dT\, \mathbb{E}_a^{[0,T]}\left(F(\tau)\delta_{bX_T}\right) = \int e^{-S_A} F(\tau)\phi_a\bar{\phi}_b,$$

where, in the right hand side,

$$\tau_j = \phi_j\bar{\phi}_j + d\phi_j\,\frac{1}{2\pi i}\,d\bar{\phi}_j.$$

Proof. First consider the case where A has positive real part. Replacing A by $A+iV$ in $A^{-1}_{jk} = \int e^{-S_A}\phi_j\bar{\phi}_k$ gives, for V real,

$$(A+iV)^{-1}_{rs} = \int e^{-S_A} e^{-\sum iV(l)\tau_l}\phi_r\bar{\phi}_s.$$

By combining this with (1.10) we obtain a special case of the theorem in which F has the form

$$F(t) = e^{i\sum k_l t_l}.$$

To pass to the general case where F is smooth with compact support, write

$$F(t) = \int d^N k\, \hat{F}(k_1,\ldots,k_N)e^{ik\cdot t}.$$

By the positive real part assumption on A, integrals are absolutely convergent so we can interchange integrals

$$\int_0^\infty dT\, \mathbb{E}_a^{[0,T]}\left(F\delta_{bX_T}\right) = \int d^N k\, \hat{F}(k) \int_0^\infty dT\, \mathbb{E}_a^{[0,T]}\left(e^{ik\cdot\tau}\delta_{bX_T}\right)$$
$$= \int d^N k\, \hat{F}(k) \int e^{-S_A} e^{ik\cdot\tau}\phi_a\bar{\phi}_b = \int e^{-S_A} F(\tau)\phi_a\bar{\phi}_b.$$

The assumption that A has positive real part is no loss of generality because both sides of the theorem are unchanged if we replace A by $A+\kappa I$ and $F(t)$ by $F(t)\exp(\kappa\sum t_i)$. For $\kappa \gg 1$, $A+\kappa I$ has positive real part. □

The τ isomorphism was clearly contained in a 1983 paper [Lut83] by the theoretical physicist Luttinger who argued that it could be used to derive corrections to all orders in powers of $\frac{1}{T}$ to the large deviation theory of Donsker and Varadhan. In Luttinger's paper the differential forms were not yet recognized as such and instead were formal anticommuting generators of a Grassman algebra. He credits McKane [McK80] and independently Parisi-Sourlas [PS80] for the invention of anticommuting numbers in this context. LeJan [LJ87, LJ88] pointed out to me that the anticommuting numbers are differential forms. Luttinger's idea was not taken very seriously by mathematicians because some of his manipulations could not cross the cultural divide. Accordingly in [BMM91] we thought it would be useful to verify that Luttinger's idea was correct in the simplest case of finite state Markov processes (with A symmetric). There remains the very interesting open problem to take Luttinger's idea beyond this finite state case.

1.4 The Self-Repelling Walk and the τ isomorphism

Let $\beta \in \mathbb{C}$ and apply the τ isomorphism to

$$F(t) = e^{-\lambda\sum t_j^2 - \beta\sum_j t_j},$$

This is not a function of compact support but Theorem 1 is still applicable because one can enlarge the class of allowed F by taking limits. The left hand side is an integral and we can take a limit for any sequence F_n of compactly supported functions which are integrably dominated. The right hand side is a finite sum of integrals whose integrands are linear in F_n or in a derivative of F_n, because the definition of a function of a form is a finite Taylor series. Therefore we can take limits under the integrals on the right hand side provided F_n and a finite number of derivatives of F_n are appropriately bounded.

On the left hand side of the τ isomorphism we find

$$G(\beta, b) = \int_0^\infty dT\, e^{-\beta T}\, \mathbb{E}_a^{[0,T]}\left(e^{-\lambda I}\delta_{b,X_T}\right).$$

The program now is to study

$$\sum_b \int e^{-S}\phi_a\bar{\phi}_b \text{ and } \sum_b b^2 \int e^{-S}\phi_a\bar{\phi}_b \tag{1.11}$$

with

$$S = \phi A \bar{\phi} + \ d\phi \, \tilde{A} \, d\bar{\phi} + \sum_j (\lambda \tau_j^2 + \beta \tau_j) \tag{1.12}$$

$$\tau_j = \phi_j \bar{\phi}_j + \ d\phi_j \, \frac{1}{2\pi i} \, d\bar{\phi}_j \qquad \Rightarrow \quad \tau_j^2 = |\phi_j|^4 + 2|\phi_j|^2 d\phi_j \frac{1}{2\pi i} d\bar{\phi}_j. \tag{1.13}$$

By the τ isomorphism (1.11) is equal to $\sum_b G(\beta, b)$ and $\sum_b G(\beta, b) b^2$. From these we obtain

$$\mathbb{E}_a^{[0,T]} \left(\exp(-\lambda I) \right) \ \text{and} \ \mathbb{E}_a^{[0,T]} \left(\exp(-\lambda I) X_T^2 \right)$$

by inverting the Laplace transform in T. Recall that the desired $\langle X_T^2 \rangle$ is the ratio. All of this must be carried out uniformly in the number N of sites in Λ. The big chip on our side of the table is that S is almost Gaussian, because we assume $0 < \lambda \ll 1$.

1.5 Laplace Approximation

The discussion in this section motivates but is not directly used in the remaining arguments so we are going to concentrate on the main ideas and omit many details. The objective is to explain the Feynman expansion.

We make the inessential simplification of only writing formulas for the case ϕ real. Let

$$I(\alpha) = \int_U d^N \phi \, e^{-\alpha S(\phi)}$$

Laplace gave the following result for the asymptotic evaluation of $I(\alpha)$. Suppose $S(\phi)$ is a smooth function of real variables $\phi = (\phi_1, \ldots, \phi_N)$ which has a non-degenerate global minimum at $\phi = 0$. Let U be a bounded open set containing 0, then

$$I(\alpha) \sim e^{-\alpha S(0)} \int_{\mathbb{R}^N} d^N \phi \, e^{-\frac{1}{2} \alpha \, \phi A \phi}$$

where $A = S^{(2)}(0)$ is the matrix of second derivatives of S at 0 and $I(\alpha) \sim f(\alpha)$ means that $I(\alpha)/f(\alpha) = 1 + O(\alpha^{-1})$ as $\alpha \to \infty$.

The proof is based on the Taylor approximation

$$S(\phi) \approx S(0) + \frac{1}{2} \, \phi A \phi,$$

noting that there is no linear term because $\phi = 0$ is the minimum. This approximation is accurate near $\phi = 0$. Away from $\phi = 0$ it becomes inaccurate but this does not matter because then $\exp(-\alpha S(\phi))$ and $\exp(-\alpha S(0) - \alpha/2 \phi A \phi)$ are both small relative to $\exp(-\alpha S(0))$, in fact exponentially small in α. For the same reason, we can allow U to be unbounded or be all of $\mathbb{R}^N$ if $S(\phi)$ grows mildly as $|\phi| \to \infty$.

Much more is true: one can prove that the above asymptotic evaluation is merely a first term in a systematic asymptotic expansion in powers of α^{-1}. Let p be any positive integer. Then

$$I(\alpha) \sim e^{-\alpha S(0)} \left[\int \, d\phi \, e^{-\frac{1}{2} \alpha \, \phi A \phi} \right] \, e^{\sum_{l=1}^{p} \, c_l \, \alpha^{-l}},$$

where $\sim$ now means

$$\frac{I(\alpha)}{\text{Right Hand Side}} = 1 + O(\alpha^{-p-1}) \text{ as } \alpha \to \infty.$$

For reasons explained in the next subsection we are going to refer to this as the *Feynman* expansion, even though it predates Feynman. For more information on asymptotic expansions for integrals see [Erd55] and [Hör83, Vol II].

1.6 Graphical Formula for c_l

Feynman revolutionized calculations in Quantum Electrodynamics (QED) when he introduced graphical representations. The effect in theoretical physics was not unlike the acceleration in progress after the introduction, in the Middle Ages, of symbols, such as "x", in place of words. Later it was realized that calculations in QED are combinatorially similar to evaluation of moments of Gaussian integrals. Finite dimensional Gaussian integrals are the simplest context for Feynman's idea, but one of the many virtues of the graphical representation is that it facilitates the analysis of dependence on dimension. Results classifying dependence on dimension are called Power Counting theorems.

In this same finite dimensional context one can also obtain asymptotic expansions for the case where α is imaginary and the integral is oscillatory. But even when the coefficients are uniform in N, there is no satisfying theory for $N = \infty$ oscillatory integrals. Oscillatory integrals in finite dimensions are analyzed using integration by parts to prove that contributions from ranges of integration that do not include critical points are negligible ($O(\alpha^{-p})$ for all p). The problem is a lack of understanding of the N dependence in this procedure.

Write the exponent in $I(\alpha)$ as $-\alpha S(0) - \alpha\phi A\phi - \alpha r(\phi)$, where $A = \frac{1}{2}S''(0)$. Then

$$I(\alpha) = e^{-\alpha S(0)}\Omega\frac{1}{\Omega}\int d\phi\, e^{-\frac{\alpha}{2}\phi A\phi}e^{-\alpha r(\phi)},$$

where Ω^{-1} normalizes the Gaussian density to have total mass 1. Let $C = A^{-1}$. For any polynomial $P(\phi)$,

$$\frac{1}{\Omega}\int d\phi\, e^{-\frac{\alpha}{2}\phi A\phi}P(\phi) = e^{\frac{1}{2}\alpha^{-1}\Delta_C}P\Big|_{\phi=0},$$

where

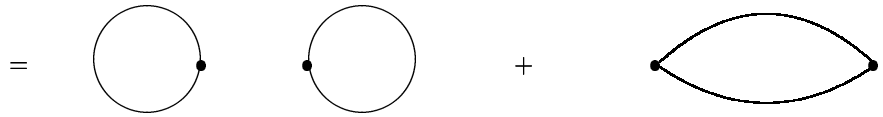
$$\Delta_C = \sum_{i,j}\frac{\partial}{\partial\phi_i}C_{ij}\frac{\partial}{\partial\phi_j}$$

and the exponential is defined by its power series, which terminates after finitely many terms when applied to the polynomial. This formula has a natural graphical representation illustrated by the following example.

Example. Set $\alpha = 1$ and $P = (1/2)(\phi_a^2/2)(\phi_b^2/2)$, then $\exp(\frac{1}{2}\Delta_C)P$ at $\phi = 0$ is

$$\frac{1}{2!}\frac{1}{2}\Delta_C\frac{1}{2}\Delta_C P = \frac{1}{8}C_{aa}\,C_{bb} + \frac{1}{4}C_{ab}\,C_{ab}$$

which is represented by



$\frac{1}{8} C_{aa}\, C_{bb}$ $\qquad\qquad$ $\frac{1}{4}\, C_{ab}\, C_{ab}$

Thus to each index a and b in the polynomial is associated a vertex and to each Laplacian is associated a line. Each graph records how the Laplacians acted according to the product rule of differentiation. If one does not put labels on the lines to record which Laplacian is associated to which line then one must instead have combinatoric coefficients because a single graph is representing more than one term from the product rule of differentiation. These coefficients can be read off from the automorphism group of the graph, see below.

Although $\exp(-\alpha r(\phi))$ is not a polynomial, if it is regarded as a formal power series in ϕ, and $\exp(\frac{1}{2}\alpha^{-1}\Delta_C)$ is applied to each term, then it is still true that

$$e^{\frac{1}{2}\alpha^{-1}\Delta_C}\left(e^{-\alpha r(\phi)}\right)\Big|_{\phi=0}$$

is a correct asymptotic expansion *in powers of* α^{-1}. As a partial justification of this claim, note that if we change variables by replacing ϕ by $\alpha^{-1/2}\phi$ in $I(\alpha)$, then, since $\alpha r(\alpha^{-1/2}\phi)$ is $\alpha O(\alpha^{-3/2}\phi^3)$, only powers of $\alpha^{-1/2}$ can appear and since $\exp(-\phi A\phi/2)$ is even in ϕ, only even powers survive.

We have

$$e^{-\alpha r(\phi)} = \sum_{n=0}^{\infty} \frac{1}{n!}(-\alpha r(\phi))^n$$

$$r(\phi)^n = \sum_{k_1,\ldots,k_n \geq 3} \frac{1}{k_1!\cdots k_n!} \prod_{j=1}^{n} S^{(k_j)}(0)\phi^{k_j}.$$

In order to evaluate

$$e^{\frac{1}{2}\alpha^{-1}\Delta_C}\left(\prod_{j=1}^{n} S^{(k_j)}\phi^{k_j}\right)\Big|_{\phi=0}$$

we expand the exponential, and observe that the only term that contributes is the one containing the right number of derivatives, that is, the one with Δ_C^m, where $2m = k_1+\cdots+k_n$. We apply the Leibnitz rule for the product over j. We can keep track of which derivative acts on which factor by drawing a graph whose vertices — labeled $\{1,\ldots,n\}$ — represent the n factors, and edges — labeled $\{1,\ldots,m\}$ — represent the Laplacians. Vertex j has degree k_j. We define the weight of a graph by letting each edge have weight $-\alpha^{-1}A^{-1}$, and a vertex of degree d have weight $\alpha S^{(d)}$. We can see that graphs with weight of order α^{-l} are the ones for which

$$\#\text{edges} - \#\text{vertices} = l.$$

The weight of a graph is independent of the labeling of vertices and edges, so we can in fact sum over unlabeled graphs, and it turns out that the combinatorial factor for an unlabeled graph G is $1/|\mathrm{Aut}(G)|$, where $\mathrm{Aut}(G)$ is the automorphism group of G.

We get an asymptotic expansion

$$\int d\phi\, e^{-\alpha S(\phi)} \sim e^{-\alpha S(0)} \Omega \sum_{l \geq 0} d_l \alpha^{-l},$$

where the coefficient d_l is a sum over weights of all graphs with $\#\text{edges} - \#\text{vertices} = l$, and each vertex having degree at least 3.

As a startling example of the way in which algebraic properties can be coded by graphs we remark that

$$\sum d_l \alpha^{-l} = \exp\left(\sum_{l \geq 1} c_l \alpha^{-l} \right),$$

where c_l is determined in exactly the same way as d_l, but summing only over *connected* graphs. The equality of the two sides is in the sense of formal power series in α^{-1}.

Studying the problem of when these expansions are asymptotic uniformly in N leads to the realm of the Glimm-Jaffe-Spencer cluster expansions. Some of the important original contributions are in [VW73, GJS74, GJS76a, GJS76b] and these are reviewed in the book [GJ87]. To a large extent these references are written for people who know or want to learn about Constructive Field Theory, but the scope of the cluster expansion is wider. Cluster expansions are a natural representation for an important class of near-Gaussian integrals so that one can analyze the dependence on N and find sufficient conditions for the Feynman expansion to be a valid asymptotic expansion when $N = \infty$.

In these papers on cluster expansions there is an emphasis on integrals whose non-Gaussian part $r(\phi)$ is *additive*. This means that it has the form

$$r(\phi) = \sum_i v(\phi_i)$$

Our self-repelling case (1.12) has this form, but it fails another important sufficient condition, which is that the spectrum of A should be a subset of the non-negative reals that is bounded away from zero uniformly in N. For the simple cubic lattice A is the lattice Laplace operator. For the infinite lattice the spectrum of this operator is not bounded away from zero because it is diagonalized by the Fourier transform to multiplication by a function of k that resembles k^2 for k small. Although these are only "sufficient" conditions, the failure of the spectral condition reflects a genuine problem which will be discussed further in Section 2.

Some more details on the Feynman expansion will be included in [Bry01]. The chapter on perturbation theory in the theoretical physics text [ID89] may be useful for more background.

1.7 Supersymmetry and Proof of (1.8)

Following ideas in [AB84] we give a proof that $\int_{\mathbb{C}^N} e^{-S_A} = 1$. There is a flow acting on $\mathbb{C}^N$ by

$$\phi_j \longmapsto e^{-2\pi i \theta} \phi_j$$

The flow is generated by a vector field X, such that $X(\phi_j) = -2\pi i\phi_j$, and $X(\bar{\phi}_j) = 2\pi i\bar{\phi}_j$. The action by pullback of the flow on forms is

$$d\phi_j \longmapsto d(e^{-2\pi i\theta}\phi_j) = e^{-2\pi i\theta}\, d\phi_j \text{ and } d\bar{\phi}_j \longmapsto e^{2\pi i\theta}\, d\bar{\phi}_j.$$

The infinitesimal flow $\mathcal{L} = \mathcal{L}_X$, the *Lie derivative*, is obtained by differentiating with respect to the flow at $\theta = 0$. E.g.,

$$\mathcal{L}\, d\phi_j = \frac{d}{d\theta} d(e^{-2\pi i\theta}\phi_j)\big|_{\theta=0} = -2\pi i\, d\phi_j.$$

A form ω is *invariant* iff $\mathcal{L}\omega = 0$. The interior product $\underline{i} = \underline{i}_X$ with the vector field X is an antiderivation that acts on forms and

$$\underline{i}\, d\phi_j = -2\pi i\phi_j \text{ and } \underline{i}\, d\bar{\phi}_j = 2\pi i\bar{\phi}_j.$$

By Cartan's formula, $\mathcal{L} = d\,\underline{i} + \underline{i}\, d$, $\mathcal{L}$ has a square root called the *supersymmetry generator*:

$$Q = d + \underline{i}.$$

We have $\mathcal{L} = (d + \underline{i})(d + \underline{i})$ because $d^2 = 0$ and $\underline{i}^{\,2} = 0$. A form ω that satisfies $Q\omega = 0$ is called supersymmetric or *closed*. For any smooth form u which decays appropriately at ∞ we have $\int Qu = 0$, because we have defined integration to project out all but the form of top degree $2N$ and the degree of $\underline{i}\, u$ is at most $2N - 1$. Furthermore $\int du = 0$ by Stoke's theorem.

Using $2\pi i\, \phi_j\bar{\phi}_k + d\phi_j\, d\bar{\phi}_k = Q\ \phi_j\, d\bar{\phi}_k$ one finds that S_A is exact and then it follows that it is closed because $Q^2 = \mathcal{L}$ and S_A is invariant. From the derivation property of Q, if ω is even and supersymmetric, so is e^{ω}.

Let u be an odd invariant form, then

$$\int e^{-S_A + tQu} = \int e^{-S_A},$$

provided the deformation preserves integrability. This is because

$$\frac{d}{dt}\int e^{-S_A+tQu} = \int e^{-S_A+tQu} Qu = \int Q\left(e^{-S_A+tQu}u\right) = 0.$$

The second step follows from the fact that $\exp(-S_A + tQu)$ is supersymmetric. Also we can differentiate the exponential with respect to t in the usual way because Qu and S_A are even and therefore commute.

We can choose u so that Qu cancels the off-diagonal parts of S_A. For $0 \le t \le 1$ this deformation preserves integrability because the exponent comes from a convex combination of A and the diagonal part of A. By setting $t = 0$ we deform away the off-diagonal part of A so that proving

$$\int e^{-S_A} = 1$$

is reduced to a product of the one complex dimensional cases which we have already established. □

There are simpler ways to prove this particular result but supersymmetry records special features of self-interacting random walk that are also important in other places in our analysis. The reader might like to prove by similar methods that

$$\int e^{-S_A} F(\tau_1, \ldots, \tau_n) = F(0, \ldots, 0) \tag{1.14}$$

provided the integral is comfortably convergent.

2 Mehler's formula and the Renormalization Group

At the end of Section 1.6 we mentioned that the principle tool for proving that the Feynman expansion is uniformly asymptotic — the cluster expansion — fails when the Hessian at the minimum has eigenvalues that creep down towards zero as the dimension of integration increases. Consider, for example, the functional of real ϕ,

$$S(\phi) = \frac{1}{2} \sum_{\substack{x,y \in \mathbb{Z}^D \\ \text{nearest neighbors}}} [\phi_x - \phi_y]^2 + \sum_{x \in \mathbb{Z}^D} \phi_x^4 \tag{2.1}$$

with the boundary condition $\phi_x = 0$ for all x outside some bounded set $\Lambda \subset \mathbb{Z}^D$. The minimum is at $\phi = 0$ and the Hessian is the finite difference Laplacian with zero boundary conditions at $\partial\Lambda$. As Λ is enlarged and the boundary $\partial\Lambda$ recedes, the configurations where ϕ_x is almost constant in x and tends slowly to zero near $\partial\Lambda$ could become less Gaussian because the quadratic part of $S(\phi)$ is almost zero on these configurations. The Feynman expansion, being based on approximation by the Gaussian part, is not uniformly asymptotic. Actions such as (2.1) which have no uniform spectral gap in the Hessian are called *massless*.

Instead of approximately evaluating the whole integral by a single application of the Feynman expansion, one holds fixed the slowly varying part of the configuration ϕ_x and applies the Feynman expansion to approximately integrate out the fluctuations on the smaller length scales. This procedure is iterated until eventually the whole integral has been evaluated. This idea was proposed by K. Wilson.

"Integrating out" corresponds in probability to the notion of conditional expectation with respect to a sigma algebra, so Wilson's idea is to calculate a conditional expectation approximately by the Feynman expansion and eventually to achieve a complete expectation by repeating this operation. He chooses the successive sigma algebras to be related by scaling so that the combined operation of conditional expectation followed by rescaling turns out to be a semigroup or at least very close to being a semigroup. The combined operation is called an *RG Transformation*, where RG stands for Renormalization Group. However it is at best only a semigroup.

In the example above the lattice breaks the scale invariance. The rescaling after a conditional expectation will map $\mathbb{Z}^D$ to a finer lattice $(\epsilon\mathbb{Z})^D$ with $\epsilon < 1$. It is easier to formulate the RG for a continuum field ϕ_x with $x \in \mathbb{R}^D$ because scaling preserves $\mathbb{R}^D$. After many applications of the RG transformation the lattice starts to look like the continuum so our emphasis on a continuum formulation is not misleading.

In the simple cases, which are called "Gaussian Fixed Points", this continuum RG is closely connected to the Hermite semigroup and Mehler's formula. Since these are familiar topics to many mathematicians we use them as the entrance to the Wilson Renormalization Group.

The results of this section are a general introduction and are not immediately applicable to the self-repelling walk problem because of the lattice. In Section 3 we return to the self-avoiding walk.

2.1 Gaussian measures

A *probability* measure μ on $\mathbb{R}^N$ is *Gaussian* if it has the form

$$\mu(d\phi) = \text{const. } e^{-\frac{1}{2}\phi A \phi}\, d\phi_1 \dots d\phi_N \tag{2.2}$$

for some real positive-definite symmetric matrix A. A *continuum* of random variables ϕ_x labeled by $x \in \mathbb{R}^D$ is said to be Gaussian if the joint probability of every finite subset of the variables is Gaussian. Such a continuum of variables ϕ_x is called a *Gaussian Random Field.* As usual the restriction to real valued ϕ is an inessential simplification.

The covariance C is defined by

$$\int \mu(d\phi)\, \phi_j \phi_k = C_{jk}.$$

For a Gaussian random field C is a function on $\mathbb{R}^D \times \mathbb{R}^D$ given by $\int \mu(d\phi)\, \phi_x \phi_y = C(x,y)$.

- The joint distribution of any finite subset $\{\phi_{x_1}, \dots, \phi_{x_n}\}$ of the continuum Gaussian random field is determined by the covariance C because the A_{ij} in (2.2) is the inverse of $C(x_i, x_j)$. Therefore we write μ_C. Any continuous positive-definite function $C(x,y)$ defines all the finite joint probabilities of some Gaussian random field.

- Gaussian measures are closed under convolution:

$$\mu_C * \mu_{C'} = \mu_{C+C'}.$$

- μ_C defines a Laplacian Δ_C on a domain of bounded $\mathcal{C}^2$ functions by

$$\mu_{tC} * f = f + \frac{t}{2}\, \Delta_C\, f + O(t^2). \tag{2.3}$$

  In other words, $\frac{1}{2}\Delta_C$ is the infinitesimal generator of the convolution semigroup $\{\mu_{tC}\}$. When f is a smooth function of only finitely many of the variables ϕ_x, say ϕ_x with $x \in \{x_1, \dots, x_N\}$, then this definition implies that Δ_C reduces to

$$\Delta_C = \sum_{j,k} \partial_j C(x_j, x_k) \partial_k.$$

  with ∂_j being the ordinary partial derivative with respect to ϕ_{x_j}. For example $\Delta_C \phi_x^4 = 12 C(x,x) \phi_x^2$.

- When f is a polynomial
$$\mu_C * f = e^{\frac{1}{2}\Delta_C} f, \tag{2.4}$$
and the right hand side can be computed by expanding the exponential in a power series, in which only finitely many terms contribute. This is called *Wick's Theorem.*

- If $C_{[0,T)}(x,y)$ with $0 < T \leq \infty$ is smooth in x, y then ϕ_x can be realized in $\mathcal{C}^p$ for any $p = 0, 1, \ldots$. See [RW00, Pages 55 – 71].

[Sim79] and [RW00, Pages 55 – 71] are useful references for random Gaussian fields.

2.2 The Hermite Semigroup

We start with the simple example $N = 1$. Consider the Gaussian measure on $\mathbb{R}$ whose (co)variance is the number defined by

$$C = C_{[a,b)} = \int_{[a,b)} e^{-2s}\, ds,$$

so that $C_{[a,b)} + C_{[b,c)} = C_{[a,c)}$. By the convolution property $\mu_{[a,b)} * \mu_{[b,c)} = \mu_{[a,c)}$. Define the scaling operator

$$\mathcal{S}_t f(\phi) = f(e^{-t}\phi).$$

Then, using $e^{-2t} C_{[a,b)} = C_{[a+t,b+t)}$, we prove that $\mu_{[a,b)} * \mathcal{S}_t = \mathcal{S}_t \mu_{[a+t,b+t)} *$ by

$$\begin{aligned}
\mu_{[a,b)} * \mathcal{S}_t f(x) &= \frac{1}{\sqrt{2\pi C_{[a,b)}}} \int d\phi\, e^{-\frac{\phi^2}{2C_{[a,b)}}} f(e^{-t}x - e^{-t}\phi) \\
&= \frac{1}{\sqrt{2\pi C_{[a+t,b+t)}}} \int d\phi\, e^{-\frac{\phi^2}{2C_{[a+t,b+t)}}} f(e^{-t}x - \phi) \\
&= \mathcal{S}_t \mu_{[a+t,b+t)} * f(x).
\end{aligned}$$

This implies that the family of operators $\mathbb{T}_t$ defined on bounded continuous functions by

$$\mathbb{T}_t f = \mathcal{S}_t \mu_{[0,t)} * f \tag{2.5}$$

is a semigroup:

$$\begin{aligned}
\mathbb{T}_s \mathbb{T}_t f &= \mathcal{S}_s \mu_{[0,s)} * \mathcal{S}_t \mu_{[0,t)} * f = \mathcal{S}_s \mathcal{S}_t \mu_{[t,s+t)} * \mu_{[0,t)} * f \\
&= \mathcal{S}_{s+t} \mu_{[0,s+t)} * f \\
&= \mathbb{T}_{s+t} f.
\end{aligned}$$

The generator L of the semigroup is defined on a domain of twice differentiable functions by differentiating $\mathbb{T}_t$ with respect to t at $t = 0$:

$$\lim_{t\to 0} \frac{1}{t}(\mathcal{S}_t \mu_{[0,t)} * f - f) = \lim_{t\to 0} \mathcal{S}_t \frac{1}{t}(\mu_{[0,t)} * f - f) + \lim_{t\to 0} \frac{1}{t}(\mathcal{S}_t f - f).$$

Since $C_{[0,t)} = t + o(t)$, the first limit is $f''(\phi)/2$, while the second one is $-\phi f'(\phi)$. Therefore

$$L = \frac{1}{2}\frac{d^2}{d\phi^2} - \phi \frac{d}{d\phi}, \tag{2.6}$$

and from the semigroup property one shows that the partial differential equation

$$\partial_t f = \left(\frac{1}{2}\frac{\partial^2}{\partial\phi^2} - \phi\frac{\partial}{\partial\phi}\right) f$$

with bounded smooth initial data f_0 is solved by

$$f = \mathcal{S}_t\mu_{[0,t)} * f_0.$$

This is half of *Mehler's Formula*. The other half is another representation for the solution f as a sum over a complete set of eigenfunctions for the operator L. These eigenfunctions are the subject of the next section.

2.3 Eigenfunctions

Let $\Delta_{[a,b)}$ be the Laplacian Δ_C with $C = C_{[a,b)}$. Define a polynomial of degree n by

$$:\phi^n: \; = e^{-\frac{1}{2}\Delta_{[0,\infty)}}\phi^n. \tag{2.7}$$

For example,

$$:\phi^4: \; = \phi^4 - 6C_{[0,\infty)}\phi^2 + 3C^2_{[0,\infty)} \tag{2.8}$$

This is called a Wick ordered monomial in physics. It is an eigenfunction,

$$\mathbb{T}_t :\phi^n: \; = e^{-nt} :\phi^n: , \tag{2.9}$$

because

$$\begin{aligned}\mathcal{S}_t\mu_{[0,t)} * :\phi^n: \; &= \mathcal{S}_t e^{\frac{1}{2}\Delta_{[0,t)}} :\phi^n: \; = \mathcal{S}_t e^{-\frac{1}{2}\Delta_{[t,\infty)}}\phi^n = e^{-\frac{1}{2}\Delta_{[0,\infty)}}\mathcal{S}_t\phi^n \\ &= e^{-nt} :\phi^n: .\end{aligned}$$

By differentiating with respect to t at 0 we find that $:\phi^n:$ is an eigenfunction of L with eigenvalue $-n$. By two integrations by parts L is a symmetric operator on a domain of smooth functions in $L^2(\mathbb{R}, \mu_{[0,\infty)})$, therefore $:\phi^n:$ are orthogonal polynomials in this space. The orthogonality and degree n identifies $:\phi^n:$ as a constant times the n^{th} Hermite polynomial. Besides being orthogonal, the Hermite polynomials are also known to be complete. Therefore one can define $\exp(-tL)$ by the spectral theorem and write an expansion for its kernel in Hermite polynomials. Thus there is an expansion for $\exp(-tL) = \mathcal{S}_t\mu_{[0,t)}*$ in terms of Hermite polynomials. This is the other half of the Mehler formula, but we will not need it for these lectures.

An important property of these polynomials is that they act like ordinary powers when differentiated:

$$\begin{aligned}\frac{d}{d\phi} :\phi^n: \; &= \frac{d}{d\phi} e^{-\Delta_{[0,\infty)}}\phi^n = e^{-\Delta_{[0,\infty)}}\frac{d}{d\phi}\phi^n = e^{-\Delta_{[0,\infty)}} n\phi^{n-1} \\ &= n :\phi^{n-1}: .\end{aligned} \tag{2.10}$$

2.4 Invariant Measure

The measure $\mu_{[0,\infty)}$ is invariant for the semigroup, i.e.

$$\int \mu_{[0,\infty)}(d\phi)\, \mathbb{T}_t F = \int \mu_{[0,\infty)}(d\phi)\, F.$$

This can be seen from the calculation

$$\begin{aligned}\int \mu_{[0,T-t)}(d\phi)\, \mathbb{T}_t F &= \mu_{[0,T-t)} * \mathbb{T}_t F\big|_{\phi=0} = \mu_{[0,T-t)} * \mathcal{S}_t \mu_{[0,t)} * F\big|_{\phi=0} \\ &= \mathcal{S}_t \mu_{[t,T)} * \mu_{[0,t)} * F\big|_{\phi=0} = \mathcal{S}_t \mu_{[0,T)} * F\big|_{\phi=0} = \mu_{[0,T)} * F\big|_{\phi=0} \\ &= \int \mu_{[0,T)}(d\phi)\, F.\end{aligned}$$

Now we let $T \to \infty$ to obtain invariance.

2.5 First Order Perturbation Theory

Applying the semigroup $\mathbb{T}_t$ gradually builds an integration with respect to the Gaussian measure $\mu_{[0,T)}$ because

$$\int \mu_{[0,T)} e^{-V} = \int \mu_{[0,T-t)} \mathbb{T}_t e^{-V} = \mathbb{T}_T e^{-V}\Big|_{\phi=0}.$$

When t increases to T, $\mu_{[0,T-t)}$ tends to a point mass measure at zero. Define the *Effective Interaction* $W = W_t$ by

$$e^{-W} = \mathbb{T}_t e^{-V}$$

Then the gradual integration becomes

$$\int \mu_{[0,T)} e^{-V} = \int \mu_{[0,T-t)} e^{-W_t}$$

In this way integration is transformed into the study of the evolution of the Effective Interaction W. Since this simple formula is the result of specializing Wilson's constructions to one dimensional integrals, we will use the term *Renormalization Group* (RG) instead of "Gradual Integration".

First Order Effective Interaction. Suppose, for example, that

$$V = \lambda : \phi^4 : .$$

Calculating to first order in V,

$$\mathbb{T}_t e^{-V} = \mathbb{T}_t(1 - V + O(V^2)) = e^{-\mathbb{T}_t V} + O(V^2).$$

We find by the eigenfunction property, that

$$W = \mathbb{T}_t V + O(V^2) = \lambda e^{-4t} : \phi^4 : + O(V^2).$$

Thus to first order in V, integration is equivalent to a flow of the coupling constant:

$$\lambda \to \lambda e^{-4t}.$$

2.6 Infinitely Many Dimensions

The semigroup $\mathbb{T}_t$, its eigenfunctions and other properties generalize to the infinite dimensional context of a Gaussian random field ϕ_x as follows: Define the action of scaling $\mathcal{S}_t$ on ϕ by

$$(\mathcal{S}_t\phi)_x = e^{-[\phi]t}\phi_{e^{-t}x}$$

where $[\phi] > 0$ is a constant, which we will call the *dimension* of ϕ. A different operator denoted by the same symbol $\mathcal{S}_t$ is *defined* on covariances by

$$\mathcal{S}_t C(x,y) = e^{-2[\phi]t} C(e^{-t}x, e^{-t}y),$$

Let $\dot{C}(x,y)$ be any smooth rapidly decaying positive-definite function of $|x-y|$. Define

$$C_{[a,b)}(x,y) = \int_{[a,b)} dt\, \mathcal{S}_t\dot{C}(x,y) = \int_{[a,b)} dt\, e^{-2[\phi]t}\dot{C}(e^{-t}x, e^{-t}y). \tag{2.11}$$

Since $C_{[a,b)}$ is an integral over scalings of a positive-definite function, it is itself positive-definite. Therefore it defines a Gaussian measure $\mu_{[a,b)}$. $\mathbb{T}_t$ continues to be defined by (2.5). At first we take its domain to be the linear subspace of bounded continuous *cylinder* functions, where a function of the random variables ϕ_x is said to be a cylinder function if it is a function of a finite subset of these variables. However one can prove that this domain is dense in $L^2(\mu_{[0,\infty)})$ and $\mathbb{T}_t$ is a contraction semigroup on this domain so it extends to all of $L^2(\mu_{[0,\infty)})$.

Other interesting facts are that $\mathbb{T}_t$ is a contraction on L^p for $1 \le p < \infty$ and it is even hypercontractive. This may be useful in future developments of the RG.

Choice of $[\phi]$. The parameter $[\phi]$ determines the long distance decay of $C_{[0,\infty)}$. Using the notation $\dot{C}(x,y) = f(|x-y|)$ and the substitution $s = e^{-t}|x-y|$ we have

$$\begin{aligned} C_{[0,\infty)}(x,y) &= \int_0^\infty e^{-2[\phi]t} f(e^{-t}|x-y|)dt \\ &= |x-y|^{-2[\phi]} \int_0^{|x-y|} s^{2[\phi]-1} f(s)ds \\ &= O(|x-y|^{-2[\phi]}) \text{ as } |x-y| \to \infty. \end{aligned}$$

This covariance is *smooth everywhere including at* $x = y$ because $\dot{C}(x,y)$ is smooth, the integral is over scalings by factors $\exp(t) \ge 1$ and the rapid decay of $\dot{C}(x,y)$ makes the integral over scalings convergent. The way to get a singularity at $x = y$ is to integrate over $t \in (-\infty, 0)$. Then one finds a short distance singularity

$$C_{[-\infty,0)}(x,y) = O(|x-y|^{-2[\phi]}), \tag{2.12}$$

as $|x-y| \to 0$. Finally consider

$$C_{(-\infty,\infty)}(x,y) = \text{const}|x-y|^{-2[\phi]}. \tag{2.13}$$

This is scale invariant, $\mathcal{S}_t C_{(-\infty,\infty)} = C_{(-\infty,\infty)}$, and singular at both long and short distances because we integrated over all scalings.

The self-repelling walk is a perturbation of a Gaussian measure whose covariance is $O(|x-y|^{2-D})$ as $x-y \to \infty$, so we will set

$$[\phi] = D/2 - 1. \tag{2.14}$$

In four dimensions $[\phi] = 1$.

2.7 First Order Perturbation Theory in Infinite Dimensions

Again, we consider the RG formula

$$\int \mu_{[0,T)} e^{-V} = \int \mu_{[0,T-t)} e^{-W_t}, \tag{2.15}$$

but now we choose

$$V = \lambda \int_\Lambda dx : \phi_x^4 : .$$

By the definition (2.7), $: \phi_x^4 :$ is given by (2.8) with $C_{[0,\infty)}(x,x)$ in place of $C_{[0,\infty)}$. Thus $: \phi_x^4 :$ is dx-integrable on any compact set Λ and so V exists.

The eigenfunction property (2.9) becomes $\mathbb{T}_t : \phi_x^4 : \; = e^{-4[\phi]t} : \phi_{e^{-t}x}^4 : .$ By a change of variables we have

$$\mathbb{T}_t V = \lambda \int_\Lambda dx\, e^{-4[\phi]t} : \phi_{e^{-t}x}^4 : \; = \lambda e^{(-4[\phi]+D)t} \int_{\mathcal{S}_t \Lambda} dx : \phi_x^4 : ,$$

giving

$$W = \lambda e^{(4-D)t} \int_{\mathcal{S}_t \Lambda} dx : \phi_x^4 : \; + O(V^2),$$

In the same way

$$\mathbb{T}_t \left(\beta \int_\Lambda dx : \phi_x^2 : \right) = \beta e^{2t} \int_{\mathcal{S}_t \Lambda} dx : \phi_x^2 :$$
$$\mathbb{T}_t \left(\gamma \int_\Lambda 1\, dx \right) = \gamma e^{Dt} \int_{\mathcal{S}_t \Lambda} 1\, dx.$$

These are not eigenfunctions of $\mathbb{T}_t$ because of the change in Λ, but they are still singled out for major parts by the non-negative exponents in t.

2.8 Effective Coupling Constants

Domain of t: The rescaling $\Lambda \to \mathcal{S}_t \Lambda$ shrinks Λ until at a time T defined by

$$\text{Volume}(\mathcal{S}_T \Lambda) = 1, \tag{2.16}$$

it becomes a unit volume. This is very useful because problems of uniformity in Λ have disappeared! Where have they gone to? They are hidden in the dificulty of controlling the action of $\mathbb{T}_t$ on the Effective Interaction for t in the long interval $[0,T]$.

Suppose

$$V = \int_\Lambda dx \, \{\lambda : \phi_x^4 : \, + \beta : \phi_x^2 : \} \tag{2.17}$$

It is determined by the parameters $\vec{\lambda} = (\Lambda, \lambda, \beta)$ so we shall write $V(\vec{\lambda})$. We have shown that upon neglecting $O(V^2)$, the effective interaction W has the same form $V(\vec{\lambda})$ but with parameters $\vec{\lambda}$ called *effective coupling constants* that depend on t according to the *flow* equations

$$\Lambda \to \mathcal{S}_t \Lambda, \qquad \begin{cases} \frac{d}{dt}\lambda = (4-D)\lambda \\ \frac{d}{dt}\beta = 2\beta \end{cases}$$

We will be interested in cases where $\lambda(t)$ remains small for $t \leq T$. We can see that this should be true for all $D > 4$. For $D = 4$ the first order flow for λ is $d\lambda/dt = 0$, but as we will see in Subsection 2.9, there is a correction from second order so that the equation becomes

$$\frac{d\lambda}{dt} = -c\lambda^2$$

with $c > 0$. This is solved by

$$\lambda = \frac{\lambda_0}{1 + c\lambda_0 t}.$$

Thus, second order perturbation theory predicts that $\lambda(t) \to 0$.

2.9 Second Order Perturbation Theory

From Mehler's formula, the evolution of the interaction is given by

$$\partial_t e^{-W} = L e^{-W}. \tag{2.19}$$

but now we should pause to explain the operator L more fully in this infinite dimensional context.

First recall from (2.6) that in one dimension L is a sum of a Laplacian and $-\phi d/d\phi$ which is the vector field generating scaling:

$$\frac{d}{dt} f(e^{-t}\phi) = -\phi \frac{d}{d\phi} f(e^{-t}\phi)$$

Recall that $\phi_x \in \mathcal{C}^k(\Lambda)$ with any k. Fix some k and let F be a smooth functional on the Banach space $\mathcal{C}^k(\Lambda)$. Then, for a fixed ϕ, the second derivative of F is a bounded bilinear function $f, g \mapsto F_2(\phi; f, g)$ on $\mathcal{C}^k(\Lambda) \times \mathcal{C}^k(\Lambda)$. From the definition of the Laplacian (2.3) one finds[2] that

$$\Delta_C F(\phi) = \int \mu_C(d\zeta) F_2(\phi; \zeta, \zeta)$$

[2] Essentially by inserting in the definition the Taylor expansion of $F(\phi + s\zeta)$ in powers of $s = 1$ to order 2. The remainder after order 2 makes no contribution provided the third derivative of F does not grow faster than a Gaussian with $\|\phi\|_{\mathcal{C}^k(\Lambda)}$.

For example

$$\frac{1}{2}\Delta_C \iint \phi_x f(x,y)\phi_y \, dx\, dy = \int \mu_C(d\zeta) \iint \zeta_x f(x,y)\zeta_y \, dx\, dy$$
$$= \iint f(x,y)C(x,y)\, dx\, dy \tag{2.20}$$

so, less precisely,

$$\frac{1}{2}\Delta_C = \iint \frac{\delta}{\delta\phi_x} C(x,y) \frac{\delta}{\delta\phi_y}\, dx\, dy$$

L is the sum of $\Delta_{\dot{C}}/2$ and the scaling part defined by

$$F \to \frac{d}{dt}\bigg|_{t=0} F(\exp(-[\phi]t)\phi_{\exp(-t)x})$$

In the one dimensional ϕ case (2.19) implies that

$$\partial_t W = LW - \frac{1}{2} W_\phi \cdot W_\phi. \tag{2.21}$$

and following our definitions this is also true for the continuum ϕ_x case with

$$W_\phi \cdot W_\phi := \int \mu_{\dot{C}}(d\zeta) W_1(\phi;\zeta) W_1(\phi;\zeta)$$

As in (2.20) the $\int \mu_{\dot{C}}(d\zeta)$ will become a factor $\dot{C}(x,y)$ that integrates against the two functional derivatives.

The corresponding integral equation is

$$W(t) = \mathbb{T}_t V - \frac{1}{2}\int_{[0,t)} \mathbb{T}_{t-s}(W_\phi(s)\cdot W_\phi(s))\, ds,$$

which can be checked using Mehler's formula. By iterating, we obtain a solution to any order in V. At second order

$$W = \mathbb{T}_t V - \frac{1}{2}\int_{[0,t)} \mathbb{T}_{t-s}\left[(\mathbb{T}_s V)_\phi \cdot (\mathbb{T}_s V)_\phi\right] ds + O(V^3).$$

In order to calculate the integral we use the trick:

$$\frac{d}{d\phi} f(\phi)g(\phi) = \left(\frac{\partial}{\partial\phi^{(1)}} + \frac{\partial}{\partial\phi^{(2)}}\right) f(\phi^{(1)})g(\phi^{(2)})\bigg|_{\phi^{(1)}=\phi^{(2)}=\phi},$$

and assuming that V is a polynomial we write the integrand as

$$\mathcal{S}_{t-s} \exp\left(\frac{1}{2}\Delta^{(1,1)}_{[0,t-s)} + \frac{1}{2}\Delta^{(2,2)}_{[0,t-s)} + \Delta^{(1,2)}_{[0,t-s)}\right) (\mathbb{T}_s V)_{\phi^{(1)}} \cdot (\mathbb{T}_s V)_{\phi^{(2)}}\Big|_{\phi^{(1)}=\phi^{(2)}=\phi}$$
$$= \mathcal{S}_{t-s} e^{\Delta^{(1,2)}_{[0,t-s)}} \left(e^{\frac{1}{2}\Delta^{(1,1)}_{[0,t-s)}}(\mathbb{T}_s V)_{\phi^{(1)}}\right) \cdot \left(e^{\frac{1}{2}\Delta^{(2,2)}_{[0,t-s)}}(\mathbb{T}_s V)_{\phi^{(2)}}\right)\Big|_{\phi^{(1)}=\phi^{(2)}=\phi}$$
$$= e^{\Delta^{(1,2)}_{[s-t,0)}} (\mathbb{T}_t V)_{\phi^{(1)}} \cdot (\mathbb{T}_t V)_{\phi^{(2)}}\Big|_{\phi^{(1)}=\phi^{(2)}=\phi}$$
$$= \sum_{j=0}^{\infty} \frac{1}{j!}\left[\Delta^{(1,2)}_{[s-t,0)}\right]^j (\mathbb{T}_t V)_{\phi^{(1)}} \cdot (\mathbb{T}_t V)_{\phi^{(2)}}\Big|_{\phi^{(1)}=\phi^{(2)}=\phi}.$$

The $\cdot$ in the last expression is with respect to $\mathcal{S}_{s-t}\dot{C}$, which comes from the fact that we interchanged $\mathbb{T}_{t-s}$ and the ϕ-derivative. The j-th term is

$$\frac{1}{j!}(\mathbb{T}_t V)_{\phi,\dots,\phi}(\mathcal{S}_{s-t}\dot{C}) \otimes C^{\otimes j}_{[s-t,0)}(\mathbb{T}_t V)_{\phi,\dots,\phi},$$

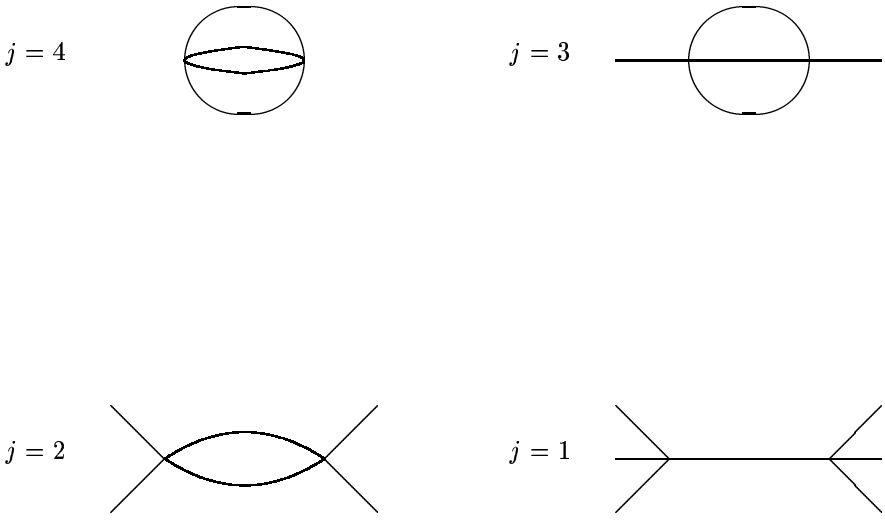
where there are $j+1$ partial ϕ-derivatives. Integrating we get the formula

$$W = \mathbb{T}_t V - \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{j!}\,(\mathbb{T}_t V)_{\phi,\dots,\phi}\, C^{\otimes j}_{[-t,0)}\,(\mathbb{T}_t V)_{\phi,\dots,\phi} + O(V^3). \tag{2.22}$$

For example, when V is the integral of $:\phi^4_x:$, $\mathbb{T}_t$ sends $:\phi^4_x:$ to $\exp(-4[\phi]t):\phi^4_{\exp(-t)x}:$ and by (2.10) the functional derivatives represented by the subscripts ϕ act by

$$:\phi^4_x: \;\mapsto 4:\phi^3_x: \;\mapsto 12:\phi^2_x: \;\text{ etc}$$

Thus there are nonvanishing terms $j = 4, 3, 2, 1$. The general structure of these terms is indicated by the following Feynman diagrams



The edges that end on one vertex but not two are called *external legs.* For example the $j = 2$ diagram has two vertices, each of which has two external legs. The two external legs at the left vertex represent a factor $:\phi^2_x:$ and the two at the right vertex represent a factor $:\phi^2_y:$. The two edges joining the vertices represent $C^2_{[-t,0)}(x-y)$. This diagram in $D = 4$ contributes to W

$$J = -36\lambda^2 \iint :\phi^2_x: C^2_{[-t,0)}(x-y):\phi^2_y:\, dx\, dy$$

where the integrals are over $\mathcal{S}_t\Lambda$.

(2.22) is an approximation to the Effective Interaction, but is this approximation well behaved for $t = T$ large? Leave aside any deplorable tendencies of effective coupling constants to grow and consider only the integral in J. As $t \to \infty$, $u(x-y) := C^2_{[-t,0)}(x-y)$ grows a nonintegrable singularity

$$u_\infty(x-y) = O\left(\frac{1}{|x-y|^{2j}}\right).$$

at $x - y = 0$ because of (2.12).

There is a marvelous remedy which *simultaneously* improves the behavior of λ at large t and removes the singularity in J: we split J into a regular and a singular part:

$$J = J_R + J_S$$

corresponding to $AB = A(B-A) + A^2$ with $A =: \phi^2_x :$ and $B =: \phi^2_y :$. Then J_R has a $t \to \infty$ limit because $|A - B| \le \sup|\nabla A||x-y|$. The divergence as $t \to \infty$ is now in

$$\begin{aligned} J_S &= -36\lambda^2 \iint (:\phi^2_x:)^2 C^2_{[-t,0)}(x-y)\,dx\,dy \\ &\approx -36\lambda^2 \left(\int C^2_{[-t,0)}(y)\,dy\right) \int (:\phi^2_x:)^2\,dx \end{aligned}$$

which is called a *local part*[3].

If we were to leave out the local part then the resulting $W_{\rm reg} = W + J_S$ would no longer be a second order solution to the flow equation (2.21). By differentiating $W_{\rm reg}$ with respect to t we find that $W_{\rm reg}$ is an $\mathcal{O}(\lambda^2)$ solution to the flow equation with the additional term

$$-36\frac{d}{dt}\left(\int C^2_{[-t,0)}(y)\,dy\right)\int (:\phi^2_x:)^2\,dx = -72a\int (:\phi^2_x:)^2\,dx \tag{2.23}$$

$$\text{with } a = \int C_{[-t,0)}(y)\dot{C}(y)\,dy \tag{2.24}$$

where *a has a limit* as $t \to \infty$. Suppose that in addition to omitting J_S we also alter the flow of λ to

$$\frac{d}{dt}\lambda = (4-D)\lambda - 72a\lambda^2, \qquad D = 4 \tag{2.25}$$

Then (2.23) is cancelled by $d\lambda/dt \int :\phi^4:$. The reader who notices that the cancellation is incomplete because $(:\phi^2_x:)^2$ is not the same as $:\phi^4_x:$ is correct: the two polynomials differ in quadratic and constant terms. However if we start with a general quartic polynomial

$$V = \int_\Lambda (\lambda :\phi^4_x: + \beta :\phi^2_x: + \gamma)\,dx$$

then the flow equations for β, γ can also be modified to as to get a complete cancelation of (2.23).

[3]We wrote $\approx$ because integrals are over $\mathcal{S}_t\Lambda$ and we do not have exact translation invariance. This is not fatal to the argument because $C_{[-t,0)}(x-y)$ decays to zero on length scale one which is much less than the diameter of $\mathcal{S}_t\Lambda$ until $t = O(T)$.

The diagrams in W with $j = 3, 4$ also diverge as $t \to \infty$, but these can also be split into regular and local parts. Again the local parts are discarded and the flow equations are altered. One finds from $j = 3$ that one more term $\zeta \int_\Lambda (\partial\phi_x)^2\, dx$ must be added to V and one has

Theorem 2. *Let*

$$V_t = \int_{\mathcal{S}_t \Lambda} \left[\lambda_t : \phi_x^4 : + \beta_t : \phi_x^2 : + \zeta_t : (\partial\phi_x)^2 : + \gamma_t \right] dx$$

Define $W_{reg,t}$ by (2.22) with $\mathbb{T}_t V$ replaced by V_t and omitting all local parts that diverge as $t \to \infty$. Then the flow equations for the effective coupling constants $\vec{\lambda} = (\lambda_t, \beta_t, \zeta_t, \gamma_t)$ can be chosen so that $W_{reg,t}$ solves (2.21) to order $\mathcal{O}(\vec{\lambda})^2$. $W_{reg,T}$ is bounded in $L^2(\mu_{[0,\infty)})$ uniformly in Λ provided the effective coupling constants $\lambda, \beta, \zeta, \gamma$ are uniformly bounded at $t = T$.

The $L^2(\mu_{[0,\infty)})$ part of Theorem 2 is obtained by using Wick' theorem (2.4) to evaluate the integral of the polynomial $W_{\rm reg}^2$.

This is only a statement of Wilson's results in the sense that no creative idea needs to be added to his discussion to prove it. A deeper result, which was his main point, but which he did not prove, is that the second order $W_{\rm reg}$ is a uniform in Λ approximation to the exact effective action if the effective coupling constants $\vec{\lambda}_T$ are small, when the effective coupling constants obey ordinary differential equations which *include corrections from all orders in* λ. Explicit expressions for these corrections to all orders are hardly to be expected but it is not necessary to have them: instead one can prove that the invariant manifolds and fixed points for the flow of $\vec{\lambda}$ are qualitatively stable stable under a class of such perturbations.

In Section 4 we will define "uniform approximation to the exact effective action". A good definition controls the error so that at $t = T$ one can prove that omitting W_T has negligible effect on correlations.

Already in the first order flow equation (2.18) we see that β is unstable — for generic choice of initial coupling constants it will not remain small. Wilson claimed that there is an invariant manifold $\beta = \beta^{\rm c}(\lambda, \zeta)$ of initial data such that β does not grow. γ is also unstable and so if it is to remain small we must also choose the initial γ dependent on (λ, ζ), but since $\exp(-\gamma \int dx)$ factors out of the integral we can choose it to be initially zero and then it evaluates the integral to second order in $\lambda_T, \beta_T, \zeta_T$ by

$$\int \mu_{[0,\infty)}(d\phi)\, e^{-V} = \int \mu_{[0,\infty)}(d\phi)\, e^{-W_{{\rm reg},T}} \approx e^{-\gamma_T}$$

because the rest of $W_{{\rm reg},T}$ is negligible.

Wilson's work was reviewed in [WK74]. These lectures and the paper [BCG+78] were very helpful to me during my first attempts to learn the general ideas. In these lectures we are only discussing the part of his work that concerns Gaussian fixed points. Theoretical physicists regard his work on non-Gaussian fixed points as of much greater significance. More recent reviews which may be useful for a view into the subsequent history and range of activity in the RG include [BG95, Fis98].

Gawedzki and Kupiainen were the first authors to prove Wilson's deeper conjectures: that for V as above and λ small there is an invariant manifold $\beta^{\rm c}(\lambda, \zeta)$ and that for β

chosen on this manifold the exact effective action flows under the RG to the Gaussian $\int(\partial\phi)^2$, in particular $\lambda \to 0$ and correlations are massless Gaussian in the scaling limit, [GK83, GK85]. The review [HT87] summarizes this and later work. Dimensions $4-\epsilon$ were studied in [BDH98b]. Other approaches to this class of problems are the phase-cell expansions [FMRS87] and wavelet expansions [Fed87].

Scaling limits of $\lambda\phi^4$ perturbations of lattice Gaussian measures have also been studied by completely different methods having nothing to do with the RG by Aizenman [Aiz82] and by Fröhlich [Frö82]. These methods are remarkable in not assuming that the initial perturbation is small, but they are less general in other ways.

There is an elegant and general theorem by Newman and Wright [NW81, NW82] that gives sufficient conditions for white noise scaling limits. White noise scaling limits are generic and arise from the instability of β. They are central limit type theorems. Intermediate between the white noise scaling limits and the results of Aizenman and Fröhlich are the results in [NS97].

3 Hierarchical Lattices and the Renormalization Group

The interaction

$$V(X) = \int_X \left[\lambda{:}\ \phi_x^4 {:}\ + \beta{:}\ \phi_x^2 {:}\ + \zeta{:}\ (\partial\phi_x)^2 {:}\ + \gamma\right] dx$$

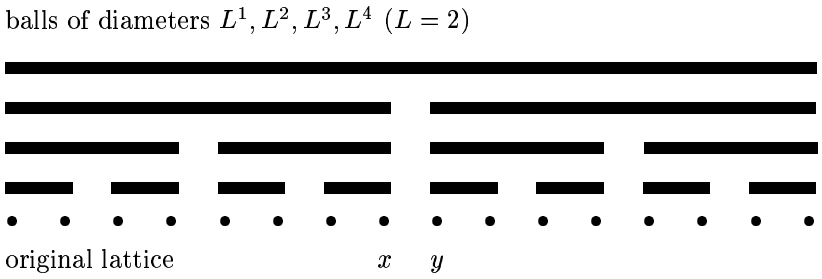
is such that $\exp(-V(X))$ is *multiplicative*,

$$\exp(-V(X \cup Y)) = \exp(-V(X))\exp(-V(Y)) \text{ for } X \cap Y = \emptyset$$

The formula (2.22) for the second order effective interaction developed in the last section shows that multiplicativity is not normally preserved by the RG. In general, one cannot control the remainder after perturbation theory without having some sort of calculus for functionals that are approximately multiplicative. We will discuss this further in Section 4. However there are special lattices called *hierarchical lattices* for which the RG preserves multiplicativity. In this section we will look at some of the main ideas in a proof that there are log corrections in the end-to-end distance of self-repelling walk on a four dimensional hierarchical lattice. The reader can refer to our papers [BIa, BIb] in places which are short on details here.

3.1 Hierarchical lattice

Fix an integer L. The picture shows the balls that define the metric. The reason why the RG will preserve multiplicativity is that no balls overlap. This is called the *ultrametric* property.



The distance between the specific points x, y in the picture is L^4 which is the diameter of the smallest ball containing them.

The dimension of this lattice is defined to be $D = 1$ because the number of points in a ball of diameter L^k is L^{kD}. We will consider a 4 dimensional lattice.

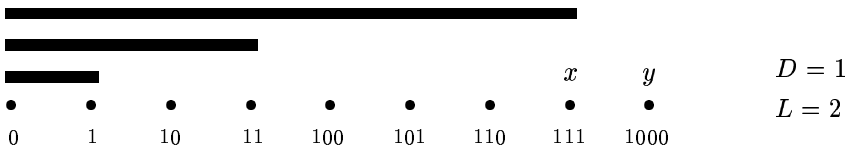
3.2 Group structure

We have defined a metric $d(x, y)$. Following [BEI92], we can make the lattice into an additive group in such a way that

$$d(x, y) = d(x - y, 0) = |x - y|.$$

Let $\mathcal{G}_1$ be the additive cyclic group of order L^D.

$$\mathcal{G}_2 \equiv \mathcal{G}_1 \oplus \mathcal{G}_1,$$
$$\mathcal{G}_3 \equiv \mathcal{G}_1 \oplus \mathcal{G}_1 \oplus \mathcal{G}_1.$$



$$x = (\ldots, 0, 1, 1, 1) \in \mathcal{G}_3,$$
$$y = (\ldots, 0, 1, 0, 0, 0) \in \mathcal{G}_4,$$
$$x - y = (\ldots, 0, 1, 1, 1, 1) \in \mathcal{G}_4.$$

$$\therefore |x - y| \equiv \inf \left\{ L^k : \mathcal{G}_k \ni x - y \right\} = L^4.$$

3.3 Scaling

Scaling is a *discrete* semigroup induced by a map L^{-1} from the infinite hierarchical lattice to itself. L^{-1} identifies all points that lie in the same ball of diameter L in the hierarchical lattice so that they become a single point in a new hierarchical lattice. Thus it is the shift

$$L^{-1}x = (\ldots, x_3, x_2, x_1), \qquad \text{for } x = (\ldots, x_3, x_2, x_1, x_0).$$

$$\therefore \mathcal{G}_k \xrightarrow{L^{-1}} \mathcal{G}_k/\mathcal{G}_1 = \mathcal{G}_{k-1}.$$

Scaling pulls a configuration $\phi_{\text{new},x}$ on the new lattice to a configuration $\phi_{\text{old},x}$ on the old lattice by making it constant on the L balls of the old lattice

$$\mathcal{S} : \phi_{\text{old},x} = L^{-[\phi]}\phi_{\text{new},L^{-1}x}$$

with $[\phi] = (D-2)/2 = 1$ in four dimensions. It acts on forms by $\mathcal{S} : d\phi_{\text{old},x} = L^{-[\phi]}d\phi_{\text{new},L^{-1}x}$. Suppose that $X = x + \mathcal{G}_1$ is some L-ball in the hierarchical lattice and F_X is a smooth form on $\mathbb{C}^X$. Then scaling induces a new form $f_x = \mathcal{S}F_X$ on $\mathbb{C}^{\{x\}}$ by

$$f_x(\phi_{\text{new},x}) = F_X(\phi_{\text{old}})$$

These definitions extend in an obvious way to include X being any finite union over of cosets $x + \mathcal{G}_1$. For example if $X = \Lambda$ so that F_X is a form on $\mathbb{C}^\Lambda$, then $\mathcal{S}F_X$ is a form on the lower dimensional $\mathbb{C}^{\Lambda/\mathcal{G}_1}$.

We *define* the action of scaling on covariances by

$$(\mathcal{S}C)(x,y) = L^{-2[\phi]}C(L^{-1}x, L^{-1}y)$$

3.4 Random walk and scaling properties

Let X_t be the process that starts at 0 and jumps by the law for $y \neq x$

$$P\left(X_{t+dt} = y \mid X_t = x\right) \propto |x-y|^{-D-2}.$$

Unlike the random walk on $\mathbb{Z}^D$ this is not taking nearest neighbor steps, but this is unavoidable: the ultrametric property implies that the hierarchical walk will remain locked inside a ball unless it has jumps on all length scales. Define *the β-potential*, also called *Green's function*,

$$G(\beta, b) = \int_0^\infty dT\, e^{-\beta T}\mathbb{E}_0^{[0,T]}\left(\delta_{b,X_T}\right).$$

The τ isomorphism associates to our random walk a Gaussian measure with covariance $G(\beta, x-y)$. We can use the RG if this covariance is an integral (or sum) over scalings as in (2.11). Part (ii) of the following proposition shows that this is the case, while part (i) shows that the hierarchical Green's function has the same homogeneity as the Green's function on the simple cubic lattice.

Proposition 3.1 (Brydges-Evans-Imbrie [BEI92]).

(i) $G(\beta = 0, b) = |b|^{2-D}$ *if* $b \neq 0$,

(ii) $G(\beta, b) = \sum_{k=0}^\infty \mathcal{S}^k\Gamma\left(L^{2k}\beta, b\right)$,

(iii) $\Gamma(\beta, b) = 0$ *if* $|b| > L$,

where

$$\Gamma(\beta, b) = \frac{1}{\beta+\gamma}\left\{\mathbb{1}_{\mathcal{G}_0}(b) - L^{-D}\mathbb{1}_{\mathcal{G}_1}(b)\right\},$$

for some constant γ. $\Gamma(\beta, b)$ *is positive semi-definite.*

Proof. See [BEI92].

This process has the usual[4] $\mathbb{E}_0^{[0,T]}\Big(|X_T|\Big) \approx \mathcal{O}\big(T^{1/2}\big)$ behavior, where $\approx$ means that there are upper and lower bounds by the right hand side.

Proof. We prove only the upper bound. For $|b| \neq 0$, the Green's function is

$$G(\beta, b) = \sum_{k=\log_L |b|}^{\infty} L^{(2-D)k} \left(\frac{1}{L^{2k}\beta + \gamma} - \frac{L^{-2}}{L^{2(k-1)}\beta + \gamma} \right)$$

Taking the inverse Laplace transform,

$$\mathbb{E}_0^{[0,T]}\Big(\delta_{b,X_T}\Big) = |b|^{-D} \sum_{k=0}^{\infty} L^{-Dk} \left(e^{-\gamma L^{-2k} T/|b|^2} - e^{-\gamma L^{-2(k-1)} T/|b|^2} \right).$$

Multiply by $|b|$ and sum over b observing that the hierarchical space of $b \neq 0$ consists of shells of radius $|b| = L^l$ with $L^{Dl} - L^{D[l-1]} = O(L^{Dl})$ elements in each shell, $l = 1, 2, \ldots$. The long time behavior of $\mathbb{E}_0^{[0,T]}\Big(|X_T|\Big)$ is established by the following two estimates.

$$\sum_{b:|b|\leq T^{1/2}} |b|^{1-D} \left(e^{-\gamma L^{-2k} T/|b|^2} - e^{-\gamma L^{-2(k-1)} T/|b|^2} \right) \leq 2 \sum_{l \leq \frac{1}{2}\log_L T} L^l = \mathcal{O}\big(T^{1/2}\big),$$

$$\sum_{b:|b|>T^{1/2}} |b|^{1-D} \left(e^{-\gamma L^{-2k} T/|b|^2} - e^{-\gamma L^{-2(k-1)} T/|b|^2} \right)$$

$$\leq L^{-2(k-1)} \mathcal{O}(T) \sum_{l > \frac{1}{2}\log_L T} L^{-l} = L^{-2(k-1)} \mathcal{O}\big(T^{1/2}\big).$$

□

To use the τ isomorphism we must approximate the infinite hierarchical lattice by a finite state space. This is done by killing the random walk on first exit from the ball $\Lambda = \mathcal{G}_n$ in the infinite lattice. The analogy with our scale integral (2.11) is made clearer by using the notation $G_{[0,n)}$ for the *Green's function killed on first exit from* $\mathcal{G}_n$.

Proposition 3.1, Part (ii), becomes [BIb]

$$G_{[0,n)}(\beta, b) = \sum_{k=0}^{n-1} \mathcal{S}^k \Gamma\big(L^{2k}\beta, b\big) + \frac{1}{L^{2n}\beta + \gamma} \mathcal{S}^n \mathbb{1}_{\mathcal{G}_0}(b). \tag{3.1}$$

In this notation $G_{[0,\infty)}$ is the Green's function of Proposition 3.1, Part (ii).

In the τ isomorphism the matrix A_{xy} is the inverse of $G_{[0,n)}(\beta, x-y)$ and the matrix indices x, y are summed over Λ. The covariance of the Gaussian $\mu_C = \mu_{[0,n)}$ is $G_{[0,n)}$. Thus

$$\int \mu_{[0,n)}\, \phi_a \bar{\phi}_b = \int_0^{\infty} dT\, e^{-\beta T} \mathbb{E}_{ab}^{[0,T]}\Big(1\Big),$$

where

$$\mathbb{E}_{ab}^{[0,T]}\Big(\bullet\Big) = \mathbb{E}_a^{[0,T]}\Big(\bullet\ \delta_{b,X_T} 1_{\{\text{exits after } T\}}\Big).$$

[4] See [BIb, Proposition 1.1] for the more precise result that the $T \to \infty$ limit exists for subsequences $L^{2n}T_0$

3.5 Interaction

Let $g(t)$ be any smooth bounded function with $g(0) = 1$. Recalling that τ_x is the time spent at lattice site x by the process X_t we set

$$g^\Lambda = \prod_{x \in \Lambda} g(\tau_x).$$

By *the τ-isomorphism*,

$$G_g(b-a) := \int_0^\infty dT\, e^{-\beta T} \mathbb{E}_{ab}^{[0,T]} \left(g^\Lambda \right) = \int \mu_{[0,n)}\, g^\Lambda \phi_a \bar{\phi}_b, \tag{3.2}$$

where on the left hand side, $\tau_x = \phi_x \bar{\phi}_x + \frac{1}{2\pi i} d\phi_x d\bar{\phi}_x$. $G_g(\beta, b-a)$ is called the *interacting Green's function.*

Since $g(0) = 1$, g has a unique representation

$$g(t) = e^{-\beta t} \{ e^{-\lambda t^2} + r(t) \}, \tag{3.3}$$

with $r(t) = \mathcal{O}(t^3)$ as $t \to 0$. For $r = 0$, we recover the self-repelling walk problem

$$\int \mu_{[0,n)}\, g^\Lambda \phi_a \bar{\phi}_b = \int_0^\infty dT\, e^{-\beta T}\, \mathbb{E}_{ab}^{[0,T]} \left(\exp(-\lambda \sum_{x \in \Lambda} \tau_x^2) \right)$$

3.6 Renormalization group

Earlier we defined $\mathbb{T}_t$ as a convolution followed by a rescaling. The following constructions have the same structure.

The convolution of two forms $A = \sum a_\alpha d\phi^\alpha$ and $B = \sum b_\alpha d\phi^\alpha$ is defined to be

$$A * B = \int_{\mathbb{C}^\Lambda} \sum_{\alpha,\beta} a_\alpha(\zeta) b_\beta(\zeta + \phi) d\zeta^\alpha (d\zeta + d\phi)^\beta$$

where the integration is only over the ζ. In particular, if F_X is a form on $\mathbb{C}^X$, then $\mu_\Gamma * F_X$ is also a form on $\mathbb{C}^X$.

The RG is the sequence of maps $\mathbb{T}^k$ where $k = 1, 2, \ldots$ and

$$\mathbb{T}F = \mathcal{S}\mu_\Gamma * F.$$

If F_X is a form on $\mathbb{C}^X$ then $\mathbb{T}F_X$ is a form on the smaller dimensional $\mathbb{C}^{X/\mathcal{G}_1}$.

Invariance: In analogy to Subsection 2.4, the decomposition (3.1) implies

$$\int_{\mathbb{C}^\Lambda} \mu_{[0,n)}\, F = \int_{\mathbb{C}^{\Lambda/\mathcal{G}_1}} \mu_{[0,n-1)}\, \mathbb{T}F, \tag{3.4}$$

We call this invariance because it suggests that the form $\mu_{[0,\infty)}$ is invariant for $\mathbb{T}$, but we cannot make this precise since we have not defined forms in infinite dimensions. Recall $\Lambda = \mathcal{G}_n$ so $\Lambda/\mathcal{G}_1 = \mathcal{G}_{n-1}$.

Independence: By (iii) of Proposition 3.1 and the ultrametric property of the hierarchical lattice, $\mathbb{T}$ acts *independently* on disjoint balls:

$$\mathbb{T}g^{\Lambda} = \mathbb{T}\big(g^{\mathcal{G}_1}\big)^{\Lambda/\mathcal{G}_1} = \big(\mathbb{T}g^{\mathcal{G}_1}\big)^{\Lambda/\mathcal{G}_1}.$$

In more detail, suppose that $x' = x + \mathcal{G}_1$ is a coset in $\Lambda/\mathcal{G}_1$, define $G_{x'} = \prod_{x \in x'} g_x$. Then

$$\mathbb{T}g^{\Lambda} = \prod_{x' \in \Lambda/\mathcal{G}_1} \mathbb{T}G_{x'}.$$

Here we see why the RG preserves multiplicativity on the hierarchical lattice.

Form invariance: $\mathbb{T}g^{x+\mathcal{G}_1}$ has the form $g'(\tau_{L^{-1}x})$ for some new function g'. This is connected with supersymmetry. In [BIa, BIb] we prove that $\mathbb{T}$ commutes with the operator Q defined in Subsection 1.7 and an even supersymmetric form on $\mathbb{C}$ is a function of τ. The supersymmetry property also implies that the normalization $g(0) = 1$ is preserved. Supersymmetry is the way this formalism remembers that it was born from a random walk.

Observable: An additional isolated ϕ_x is scaled:

$$\mathbb{T}\left(g^{x+\mathcal{G}_1}\phi_b\right) = g'(\tau_{L^{-1}x})L^{-1}\phi_{L^{-1}b}, \tag{3.5}$$

where g' is the same as in the last item. Thus $\mathbb{T}$ induces a very simple "renormalization" of the "observable": $\phi_b \to L^{-1}\phi_{b/L}$.

3.7 Recursion

The results in the last subsection allow us to regard the RG as a dynamical system on the *coordinates* (β, λ, r) that specify the interaction g through (3.3). Starting with

$$(\beta_0, \lambda_0, r_0) = (\beta, \lambda, r),$$

successive applications of $\mathbb{T}$ generate an orbit $(\beta_k, \lambda_k, r_k)$ with $0 \le k \le n-1$ in the space of interactions. Each RG map also causes a reduction $\Lambda \mapsto \mathcal{S}\Lambda$ in the state space. When $k = n-1$ the state space has shrunk to $\mathcal{G}_1$ so that the final integration is over $\mathbb{C}^{\mathcal{G}_1}$.

Using the recursion we generate a sequence of Green's functions with different interactions which are equal up to scaling: let g_k be the interaction determined by $(\beta_k, \lambda_k, r_k)$ and suppose that b is a point in the hierarchical lattice at distance $|b| > L$ from the origin. Then

$$G_{g_k}(b) = \mathcal{S}G_{g_{k+1}}(b) \tag{3.6}$$

This is an immediate consequence of the τ isomorphism, the Invariance property (3.4) and the Observable property (3.5). Although there are two factors ϕ_0 and ϕ_b in the Green's function, the RG acts on them independently because of the Independence property.

We want results for the Green's function on the infinite lattice, so we must consider the *infinite volume limit* in which the initial $\Lambda = \mathcal{G}_n$ is increased. First note that if the initial state space is enlarged then the new orbit is the same as the old orbit except it is longer — enlarging Λ merely extends the orbit. This is because the terms in the scale decomposition (3.1) are independent of the initial Λ. Only the number of terms is Λ dependent. Therefore we can define *the infinite volume orbit*

$$(\beta_k, \lambda_k, r_k)_{0 \le k < \infty}$$

as the union of all such finite Λ_0 orbits. For real parameters β, λ in the initial interaction $\exp(-\lambda\tau^2 - \beta\tau)$, the Green's function also has an infinite volume limit if β is not too negative. This follows from monotone convergence applied to the random walk representation. This existence will extend to complex domains for the initial λ and β, if we prove uniform bounds and analyticity in β, by the relative compactness of normal families of analytic functions combined with convergence on the real axis.

Define the sector in the complex plane

$$\mathcal{D}_\beta := \{\beta \neq 0 : |\arg \beta| < \frac{5\pi}{8}\}$$

and note that it has an opening angle larger than π. The domain for λ is also a sector, but in addition λ must be small:

$$\mathcal{D}_\lambda := \{\lambda \neq 0 : |\arg \lambda| < \frac{\pi}{8} \text{ and } |\lambda| < \delta\}$$

We arrange that the initial interaction is the self-repelling walk by setting $r_0 = 0$. Then $(\beta_k, \lambda_k, r_k)$ are determined as functions of $(\beta, \lambda) = (\beta_0, \lambda_0)$ by the RG map.

These arguments show that there is an orbit of interactions but give hardly any information on the orbit. However second order perturbation theory as in Subsection 2.9 can be worked out for this discrete RG. The recursion

$$\begin{cases} \lambda_{j+1} &= \lambda_j - \frac{8B\lambda_j^2}{(1+\beta_j)^2} + \epsilon_{\lambda,j}, \\ \beta_{j+1} &= L^2 \left[\beta_j + \frac{2B}{1+\beta_j}\lambda_j\right] + \epsilon_{\beta,j} \end{cases} \tag{3.7}$$

where $B = 1 - L^{-4}$, replaces the differential flow equations. For example, compare the λ_{j+1} equation with (2.25).

This recursion was derived from a recursion for $(\beta_k, \lambda_k, r_k)$ and then r_k was eliminated which is why β_k, λ_k must be regarded as functions of the initial β, λ. We are able to prove good estimates on the recursion for all k such that (β_k, λ_k) remains inside an enlarged domain obtained by including all β within distance $\frac{1}{2}$ of $\mathcal{D}_\beta$. In particular, we prove that the remainders $\epsilon_{\lambda,j}$, $\epsilon_{\beta,j}$ are analytic functions of $(\beta, \lambda) = (\beta_0, \lambda_0)$ satisfying

$$\begin{aligned} |\epsilon_{\lambda,j}| &\leq c_L |\lambda_j|^3 |1+\beta_j|^{-\frac{1}{2}}, \\ |\epsilon_{\beta,j}| &\leq c_L |\lambda_j|^2 |1+\beta_j|^{-\frac{1}{2}}. \end{aligned} \tag{3.8}$$

Proposition 3.2. *For each $\lambda \in \mathcal{D}_\lambda$ there exists $\beta^c(\lambda) = -O(\lambda)$ such that $\beta_k^c := \beta_k(\beta^c(\lambda)) \to 0$ as $k \to \infty$. Furthermore when $\beta = \beta^c(\lambda)$, $\lambda_k \to 0$.*

This shows that there are sequences that stay in the enlarged domain forever. This proposition is partly a construction of what is called the *stable manifold* of the fixed point.

In the next theorem G_λ is the *infinite volume limit* ($\Lambda = \mathcal{G}_n$, $n \to \infty$) of G_g defined in (3.2) with $g = \exp(-\beta t - \lambda t^2)$. Thus $G_0(\beta, b)$ has $g = \exp(-\beta t)$ and this is the β potential or *free* Green's function. The theorem tells us that the interacting Green's function is well approximated, particularly at large $|b|$ by a free Green's function with a different β parameter defined by

$$\beta_{\text{eff}} = \lim_{k\to\infty} L^{-2k}(\beta_k - \beta_k^c). \tag{3.9}$$

This is called an *effective beta* in accordance with a general terminology in physics describing cases where interacting systems are approximated by noninteracting systems optimizing over the parameters of the noninteracting system.

Theorem 3.3. *For all $\lambda \in \mathcal{D}_\lambda$, $G_\lambda(\beta, b)$ is analytic in $\beta^c(\lambda) + \mathcal{D}_\beta$ and*

$$\left|G_\lambda(\beta, b) - G_0(\beta_{\mathit{eff}}, b)\right| \leq \mathcal{O}(\lambda_{N(b)})\left|G_0(\beta_{\mathit{eff}}, b)\right|,$$

where $N(b) = \log_L |b|$ is the number of iterations of the RG needed to scale b into 0.

The main idea behind this result is a kind of scattering theory argument which compares the effect of RG transformations on the interacting Green's function with RG transformations on the free Green's function: namely, iterate (3.6),

$$G_\lambda(\beta, b) = L^{-2k} G_{g_k}(L^{-k} b), \tag{3.10}$$

$k = N(b) - 1$ times. This extracts the b dependence $L^{-2k} = L^2 |b|^{-2}$ and scales the starting point 0 and the endpoint b to within distance L of each other. Having brought them close to each other, the interacting Green's function is equal up to a $\mathcal{O}(\lambda_{N(b)})$ error to the free Green's function,

$$G_{g_k}(L^{-k} b) = G_0(\beta_k - \beta_k^c, L^{-k} b) + \mathcal{O}(\lambda_k). \tag{3.11}$$

By inverse RG transformations the free Green's function scales back to the free Green's function for the original starting point 0 and endpoint b.

$$L^{-2k} G_0(\beta_k - \beta_k^c, L^{-k} b) = G_0(L^{-2k}[\beta_k - \beta_k^c], b) \tag{3.12}$$

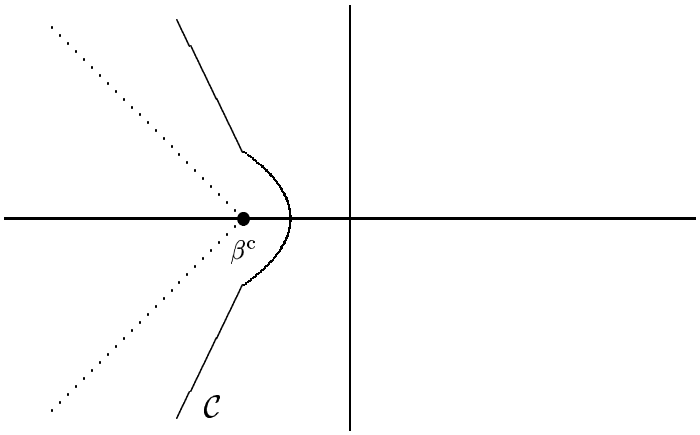
We also need lower bounds on the free Green's function in order to see that the error term is less than a free Green's function. Preliminary descriptions of these ideas appeared in [Imb94, Gol95]

3.8 End-to-end distance

By *the inverse Laplace transform*

$$\mathbb{E}_{0b}^{[0,T]}\left(\left(e^{-\lambda\tau^2}\right)^\Lambda\right) = \int_{\mathcal{C}} \frac{d\beta}{2\pi i}\, e^{\beta T} G_\lambda(\beta, b),$$

where the contour $\mathcal{C}$ of the integration is shown in the picture, using the dotted lines to display the boundary of the β domain.



The inversion of Laplace transforms is a well known hard problem. In our case beautiful results such as Tauberian theorems simply will not help, because, recalling the discussion at the end of Subsection 1.4, we have to calculate a *ratio* of two fixed T objects as $T \to \infty$, whereas Tauberian theorems are about replacing an abelian average over T by another average over T. In this subsection we try to motivate the main ideas and as usual refer the reader to [BIa, BIb] for details.

First we claim that Theorem 3.3 reduces the problem to

$$\int_{\mathcal{C}} \frac{d\beta}{2\pi i}\, e^{\beta T} G_0(\beta_{\text{eff}}, b)$$

The main reason is that the factor $\lambda_{N(b)}$ supplies additional logarithmic decay in b because the solution to the λ recursion (3.7) is

$$\lambda_k \approx \frac{\lambda}{1 + 8B\lambda k} \tag{3.13}$$

End of discussion on this claim.

Because the contour *slopes backwards* and the integral contains $\exp(\beta T)$, the dominant contribution to the integral comes from the part of the contour within $\mathcal{O}(1/T)$ of β^c. Consider the curved part of the contour. By choice of contour this is a segment of the circle $|\beta - \beta^c| = 1/T$. By solving the β recursion (3.7) we find that for β on this part of the contour

$$\beta_{\text{eff}} \approx (\beta - \beta^c) \log^{-1/4} T.$$

The term $\frac{2B}{1+\beta_j}\,\lambda_j$ in (3.7) together with (3.13) slows exponential divergence of β_k away from the stable manifold orbit β_k^c. This is responsible for the $\log^{-1/4} T$.

By inverting this change of variable we obtain

$$\beta = \beta^c + \beta_{\text{eff}} \log^{1/4} T.$$

Substitute in the inverse Laplace transform. Let $T_{\rm new} = T \log^{1/4} T$, then

$$\mathbb{E}_{0b}^{[0,T]}\Big(\big(e^{-\lambda\tau^2}\big)^{\Lambda}\Big) \approx e^{\beta^c T}\log^{1/4} T \int_{\mathcal{C}} \frac{d\beta_{\rm eff}}{2\pi i}\, e^{\beta_{\rm eff} T_{\rm new}} G_0(\beta_{\rm eff}, b)$$
$$= e^{\beta^c T}\log^{1/4} T\, \mathbb{E}_{0b}^{[0,T_{\rm new}]}\Big(1\Big).$$

Here we see that the interaction is equivalent to no-interaction, a change of normalization $e^{\beta^c T}\log^{1/4} T$ and a different time: T becomes $T_{\rm new}$. Then,

$$\langle |X_T| \rangle := \frac{\sum_b |b|\, \mathbb{E}_{0b}^{[0,T]}\Big(\big(e^{-\lambda\tau^2}\big)^{\Lambda}\Big)}{\sum_b \mathbb{E}_{0b}^{[0,T]}\Big(\big(e^{-\lambda\tau^2}\big)^{\Lambda}\Big)} \approx \frac{\sum_b |b|\, \mathbb{E}_{0b}^{[0,T_{\rm new}]}\Big(1\Big)}{\sum_b \mathbb{E}_{0b}^{[0,T_{\rm new}]}\Big(1\Big)} = \mathbb{E}_0\Big(|X_{T_{\rm new}}|\Big).$$

These are some of the main ideas in the proof of

Theorem 3.4. *Fix an integer $L \geq 2$. If λ is sufficiently small, then the self-repelling expectation $\langle\rangle$ and the simple random walk expectation E_0 are related by*

$$\langle |X_T| \rangle = \left(1 + \frac{O(\lambda)}{\ell(T^{-1})}\right) \mathbb{E}_0\Big(|X_{T_{\rm new}}|\Big), \tag{3.14}$$

where $T > 1$, $B \equiv 1 - L^{-4}$,

$$\ell(T^{-1}) = 1 + O(\lambda) + B\lambda(4\log T + \log|1 + \lambda \log T|). \tag{3.15}$$

and $T_{\rm new} = T\ell(T^{-1})^{\frac{1}{4}}$.

We have mentioned that for the hierarchical walk without interaction $\mathbb{E}_0\Big(|X_T|\Big)$ grows as $\sqrt{T}$ so Theorem 3.4 shows that the interaction changes this law to $\sqrt{T}\log^{1/8} T$.

A preliminary discussion of this theorem appeared in [Imb94]

4 Analysis of Remainders and Outlook

In this final section we look at methods for controlling the remainder after perturbation theory. For example how are the bounds on remainders (3.8) obtained? Our discussion is based on [BIa, BIb]. After considering hierarchical lattices we turn to perturbations of Gaussian random fields on $\mathbb{R}^D$.

At various places we will be assuming that the parameter L defining the hierarchical lattice is large and that the coupling constant λ that measures the strength of the repulsion of the walk is small, depending on L. However we did not include the hypothesis L large in our main Theorem 3.4 because it can be assumed with no loss of generality. For example if $L = 2$ we conglomerate p RG maps into one with a new $L = 2^p$.

4.1 Norms

Let X be a subset of the state space Λ. We consider forms on $\mathbb{C}^X$. Let

$$d\phi^\alpha = \prod_{x\in X} d\phi_x^{\alpha_x} d\bar\phi_x^{\bar\alpha_x},$$

where each α_x and $\bar\alpha_x$ can be any non-negative integer, but only zero or one give nonvanishing contributions because differentials are raised to a power using the wedge product. Thus zero means "omit this differential" and one means "retain this differential". Any smooth form F_X on $\mathbb{C}^X$ can be uniquely expanded as

$$F_X = \sum_\alpha \frac{1}{\alpha!} F_X^{(\alpha)}(\phi)\, d\phi^\alpha.$$

We have put in a factor $\alpha! = \prod_x \alpha_x!\bar\alpha_x!$ even though it equals one to underline a formal similarity with Taylor series. Thus we write $F_X^{(\alpha,\beta)}$ for a derivative of $F_X^{(\alpha)}(\phi)$ with respect to ϕ (or $\bar\phi$) and define, for $h \geq 0$,

$$\|F_X\|_h \equiv \|F_X\|_{a,h} := \sum_{\alpha,\beta} \frac{h^{\alpha+\beta}}{\alpha!\,\beta!} \sup_\phi \left|F^{(\alpha,\beta)}(\phi)\, w^{-X}(\phi)\right|,$$

where $w^{-X}(\phi) = \prod_{x\in X} w^{-1}(\phi_x)$ is a Gaussian weight

$$w(\phi_x) = e^{-a\phi_x\bar\phi_x}$$

and the supremum is taken over $\phi \in \mathbb{C}^X$. When a retains the same value throughout an equation we drop it from the notation.

The parameter a places a Gaussian bound on decay (a positive) or growth (a negative) of F_X as a function of ϕ at infinity. As $a \to -\infty$ the weight concentrates the supremum towards $\phi = 0$ so that

$$|F_X|_h := \sum_{\alpha,\beta} \frac{h^{\alpha+\beta}}{\alpha!\,\beta!} \left|F^{(\alpha,\beta)}(0)\right|$$

is the limit of $\|F_X\|_{a,h}$. These norms were defined in [BEI92] and are reviewed in [BIb].

We generally use lower case letters such as r or g for forms on $\mathbb{C}^X$ when X is the origin in the hierarchical lattice. We write $g_x \equiv g_{\{x\}}$ for a form on $\mathbb{C}^X$ when X is the set consisting of the single state $\{x\}$ and set

$$g^X := \prod_{x\in X} g_x.$$

The order of the product need not be specified if at most one of the forms g_x contains terms of odd degree.

Our default choices are $h = |\lambda|^{-1/4}$ and $a = h^{-2}$ because then, by scaling $\phi = h\phi'$,

$$\|e^{-\lambda\tau^2}\|_{a,h} = \|e^{-\tau^2}\|_{1,1} < \infty$$

For a proof of the inequality, see [BEI92].

Reviewing definitions we recall that the RG transformation maps the parameters λ, β and r to new parameters λ', β' and r' such that,

$$e^{-\beta'\tau}\left(e^{-\lambda'\tau^2}+r'\right)=\mathcal{S}\mu_\Gamma * \left(e^{-\beta\tau}[e^{-\lambda\tau^2}+r]\right)^{\mathcal{G}_1}.$$

with β' and λ' chosen so that r' is *normalized*:

$$r'(t)=\mathcal{O}\big(t^3\big). \tag{4.1}$$

As a step on the way to solving these equations for the new parameters we first consider

$$e^{-\tilde{\beta}\tau}\left(e^{-\tilde{\lambda}\tau^2}+\tilde{r}\right)=\mathcal{S}\mu_\Gamma * \left(e^{-\beta\tau}[e^{-\lambda\tau^2}+r]\right)^{\mathcal{G}_1}. \tag{4.2}$$

where $\tilde{\beta}$ and $\tilde{\lambda}$ are not yet fixed so that (4.1) is satisfied. When $r=0$ the solution for $\tilde{r}$ is

$$r'_{\text{main}} := e^{\tilde{\beta}\tau}\mathcal{S}\mu_\Gamma * \left(e^{-\beta\tau}e^{-\lambda\tau^2}\right)^{\mathcal{G}_1} - e^{-\tilde{\lambda}\tau^2}.$$

$\tilde{\beta}$ and $\tilde{\lambda}$ are chosen so that, as a formal series in λ, r'_{main} vanishes to order λ^3. This of course includes a claim that this is possible — let us return to that later.

Let $B(1/2)$ be the ball of radius $1/2$ centered on the origin in the complex plane. r'_{main} is an explicit functional so we can prove that there is a constant K_L such that for $\lambda \in \mathcal{D}_\lambda$ and $\beta \in B(1/2)+\mathcal{D}_\beta$,

$$\|r'_{\text{main}}\| := |\lambda|^{-2}\big|r'_{\text{main}}\big|_{h=1} + \big\|r'_{\text{main}}\big\|_h \le K_L|\lambda|.$$

We will comment on this more later. Recall that $\mathcal{D}_\lambda$ included a restriction $|\lambda|<\delta$. The last claim and the next lemma rest on the hypothesis that this δ is small *depending on* L. r'_{main} is the main part of r' in the following sense:

Proposition 4.1. *Let $\lambda \in \mathcal{D}_\lambda, \beta \in B(1/2)+\mathcal{D}_\beta$. For L sufficiently large, $\|r\| \le 2K_L|\lambda|$ implies $\|r'\| \le 2K_L|\lambda'|$*

This is important because it tells us that r' is inconsequential for as long as λ, β remain inside their domains. On the other hand, by second order perturbation theory, (3.7),

$$\lambda' = \lambda - \frac{8B\lambda^2}{(1+\beta)^2},$$

up to an error controlled by $|r|_{h=1} \le \lambda^2\|r\| \le 2K_L\lambda^3$, so we have a mechanism to verify that λ, β do remain inside their domains.

β generically becomes large because the recursion (3.7) has the factor L^2, but this need not invalidate perturbation theory in λ for the following reason. Recall that $\Gamma=\Gamma(\beta)$ was defined in Proposition 3.1. We take advantage of $e^{-\beta\tau}$ being Gaussian to move it past μ_Γ by the identity,

$$\mathcal{S}\mu_{\Gamma(\beta_1)} * \left(e^{-\beta\tau}\right)^{\mathcal{G}_1} F = e^{-L^2\beta\tau}\mathcal{S}\mu_{\Gamma(\beta_1+\beta)} * F, \tag{4.3}$$

for F a form on $\mathbb{C}^{\mathcal{G}_1}$. For as long as the real part of $(1+\beta)^{-1}$ remains positive and bounded, there are good estimates on convolution by μ_Γ. For example we can control an RG transformation for $\Re\beta \ge -\frac{1}{2}$.

Thus, we can reach a half plane as the domain for β, but how do we reach β in a sector $\mathcal{D}_\beta$ that has an opening angle larger than π? This is obtained by an analytic continuation in the initial β using the change of contour of integration

$$\phi_x \to e^{i\theta}\phi_x,$$
$$\bar{\phi}_x \to e^{i\theta}\bar{\phi}_x,$$

which is equivalent to

$$\beta \to e^{2i\theta}\beta,$$
$$\lambda \to e^{4i\theta}\lambda,$$
$$\Gamma \to e^{-2i\theta}\Gamma.$$

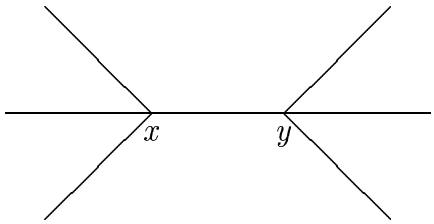
This analytic continuation is not just valid for hierarchical models. For the nearest-neighbor walk on $\mathbb{Z}^d$, it follows from

$$G_\lambda(\beta, b) = \sum_{\omega: o \to b} \prod_x \int_0^\infty dt\, e^{-(2d+\beta)t - \lambda t^2} \frac{t^{N_x - 1}}{(N_x - 1)!},$$

where N_x is the number of visits to site x, by the rotation of contour $t \to e^{2i\theta}t$.

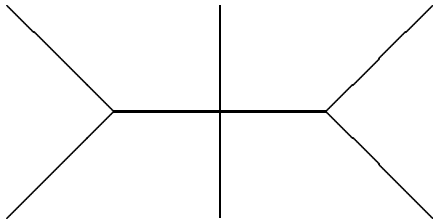
By combining these ideas there are good estimates for β in $B(1/2) + \mathcal{D}_\beta$. This domain for β is large enough to contain the global trajectories we need for our analysis of the inverse Laplace transform.

In the hierarchical model, the τ^3 correction in the second order perturbation,



vanishes because $\sum_y \Gamma(\beta, x-y) = 0$. If we did not have this property, it would be impossible to choose $\tilde{\beta}$ and $\tilde{\lambda}$ in (4.2) so as to make $\tilde{r} = \mathcal{O}(\lambda^3)$ because $\tilde{r}$ would contain this $\mathcal{O}(\lambda^2)\tau^3$ term.

Let us conclude with a brief explanation for the $\mathcal{O}(\lambda)$ bound on $\|r'_{\text{main}}\|$. The remainder after second order perturbation theory is roughly speaking third order perturbation theory times $\exp(-\lambda\tau^2)$. One of the third order Feynman diagrams is



This diagram is a polynomial in $\phi, \bar{\phi}$ and $d\phi, d\bar{\phi}$ of degree 6 because there are six external legs. Since there are three vertices there is a factor λ^3. By scaling $\phi = h\phi'$ we find that $\|\lambda^3\phi^6 \exp(-\lambda\tau^2)\|_{a,h}$ equals $|\lambda|^3 h^6 \|\phi^6 \exp(-\tau^2)\|_{1,1} \leq c|\lambda|$ because $h = |\lambda|^{-1/4}$.

4.2 Sketch of Proof of Lemma

The norm has the following properties,

1. If X and Y are disjoint, $\|F_X G_Y\|_h \leq \|F_X\|_h \|G_Y\|_h$.
2. for $a \geq 0$, $\|\mathcal{S}F_X\|_{a,h} = \|F_X\|_{a/L^2,h/L}$.
3. For $h' > h$,
$$\|F_X^{(\alpha,\beta)}\|_h \leq \frac{(\alpha+\beta)!}{(h'-h)^{\alpha+\beta}} \|F_X\|_{h'}.$$
4. Convolution by $\mu \equiv \mu_\Gamma$ is almost the identity: Let $X \subset \mathcal{G}_1$ and $a \geq 0$,
$$\|\mathcal{S}\mu * F_X - \mathcal{S}F_X\|_{2a,h} \leq c_L h^{-2} \|F_X\|_{a,2h/L}.$$
This bound holds for the complex covariance $\Gamma(\beta)$, when $\Re\beta > -1/2$.
5. Suppose $r(t)$ is a smooth function of a real variable t with $r(t) = \mathcal{O}(t^3)$ as $t \to 0$. It determines a form r on $\mathbb{C}$ by substituting $\tau := \tau_x$ in place of t, where x is the origin in Λ. Then,
$$\|r\, e^{-\lambda\tau^2}\|_h \leq \mathcal{O}(L^{-6})\, \|r\|_{Lh/2}.$$

To focus on some of the important ideas we will omit the β parameter from all formulas. This is not too disgraceful because most of it goes into the covariance by (4.3). Keep in mind the order of choice of parameters: first L is chosen large and then the domain of λ is fixed, in particular $|\lambda| \leq \delta$ with δ small depending on L.

We will focus on the $\|\tilde{r}\|_h$ part of

$$\|\tilde{r}\| = |\lambda|^{-2} |\tilde{r}|_{h=1} + \|\tilde{r}\|_h$$

By expanding $\tilde{r}$ in r in (4.2) we obtain

$$\begin{aligned}\tilde{r} &= \mathcal{S}\mu * \left(e^{-\lambda\tau^2} + r\right)^{\mathcal{G}_1} - e^{-\tilde{\lambda}\tau^2} \\ &= r'_{\text{main}} + \sum_{\substack{X\subset\mathcal{G}_1,\\ |X|\geq 1}} \mathcal{S}\mu * \left\{ r^X \left(e^{-\lambda\tau^2}\right)^{\mathcal{G}_1\setminus X}\right\}\end{aligned}$$

The first principle is that all the terms in the sum which contain two or more factors of r are inconsequential simply by choosing λ small depending on L. In more detail

$$\left\| \sum_{\substack{X\subset\mathcal{G}_1,\\ |X|\geq 2}} \mathcal{S}\left\{ r^X \left(e^{-\lambda\tau^2}\right)^{\mathcal{G}_1\setminus X}\right\}\right\|_{2h} \leq \sum_{\substack{X\subset\mathcal{G}_1,\\ |X|\geq 2}} \left\|\mathcal{S}\left\{ r^X \left(e^{-\lambda\tau^2}\right)^{\mathcal{G}_1\setminus X}\right\}\right\|_{2h}$$

and by property (2)

$$\leq \sum_{\substack{X\subset\mathcal{G}_1,\\ |X|\geq 2}} \left\| r^X \left(e^{-\lambda\tau^2}\right)^{\mathcal{G}_1\setminus X}\right\|_{2h/L}$$

followed by property (1)

$$\leq \sum_{\substack{X\subset\mathcal{G}_1,\\ |X|\geq 2}} \|r\|^X_{2h/L} \|e^{-\lambda\tau^2}\|^{\mathcal{G}_1\setminus X}_{2h/L}$$

The norms are increasing in $2h/L \leq h$ so we can bound in terms of $\|r\|_h \leq K_L|\lambda|$ and $\|\exp(-\lambda\tau^2)\|_h \leq c$. There are only an L dependent number of terms in the sum so

$$\left\| \sum_{\substack{X\subset\mathcal{G}_1,\\ |X|\geq 2}} \mathcal{S}\left\{ r^X \left(e^{-\lambda\tau^2}\right)^{\mathcal{G}_1\setminus X}\right\}\right\|_{2h} \leq c_L|\lambda|^2 \tag{4.4}$$

for some L dependent constant c_L. By property (4) we can get the same type of bound with $\mathcal{S}\mu*$ in place of $\mathcal{S}$ in (4.4). (4.4) is bounded by $c\lambda$ for any c by choosing $c_L|\lambda| < c$.

Next consider

$$\left\| \sum_{x\in\mathcal{G}_1} \mathcal{S}\left\{ r_x \left(e^{-\lambda\tau^2}\right)^{\mathcal{G}_1\setminus\{x\}}\right\}\right\|_{2h}$$

Since there are L^4 terms in the sum and $\mathcal{S}(AB) = \mathcal{S}(A)\mathcal{S}(B)$

$$\leq L^4 \left\|\mathcal{S}(r)\mathcal{S}\left(e^{-\lambda\tau^2}\right)^{\mathcal{G}_1\setminus\{x\}}\right\|_{2h}$$

By unraveling the definition of $\mathcal{S}$, the second factor is $\exp(-a(L)\lambda\tau^2)$ where $a(L) = 1 - L^{-4}$ so

$$= L^4 \|\mathcal{S}(r)e^{-a(L)\lambda\tau^2}\|_{2h}$$

As it stands, a bound by the desired $2K_L|\lambda|$ looks hopeless because of the L^4, but this is where the normalization (4.1) of r comes to the rescue. Since $r(t) = \mathcal{O}(t^3)$ we may use property (5)

$$\leq L^4\mathcal{O}(L^{-6})\|\mathcal{S}(r)\|_{Lh}$$

so that by property (2) and $\|r\|_h \leq K_L|\lambda|$

$$\left\| \sum_{x \in \mathcal{G}_1} \mathcal{S}\Big\{ r_x \big(e^{-\lambda\tau^2}\big)^{\mathcal{G}_1 \setminus \{x\}} \Big\} \right\|_{2h} \leq \mathcal{O}(L^{-2}) K_L |\lambda|$$

By property (4) we can replace $\mathcal{S}$ by $\mathcal{S}\mu*$.

Collecting these bounds we have

$$\|\tilde{r}\|_{2h} \leq K_L|\lambda| + \mathcal{O}(L^{-2})K_L|\lambda| + c_L|\lambda|^2 \leq \frac{3}{2} K_L|\lambda|$$

by choosing L large so that $\mathcal{O}(L^{-2}) \leq \frac{1}{4}$ and then the domain of λ so that $c_L|\lambda| \leq \frac{1}{4}K_L$.

These arguments have given some idea of why the norm $\|r\|_h$ is suited to analysis of the RG. We leave this argument now with the remark that the rest of the proof is an argument that the norm of $\tilde{r}$ is almost the same as the exactly normalized r' such that

$$e^{-\tilde{\beta}\tau} \left(e^{-\tilde{\lambda}\tau^2} + \tilde{r} \right) = e^{-\beta'\tau} \left(e^{-\lambda'\tau^2} + r' \right)$$

This is because $\tilde{r}$ is already normalized to order λ^2 and we can use the $|\lambda|^{-2}|\tilde{r}|_{h=1} \leq \|\tilde{r}\|$ part of the norm to show that at most $\mathcal{O}(\lambda^3)$ changes in $\tilde{\lambda}$ and $\tilde{\beta}$ suffice to get an exact normalization.

4.3 Outlook

We turn to the case where ϕ_x is a Gaussian random field on $\mathbb{R}^D$ and elaborate on the remarks below Theorem 2: in what sense is second order perturbation theory in the running coupling constant a uniform approximation to the exact effective interaction W? It may not be a good idea to try to answer this question too completely because the attempt to construct the exact effective interaction W will encounter the dreaded "Large Field problem[5]". Roughly speaking, when Λ is large, it will contain a region $X \subset \Lambda$ where ϕ is much larger than is typical for a ϕ distributed according to the Gaussian measure $\mu_{[0,\infty]}$. In such a region X, perturbation theory for W becomes very inaccurate. Furthermore estimates on $|W|$ are not a good idea in these regions because they will lose critical information on the sign of W. In these anomalous regions $\exp(-W)$ is very small because W is large *positive*.

We will not construct W, but instead write the integrand $\exp(-W)$ in a way that resembles the $\prod_x (\exp(-v_x) + r_x)$ used in the hierarchical model. Firstly, for uniformity in Λ, it is essential to represent the integrand in a way that quantifies multiplicativity: For example, suppose that Λ is partitioned into half-open unit cubes Δ. Then

$$e^{-\int_\Lambda v(\phi_x)\,dx} = \prod_{\Delta \subset \Lambda} e^{-\int_\Delta v(\phi_x)\,dx}$$

exemplifies a functional that is exactly a product of local factors. Secondly, a representation based on $\exp(-v_x) + r_x$ permits norm estimates on r whereas a norm on W runs afoul of the large field problem. Thirdly, we can examine the error after perturbation theory in terms of the difference between $\exp(-v_x) + r_x$ and some perturbative approximation. For

[5] For general background related to the "large field problem" see [VEFS93].

example we can use first order perturbation theory to construct an optimal v and then r is the remainder. Second order perturbation would involve splitting r into an explicit second order part determined by calculation and a new remainder which would carry the estimates.

Suppose we have a Gaussian random field ϕ_x, where $x \in \mathbb{R}^D$, with a covariance $C_{[0,\infty]}$ that admits a scale decomposition:

$$C_{[0,\infty)}(x,y) = \sum_{j \geq 0} \mathcal{S}^j C(x,y) \tag{4.5}$$

where $C(x,y) = C(x-y)$ is a positive-definite smooth function and

$$\mathcal{S}C(x,y) = L^{-2[\phi]} C(x/L, y/L).$$

Given an integral scale decomposition as in (2.11), we can get a discrete decomposition by breaking the range of integration into intervals $[j, j+1)$ with $j = 0, 1, \ldots$ and letting C be the integral over $[0,1)$. The *Renormalization Group (RG)* map is

$$\mathbb{T} := \mathcal{S}\mu_C *$$

We shall be interested in its action on bounded smooth functionals $Z_X(\phi)$ defined on the Banach space $\mathcal{C}^p(X)$, where X is a nice subset of $\mathbb{R}^D$. ϕ_x becomes an element of $\mathcal{C}^p(X)$ by restricting x to X.

The objective is to prove a result that takes the place of Proposition 4.1. We will stay close to the hierarchical model if we also *assume* that the covariance C that generates the scale decomposition is *finite range*:

$$C(x,y) = 0 \text{ if } |x-y| \geq 1 - \epsilon \tag{4.6}$$

The class of covariances that admit a scale decomposition into scalings of a finite range, smooth and positive-definite $C(x,y)$ must be very small, but if we relax the condition that terms in the decomposition be exact scalings of a single function by allowing

$$C_{[0,\infty)}(x,y) = \sum_{j \geq 0} \mathcal{S}^j C_j(x,y), \tag{4.7}$$

where C_j is uniformly bounded in a suitable norm, then it is much larger and this is good enough for our purposes. The main criterion in choosing the norm is that $\|C_j\|$ should control fluctuations in unit cubes Δ for the Gaussian process whose covariance is C_j:

$$P(\|\zeta\|_{\mathcal{C}^p(\Delta)} > A) \leq \exp(-\|C_j\| A^2).$$

In [Bry01] we show that there are such scale decompositions for inverses of certain elliptic partial differential operators. There are enough examples to show that scale decompositions exist for the important cases but it would be interesting to have a useful characterization of covariances that have such scale decompositions.

The next task is to describe the structure of functionals on which $\mathbb{T}$ will act. We partition $\mathbb{R}^D$ into (half open) unit cubes Δ centered on $\mathbb{Z}^D$. All sets, $X, \Lambda, \ldots$ are required to be unions of such cubes, or the null set. Let

$$V(X) := \int_X v(\phi_x)\, dx,$$

where v is a polynomial in ϕ_x and derivatives, such as

$$v(\phi_x) = \lambda : \phi_x^4 : + \beta : \phi_x^2 : + \zeta : (\partial \phi_x)^2 : . \tag{4.8}$$

Suppose $g(\Delta) := \exp(-V(\Delta))$. We define

$$g^X := \prod_{\Delta \subset X} g(\Delta).$$

For any other functional $\tilde{g}(\Delta)$ we have the binomial expansion

$$g^\Lambda = \big(g - \tilde{g} + \tilde{g}\big)^\Lambda = \sum_{X \subset \Lambda} \tilde{g}^{\Lambda \setminus X} \big(g - \tilde{g}\big)^X.$$

Let C be the $C_{j=0}$ in the decomposition (4.7). The RG map $\mathbb{T}_{j=0}$ contains the convolution $\mu_C * F(\phi) = \int d\mu_C(\zeta) F(\phi + \zeta)$. In the binomial expansion we set $g_\Delta = g_\Delta(\phi + \zeta)$ and $\tilde{g}_\Delta = \tilde{g}_\Delta(\phi)$ so that only the factors $g - \tilde{g}$ depend on the *increment* ζ and

$$\mu_C * g^\Lambda = \sum_{X \subset \Lambda} \tilde{g}^{\Lambda \setminus X} \mu_C\big((g - \tilde{g})^X\big),$$

where $(\mu_C F)(\phi) := \int d\mu_C(\zeta) F(\phi, \zeta)$ is the expectation over the increment[6] ζ. The set X has a unique decomposition into *connected components* X_k. These connected components are separated from each other by a distance greater or equal to one because X is a union of unit cubes Δ. The finite range property of the covariance C of the increment implies that increments ζ_x and ζ_y are *independent* for x, y in different connected components. Therefore

$$\mu_C * g^\Lambda = \sum_{X \subset \Lambda} \tilde{g}^{\Lambda \setminus X} \prod_k \mu_C\big((g - \tilde{g})^{X_k}\big).$$

Unlike the hierarchical model, the right hand side is no longer of form g^Λ so we must look for a more general class of functionals which will be closed under the action of $\mu_C *$.

We will use expressions of the form,

$$\mathcal{E}\mathrm{xp}[A](\Lambda) := \sum_{\pi : \text{partition of } \Lambda} \prod_{X \in \pi} A(X). \tag{4.9}$$

where $A(\emptyset) = 0$. (4.9) is called a *polymer expansion* and A is called a *polymer activity*. In these lectures polymer activities also satisfy $A(X) = 0$ whenever X is not connected.

We define $\square(X) = 1_{X \text{ is a unit cube}}$. Note that g^Λ is a special case of a polymer expansion in which $A = g\square$, because in this case the only partition contributing to the sum in (4.9) is the finest partition where all elements in the partition are unit cubes Δ. We will want A to have the property that hardly any of it lives on large sets X so that the finest partition is dominant in the polymer expansion.

[6] In probabilistic language we take the expectation conditioned on the sigma algebra generated by ϕ. A natural choice for $\tilde{g}(\Delta)$ is the conditional expectation of $g(\Delta)$.

We substitute $A(X) = \tilde{A}(X) + A(X) - \tilde{A}(X)$ where $A(X) = A(X, \phi + \zeta)$ and $\tilde{A}(X) = \tilde{A}(X, \phi)$. The result is

$$\mathcal{E}\mathrm{xp}[A](\Lambda) = \sum_{\pi} \sum_{\tilde{\pi} \subset \pi} \prod_{X \in \tilde{\pi}} \tilde{A}(X) \prod_{Y \in \pi \setminus \tilde{\pi}} (A - \tilde{A})(Y) \tag{4.10}$$

To take advantage of the independence resulting from the finite range property of C we glue polymers together into larger polymers with sets $Y \in \pi \setminus \tilde{\pi}$ buried inside them so that these larger polymers are independent. Therefore, define a map $\pi, \tilde{\pi} \mapsto \pi'$, where π' is a new partition, by declaring that

1. π' is a coarser partition than π. (This condition is going to be changed below).
2. If $Z \in \pi'$ and $Z \supset Y$ for some $Y \in \pi \setminus \tilde{\pi}$ then distance$(Y, Z^c) \geq 1$.
3. π' is the finest such partition.

The first condition means that each set in π' is a union of sets in π. The second condition will make the new poymers independent. The last condition makes π' unique. *Finest* means that there is no way to refine the partition by splitting some set Z in π' and still maintain the first two properties. Let

$$A_{\mathrm{new}}(Z) := \sum_{\pi, \tilde{\pi} \mapsto \{Z\}} \prod_{X \in \tilde{\pi}} \tilde{A}(X) \prod_{Y \in \pi \setminus \tilde{\pi}} (A - \tilde{A})(Y).$$

where π is now a partition of Z, $\tilde{\pi} \subset \pi$ and $\{Z\}$ is the partition π' of Z that consists of the single set Z. We *claim* that

$$\mathcal{E}\mathrm{xp}[A](\Lambda) = \mathcal{E}\mathrm{xp}[A_{\mathrm{new}}](\Lambda)$$
$$\mu_C * \mathcal{E}\mathrm{xp}[A](\Lambda) = \mathcal{E}\mathrm{xp}[\mu_C(A_{\mathrm{new}})](\Lambda)$$

Proof: Rewrite (4.10) as the sum over all $\pi, \tilde{\pi} \subset \pi$ that map to a fixed partition π' followed by a sum over *all* partitions π'.

$$\sum_{\pi, \tilde{\pi}} = \sum_{\pi'} \sum_{\pi, \tilde{\pi} \mapsto \pi'}$$

For $\pi' = \{Z_1, \ldots, Z_M\}$,

$$\sum_{\pi, \tilde{\pi} \mapsto \pi'} = \sum_{\pi, \tilde{\pi} \mapsto \{Z_1\}} \cdots \sum_{\pi, \tilde{\pi} \mapsto \{Z_M\}} .$$

This proves the first claim. The second claim is an immediate consequence of condition (2) in the definition of $\pi, \tilde{\pi} \mapsto \pi'$. □

Recall that there is a rescaling in $\mathbb{T}$. The problem is that the rescaling $\mathcal{S}$ shrinks unit cubes to cubes of side $L^{-1}\Delta$ centered on $(L^{-1}\mathbb{Z})^D$. We intend to forestall this trend towards finer sets by making a small change in condition (1) in the definition of $\pi, \tilde{\pi} \mapsto \pi'$ so that, before $\mathcal{S}$ is applied, the new polymer expansion is based on sets made out of cubes of side L.

To this end we partition $\mathbb{R}^D$ into *L-cubes* of side L centered on $(L\mathbb{Z})^D$. We say that Z is an *L-set* if Z is a union of L-cubes. We require that L be an integer so that each L-cube is a union of cubes Δ. For Z a union of L-cubes we define an L-polymer expansion $\mathcal{E}\mathrm{xp}_L[A](\Lambda)$ by (4.9) but with partitions π of Λ into L-subsets. We change (1) to

1. π' is a coarser partition than π *and the sets* $Z \in \pi'$ *are* L*-sets*

Then

- $\mathcal{E}\mathrm{xp}[A](\Lambda) = \mathcal{E}\mathrm{xp}_L[A_{\mathrm{new}}](\Lambda)$
- $\mu_C * \mathcal{E}\mathrm{xp}[A](\Lambda) = \mathcal{E}\mathrm{xp}_L[\mu_C(A_{\mathrm{new}})](\Lambda)$
- $\mathcal{S}\big[\mathcal{E}\mathrm{xp}_L[A](\Lambda)\big] = \mathcal{E}\mathrm{xp}[\mathcal{S}A](L^{-1}\Lambda)$

Thus the action of $\mathbb{T}$ on $\mathcal{E}\mathrm{xp}[A]$ is equivalent to the composition $\mathbb{T}_{\mathcal{E}\mathrm{xp}}$ of the maps $A \mapsto A_{\mathrm{new}}$ followed by $A \mapsto \mu_C(A)$ followed by $A \mapsto \mathcal{S}[A]$. Our conclusion is

$$\mathbb{T}\mathcal{E}\mathrm{xp}[A](\Lambda) = \mathcal{E}\mathrm{xp}[\mathbb{T}_{\mathcal{E}\mathrm{xp}}A](L^{-1}\Lambda)$$

Guided by the hierarchical case, we write

$$A(X) = \square(X)e^{-V(X)} + R(X)$$

where $V = \int v(\phi_x)dx$ and v is a polynomial such as (4.8) which is determined by running coupling constants $\vec{\lambda}$. We introduce a carefully chosen Banach space for the functionals R as follows. First, $R(X)$ must have n_{der} derivatives, R_j with $j = 0, 1, \ldots, n_{\mathrm{der}}$. These are j-multilinear functionals

$$f_1, \ldots, f_j \mapsto R_j(X, \phi; f_1, \ldots, f_j)$$

on $\mathcal{C}^p(X)$, bounded in the natural norm

$$\|R_j(X, \phi)\| := \sup_{f_1, \ldots, f_j} \big|R_j(X, \phi; f_1, \ldots, f_j)\big|$$

where each f_i is in the unit ball of $\mathcal{C}^p(X)$. We remove the ϕ dependence by taking a weighted supremum over ϕ,

$$\|R_j(X)\|_G := \sup_{\phi} \|R_j(X, \phi)\| G^{-1}(X, \phi),$$

G is a weight which controls growth in ϕ. Then we set

$$\|R(X)\|_{G,h} := \sum_{j=0}^{n_{\mathrm{der}}} \frac{h^j}{j!} \|R_j(X, \phi)\|$$

and we remove the X dependence by

$$\|R\|_{G,h,A} = \sum_{X \ni 0} \|R(X)\|_h \, A^{|X|}. \tag{4.11}$$

$|X|$ is the number of cubes Δ in X and A is a (large) positive number so that a small norm $\|R\|_{G,h,A}$ means that R is almost supported on sets X which are single cubes. There is also $|R|_{A,h}$ in which the supremum over ϕ is replaced by evaluation at $\phi = 0$. We set $\|R\| := \|R\|_{G,h,A} + \lambda^{-2}|R|_{h=1,A}$. If R is not translation invariant under translation of X and ϕ, then the $\sum_{X \ni 0}$ in (4.11) is replaced by $\sup_\delta \sum_{X \supset \Delta}$.

A functional $F(\Lambda, \phi)$ can be written in more than one way as $\mathcal{E}\mathrm{xp}[\Box e^{-V} + R](\Lambda)$ and thus does not determine $(\vec{\lambda}, R)$ unless one imposes *normalization* conditions on the first few terms of the Taylor expansion

$$\sum \frac{1}{j!} R_j(X, 0; \phi, \ldots, \phi)$$

of R. These normalization conditions have the same role as in the hierarchical case and their content is that the Taylor series of R should not contain terms such as $\int : \phi^4 :$ that can be canceled by a change of v. Therefore $\mathbb{T}$ is composed with a further map $\mathcal{E}$ that normalizes R by a shift in v. The composition $\mathcal{E}\mathbb{T}_{\mathcal{E}\mathrm{xp}}$ induces the action $(\vec{\lambda}, R) \mapsto (\vec{\lambda}', R')$ on the coordinates. The maps $\mathcal{E}$ is studied in detail in [BDH98a].

To control the remainder after *first order* perturbation theory, one defines the main part of the remainder

$$r'_{\mathrm{main}} := \mathbb{T}_{\mathcal{E}\mathrm{xp}}[A] - \tilde{A} \text{ with } A = \exp(-V)\Box$$

where $\tilde{A}$ equals $\exp(-V)\Box$ with new coupling constants

$$\lambda \to L^{-4[\phi]+D}\lambda, \quad \beta \to L^{-2[\phi]+D}\beta, \quad \zeta \to \zeta, \text{ etc.}$$

as explained in subsection 2.7.

The analogue of Proposition 4.1 is a result of the form: for L sufficiently large and $\vec{\lambda}$ sufficiently small depending on L, if $\|R\| \leq 2K|\lambda|^q$, then $\|R'\| \leq 2K_L|\lambda|^q$.

In dimension $D > 4$, a first order result like this can control the remainder for all iterations of the RG because the factor $L^{-4[\phi]+D}$ decreases λ so that the hypotheses remain true. More details will be given in [Bry01, BMS]. The papers [BDH95, BDH98b] follow the same general scheme, but with second order perturbation theory.

4.4 Strong Coupling

These lectures have concentrated entirely on the case λ small where the perturbation to the Gaussian is small. Correspondingly, for self-repelling walk, the self-repulsion is small. There are very interesting problems for the case where λ is *large*. In Constructive QFT, Glimm-Jaffe-Spencer [GJS76a, GJS76b] found convergent expansions for strong coupling $\lambda\phi^4$ and proved that $\lambda \gg 1$ forces broken symmetry: two pure phases selected by $\phi_x = \pm\,\mathrm{const}$. By the τ isomorphism, the case where ϕ is complex is relevant to the study of polymers with strong self-interactions. Strong self-interactions $\sum v(\tau_x)$ with the property that $v(t)$ has a minimum at $t > 0$ should entice a polymer into assuming a collapsed configuration instead of an extended configuration so that the expected end-to-end distance grows very slowly if at all with the number of steps. Can one use the RG to study correlations in these cases? A stab at the hierarchical case was made in [GI95].

The analysis of integrals $\int d\phi \exp(-S(\phi))$ where ϕ is complex and the minima of the action $S(\phi)$ are at $\phi \neq 0$ is hard because the symmetry $\phi_x \to e^{i\theta}\phi_x$ implies that the action has degenerate minima on an S^1 orbit. Fluctuations $\phi_x \to e^{i\theta_x}\phi_x$ where θ_x varies slowly with x are not well coerced by S when Λ is large. Nevertheless, RG expansions which are uniform in Λ have recently been constructed for the lattice model by Balaban [Bał95, Bal96a, Bal96c, Bal96b, Bal98b, Bal98c, Bal98a]. In principle his methods should apply, via the τ

isomorphism, to self-interacting walk problems. This could give very complete results on *correlations* for collapsed phases of self-interacting walks. The mean field theory of broken symmetry for the supersymmetric model is the image under the τ isomorphism of Donsker-Varadhan theory [Lut83, BMM91, BS95, BS97]. Balaban's methods are in principle capable of very accurately controlling corrections to the Donsker-Varadhan large deviation principle even for hard cases without compactness. But we need some creative simplifications to his papers in order to start moving in this direction.

References

[AB84] M. F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.

[Aiz82] M. Aizenman. Geometric analysis of φ^4 fields and Ising models, Parts I and II. *Commun. Math. Phys.*, **86**:1–48, (1982).

[Bał95] Tadeusz Bałaban. A low temperature expansion for classical N-vector models. I. A renormalization group flow. *Comm. Math. Phys.*, 167(1):103–154, 1995.

[Bal96a] Tadeusz Balaban. Localization expansions. I. Functions of the "background" configurations. *Comm. Math. Phys.*, 182(1):33–82, 1996.

[Bal96b] Tadeusz Balaban. A low temperature expansion for classical N-vector models. II. Renormalization group equations. *Comm. Math. Phys.*, 182(3):675–721, 1996.

[Bal96c] Tadeusz Balaban. The variational problems for classical N-vector models. *Comm. Math. Phys.*, 175(3):607–642, 1996.

[Bal98a] Tadeusz Balaban. The large field renormalization operation for classical N-vector models. *Comm. Math. Phys.*, 198(3):493–534, 1998.

[Bal98b] Tadeusz Balaban. A low temperature expansion for classical N-vector models. III. A complete inductive description, fluctuation integrals. *Comm. Math. Phys.*, 196(3):485–521, 1998.

[Bal98c] Tadeusz Balaban. Renormalization and localization expansions. II. Expectation values of the "fluctuation" measures. *Comm. Math. Phys.*, 198(1):1–45, 1998.

[BCG+78] G. Benfatto, N. Cassandro, G. Gallavotti, F. Nicolo, E. Olivieri, E. Presutti, and E. Scacciatelli. Some probabilistic techniques in field theory. *CMP*, 59:143–166, 1978.

[BDH95] D. C. Brydges, J. Dimock, and T.R. Hurd. The short distance behavior of ϕ^4_3. *CMP*, 172:143–185, 1995.

[BDH98a] D. Brydges, J. Dimock, and T. R. Hurd. Estimates on renormalization group transformations. *Canad. J. Math.*, 50(4):756–793, 1998.

[BDH98b] D. Brydges, J. Dimock, and T. R. Hurd. A non-Gaussian fixed point for ϕ^4 in $4-\epsilon$ dimensions. *Comm. Math. Phys.*, 198(1):111–156, 1998.

[BEI92] D. C. Brydges, S. Evans, and J. Imbrie. Self avoiding walk on a hierarchical lattice in four dimensions. *Annals of Probability*, 20:82–124, 1992.

[BG95] Giuseppe Benfatto and Giovanni Gallavotti. *Renormalization group*. Princeton University Press, Princeton, NJ, 1995. Princeton Paperbacks.

[BIa] D. C. Brydges and J. Imbrie. End to end distance from the Green's function for a hierarchical self-avoiding walk in four dimensions. In preparation.

[BIb] D. C. Brydges and J. Imbrie. The Greens function for a hierarchical self-avoiding walk in four dimensions. In preparation.

[BLGZJ73] E. Brezin, J. C. Le Guillou, and J. Zinn-Justin. Approach to scaling in renormalized perturbation theory. *PR*, D8:2418–2430, 1973.

[BMM91] D. C. Brydges and I. Munoz-Maya. An application of Berezin integration to large deviations. *Journal of Theoretical Probability*, 4:371–389, 1991.

[BMS] D. Brydges, P. Mitter, and B. Scoppola. Cluster expansions without analyticity. In preparation.

[Bry01] D. C. Brydges. Taming functional integrals. Nachdiplomvorlesung, Wintersemester 2000/01, ETH, Zürich, Mathematik, 2001.

[BS95] D.C. Brydges and G. Slade. The diffusive phase of a model of self-interacting walks. *Probab. Th. Rel. Fields*, **103**:285–315, (1995).

[BS97] Erwin Bolthausen and Uwe Schmock. On self-attracting d-dimensional random walks. *Ann. Probab.*, 25(2):531–572, 1997.

[Erd55] A. Erdelyi. *Asymptotic Expansions*. Dover, New York, 1955.

[Fed87] P. Federbush. Quantum field theory in 90 minutes. *Bull. AMS*, 17:93–103, 1987.

[Fis98] Michael E. Fisher. Renormalization group theory: its basis and formulation in statistical physics. *Rev. Modern Phys.*, 70(2):653–681, 1998.

[FMRS87] J. Feldman, J. Magnen, V. Rivasseau, and R. Seneor. Construction and Borel summability of infrared ϕ^4_4 by a phase space expansion. *CMP*, 109:437–480, 1987.

[Frö82] Jürg Fröhlich. On the triviality of $\lambda\varphi^4_d$ theories and the approach to the critical point in $d> atop(- - -)4$ dimensions. *Nuclear Phys. B*, 200(2):281–296, 1982.

[GI95] Steven E. Golowich and John Z. Imbrie. The broken supersymmetry phase of a self-avoiding random walk. *Comm. Math. Phys.*, 168(2):265–319, 1995.

[GJ87] J. Glimm and A. Jaffe. *Quantum Physics, A Functional Integral Point of View*. 2nd edition. Springer, Berlin, 1987.

[GJS74] J. Glimm, A. Jaffe, and T. Spencer. The Wightman axioms and particle structure in the $P(\phi)_2$ quantum field model. *Ann. Math.*, 100:585, 1974.

[GJS76a] James Glimm, Arthur Jaffe, and Thomas Spencer. A convergent expansion about mean field theory. I. The expansion. *Ann. Physics*, 101(2):610–630, 1976.

[GJS76b] James Glimm, Arthur Jaffe, and Thomas Spencer. A convergent expansion about mean field theory. II. Convergence of the expansion. *Ann. Physics*, 101(2):631–669, 1976.

[GK83] K. Gawedzki and A. Kupiainen. Block spin renormalization group for dipole gas and $(\bigtriangledown\phi)^4$. *Ann. Phys.*, 147:198, 1983.

[GK85] K. Gawędzki and A. Kupiainen. Massless lattice φ_4^4 theory: Rigorous control of a remormalizable asymptotically free model. *Commun. Math. Phys.*, **99**:199–252, (1985).

[Gol95] S. E Golowich. The self-avoiding walk in four dimensions. In V. Rivasseau, editor, *Constructive Physics*, pages 301–309, Berlin, Heidelberg, New York, 1995. Springer-Verlag.

[Hör83] Lars Hörmander. *The analysis of linear partial differential operators. II.* Springer-Verlag, Berlin, 1983. Differential operators with constant coefficients.

[HS92] T. Hara and G. Slade. Self-avoiding walk in five or more dimensions. I. The critical behaviour. *Commun. Math. Phys.*, **147**:101–136, (1992).

[HT87] Takashi Hara and Hal Tasaki. Rigorous renormalization group analysis of ϕ^4 field theory. *Progr. Theoret. Phys. Suppl.*, (92):26–45, 1987.

[ID89] Claude Itzykson and Jean-Michel Drouffe. *Statistical field theory. Vol. 1.* Cambridge University Press, Cambridge, 1989. From Brownian motion to renormalization and lattice gauge theory.

[Imb94] J. Imbrie. End-to-end distance for a four dimensional self-avoiding walk. In J.S. Feldman, R. Froese, and L.M. Rosen, editors, *Mathematical Quantum Field Theory I: Field Theory and Many-Body Theory*, pages 171–189, Providence, Rhode Island, 1994. CRM Proceedings & Lecture Notes, Volume 7, 1994, American Mathematical Society.

[LJ87] Y. Le Jan. Temps local et superchamp, Seminaire de Probabilités xxi. In *Lecture Notes in Mathematics, Vol 1247*. Springer Verlag, New York, Berlin, 1987.

[LJ88] Y. Le Jan. On the Fock space representation of functionals of the occupation field and their renormalisation. *J. Funct. Anal.*, 80:88–108, 1988.

[Lut83] J. M. Luttinger. The asymptotic evaluation of a class of path integrals. II. *J. Math. Phys.*, 24(8):2070–2073, 1983.

[McK80] A.J. McKane. Reformulation of $n \to 0$ models using anticommuting scalar fields. *Physics Lett. A*, 76:22, 1980.

[MS93] N. Madras and G. Slade. *The Self-Avoiding Walk*. Birkhäuser, Boston, 1993.

[NS97] A. Naddaf and T. Spencer. On homogenization and scaling limit of some gradient perturbations of a massless free field. *CMP*, 183:55–84, 1997.

[NW81] C. W. Newman and A. L. Wright. An invariance principle for certain dependent sequences. *Ann. Probab.*, 9:671–675, 1981.

[NW82] C. W. Newman and A. L. Wright. Associated random variables and martingale inequalities. *Z. Wahrsch. Verw. Gebiete*, 59:361–371, 1982.

[PS80] G. Parisi and N. Sourlas. Self-avoiding walk and supersymmetry. *J. Phys. Lett.*, 41:1403–1406, 1980.

[RW00] L. C. G. Rogers and David Williams. *Diffusions, Markov processes, and martingales. Vol. 1.* Cambridge University Press, Cambridge, 2000. Foundations, Reprint of the second (1994) edition.

[Sim79] B Simon. *Functional integration and quantum physics.* Academic Press, New York, 1979.

[VEFS93] A. Van Enter, R. Fernandéz, and A. Sokal. Regularity properties and pathologies of position space renormalization-group transformations: scope and limitations of Gibbsian theory. *J. Stat. Phys.*, 72:879–1167, 1993.

[VW73] G. Velo and A. Wightman, editors. *Constructive quantum field theory.* Springer-Verlag, Berlin, 1973. The 1973 "Ettore Majorana" International School of Mathematical Physics, Erice (Sicily), 26 July–5 August 1973, Lecture Notes in Physics, Vol. 25.

[WK74] K. G. Wilson and J. Kogut. The renormalization group and the ϵ expansion. *Phys. Rep. (Sect C of Phys Lett.)*, 12:75–200, 1974.