Pisot numeration systems and beyond

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Automata Theory and Symbolic Dynamics Workshop

Pisot definitions

Pisot number

Pisot-Vijayaraghavan number An algebraic integer is a Pisot number if its algebraic conjugates λ (except itself) satisfy

$$|\lambda| < 1$$

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Example The golden ratio $\frac{1+\sqrt{5}}{2}$ is the dominant root of

$$X^2 - X - 1$$

The Tribonacci number is the dominant root of

$$X^3 - X^2 - X - 1$$

Pisot definitions

Let σ be a substitution over the alphabet $\mathcal{A}$

Pisot substitution σ is primitive and its Perron–Frobenius eigenvalue (for its incidence matrix) is a Pisot number

Irreducible Pisot substitution The characteristic polynomial of its incidence matrix is irreducible

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Irreducible Pisot substitution The characteristic polynomial of its incidence matrix is irreducible

Tribonacci substitution

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

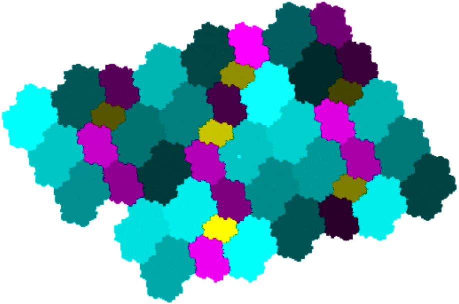
$$\text{Its incidence matrix is } M_\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Its characteristic polynomial is $X^3 - X^2 - X - 1$. Its Perron-Frobenius eigenvalue $\beta > 1$ is a Pisot number

The Pisot conjecture

Substitutive structure + Algebraic assumption (Pisot)

= Order (discrete spectrum)



Discrete spectrum = translation on a compact group

Substitutive dynamical systems

Let σ be a primitive substitution over $\mathcal{A}$. Let $\omega = (\omega_n)$ with $\sigma(\omega) = \omega$ be an infinite word generated by σ. Let S be the shift

$$S((\omega_n)_n) = (\omega_{n+1})_n$$

The symbolic dynamical system generated by σ is (X_σ, S)

$$X_\sigma := \overline{\{S^n(\omega);\ n \in \mathbb{N}\}} \subset \mathcal{A}^{\mathbb{N}}$$

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Question Under which conditions is it possible to give a geometric representation of a substitutive dynamical system as a translation on a compact abelian group ? (discrete spectrum)

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The Pisot Conjecture Dates back to the 80's

[Bombieri-Taylor, Rauzy,Thurston]

If σ is a Pisot irreducible substitution, then (X_σ, S) has discrete spectrum

Substitutive dynamical systems

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The symbolic dynamical system generated by σ is (X_σ, S)

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Example In the Fibonacci case

$$\sigma \colon a \mapsto ab, b \mapsto a$$

(X_σ, S) is isomorphic to $(\mathbb{R}/\mathbb{Z}, R_{\frac{1+\sqrt{5}}{2}})$

$$R_{\frac{1+\sqrt{5}}{2}} \colon x \mapsto x + \frac{1+\sqrt{5}}{2} \bmod 1$$

Substitutive dynamical systems

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The symbolic dynamical system generated by σ is (X_σ, S)

$$X_\sigma := \overline{\{S^n(\omega);\ n \in \mathbb{N}\}} \subset \mathcal{A}^{\mathbb{N}}$$

The Pisot Conjecture

If σ is a Pisot irreducible substitution, then (X_σ, S) has discrete spectrum

The conjecture is proved for two-letter alphabets

[Host, Barge-Diamond, Hollander-Solomyak]

Tribonacci's substitution [Rauzy '82]

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

$$1213121121312121 3 \cdots$$

Question Is it possible to give a geometric representation of the associated substitutive dynamical system X_σ as a translation on an abelian compact group ?

Yes ! (X_σ, S) is isomorphic to a translation on the two-dimensional torus

Question How to produce explicitly a fundamental domain for this translation ?

Rauzy fractal G. Rauzy introduced in the 80's a compact set with fractal boundary that tiles the plane which provides a geometric representation of (X_σ, S)
$\rightsquigarrow$Thurston for beta-numeration

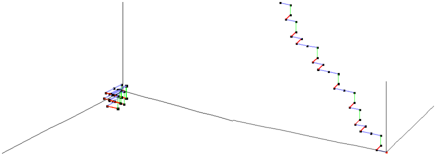
The Tribonacci fractal as a geometric representation

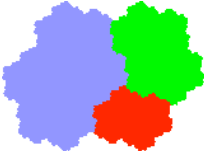
Consider the Tribonacci substitution $\sigma : 1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 1$
One represents $\sigma^\infty(1)$ as a broken line by abelianization

$$f : \{1,2,3\}^* \to \mathbb{Z}^3,\ 1 \mapsto \vec{e}_1,\ 2 \mapsto \vec{e}_2,\ 3 \mapsto \vec{e}_3,$$

$$f(w) = |w|_1 \vec{e}_1 + |w|_2 \vec{e}_2 + |w|_3 \vec{e}_3,$$

that we will be projected according to the eigenspaces of M_σ





A geometric representation of substitutive dynamical systems

Abelianisation Let d stand for the cardinality of $\mathcal{A}$

$$f\colon w \in \mathcal{A}^\star \mapsto (|w|_1, |w|_2, \cdots, |w|_d) \in \mathbb{N}^d$$

Let σ be a Pisot unimodular substitution

Let $\omega \in \mathcal{A}^{\mathbb{N}}$ with $\sigma(\omega) = \omega$

Let π_c denote the projection onto the contracting eigenplane of σ along its expanding eigenline

The Rauzy fractal of σ is defined as

$$\mathcal{R}_\sigma := \overline{\{\pi_c \circ f(\omega_0 \cdots \omega_{n-1}) \mid n \in \mathbb{N}\}}$$

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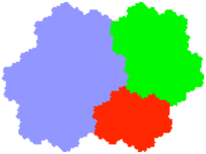
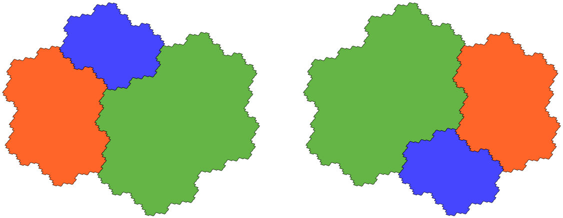
It is subvided into d pieces

$$\mathcal{R}_\sigma(i) := \overline{\{\pi_c \circ f(\omega_0 \cdots \omega_{n-1}) \mid \ \omega_n = i,\ n \in \mathbb{N}\}}$$



Rauzy fractal and dynamics

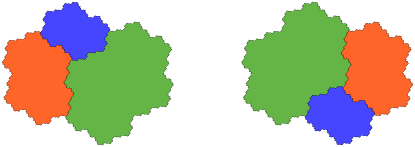
One first defines an exchange of pieces acting on the Rauzy fractal



Rauzy fractal and dynamics

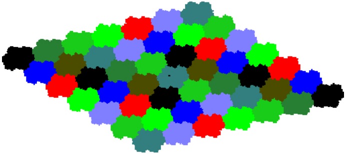
One first defines an exchange of pieces acting on the Rauzy fractal.

This due to the fact that the subtiles are disjoint in measure



This exchange of pieces factorizes into a translation of $\mathbb{T}^2$

This due to the fact that the Rauzy fractal tiles periodically the plane



Pisot substitution conjecture

Word substitutions Let σ be a Pisot irreducible substitution, then

- (X_σ, S) has discrete spectrum
- its associated Rauzy fractal provides a periodic tiling

β-numeration Let β be a Pisot number The β-shift X_β has discrete spectrum

Example Tribonacci numeration with β Pisot root of $X^3 - X^2 - X - 1$

β-numeration 111 is not allowed

Tribonacci dynamics

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

Theorem [Rauzy'82] (X_σ, S) is measure-theoretically isomorphic to the translation R_β on the two-dimensional torus $\mathbb{T}^2$

$$R_\beta : \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + (1/\beta, 1/\beta^2)$$

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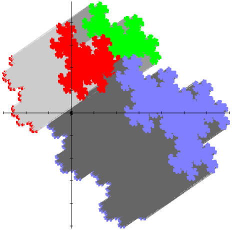
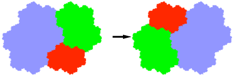
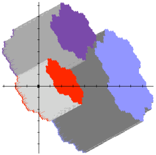
Markov partition for the toral automorphism $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$



In terms of entropy

Substitution σ	β numeration	
$(X_\sigma, S) \sim (\mathcal{R}_\sigma, E)$	$(X_\beta, +1)$	0 entropy
exchange of pieces	odometer	
(X_σ, σ)	$(X_\beta^r, S) \sim ([0,1], T_\beta)$	positive entropy
$(\mathbb{T}^d, M_\sigma)$	(X_β, S) 2-sided β-shift	





Beyond the Pisot conjecture

How to reach nonalgebraic parameters ?

Theorem [Rauzy'82] (X_σ, S) is measure-theoretically isomorphic to the translation R_β on the two-dimensional torus $\mathbb{T}^2$

$$R_\beta : \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + (1/\beta, 1/\beta^2)$$

- We want to find symbolic realizations for toral translations
- We want to reach nonalgebraic parameters by considering convergent products of matrices
- We want to consider not only a substitution but a sequence of substitutions

How to reach nonalgebraic parameters ?

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$\rightsquigarrow$ Multidimensional continued fractions algorithms

S-adic expansions

Definition An infinite word ω is said *S*-adic if there exist

- a finite set of substitutions $\mathcal{S}$
- an infinite sequence of substitutions $(\sigma_n)_{n\geq 1}$ with values in $\mathcal{S}$

such that

$$\omega = \lim_{n\to+\infty} \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(0)$$

Arnoux-Rauzy words

$$\sigma_1: \begin{array}{ll} 1 & \mapsto 1 \\ 2 & \mapsto 21 \\ 3 & \mapsto 31 \end{array} \qquad \sigma_2: \begin{array}{ll} 1 & \mapsto 12 \\ 2 & \mapsto 2 \\ 3 & \mapsto 32 \end{array} \qquad \sigma_3: \begin{array}{ll} 1 & \mapsto 13 \\ 2 & \mapsto 23 \\ 3 & \mapsto 3 \end{array}$$

$$\omega = \lim_{n \to \infty} \sigma_{i_0} \sigma_{i_1} \cdots \sigma_{i_n}(1)$$

and every letter in $\{1, 2, 3\}$ occurs infinitely often in $(i_n)_{n \geq 0}$

Example The Tribonacci substitution and its fixed point

• There exist AR words that are (measure-theoretically) weak mixing [Cassaigne-Ferenczi-Messaoudi]

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Example The Tribonacci substitution and its fixed point

- The set of the letter density vectors of AR words has zero measure [Arnoux-Starosta]
- There exist AR words that are (measure-theoretically) weak mixing [Cassaigne-Ferenczi-Messaoudi]

Toward S-adic Rauzy fractals

Inspired by Rauzy's program : generalization of the
Sturmian case

Our program is to associate with any translation acting on $\mathbb{T}^d$ (i.e., with any line in $\mathbb{R}^d$)

- an S-adic sequence

$$\omega = \lim_{n \to +\infty} \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(0)$$

- a Rauzy fractal whose coded trajectories correspond to the S-adic system, i.e.,
  - its subpieces are disjoint in measure
  - it tiles periodically
- and to be able to get information on the topology of the Rauzy fractal

S-adic expansions

$$\omega = \lim_{n \to +\infty} \sigma_1 \sigma_2 \cdots \sigma_n(0)$$

Algebraically Generalized Perron–Frobenius eigendirection

One considers an infinite product of matrices

$$M_1 \cdots M_n \cdots$$

with entries in $\mathbb{N}$

Does there exist a vector $\vec{v}$ such that

$$\bigcap_k M_1 \cdots M_k \left(\mathbb{R}_+^d\right) = \mathbb{R}_+ \vec{v} \quad ?$$

S-adic expansions

$$\omega = \lim_{n \to +\infty} \sigma_1 \sigma_2 \cdots \sigma_n(0)$$

Arithmetically Weak and strong convergence of multidimensional continued fraction algorithms

Theorem There exists $\delta > 0$ s.t. for almost every (α, β), there exists $n_0 = n_0(\alpha, \beta)$ s.t. for all $n \geq n_0$

$$|\alpha - p_n/q_n| < \frac{1}{q_n^{1+\delta}}, \; |\beta - r_n/q_n| < \frac{1}{q_n^{1+\delta}}$$

where p_n, q_n, r_n are produced by Brun or by Jacobi-Perron algorithm

Brun [Ito-Fujita-Keane-Ohtsuki '96]

Jacobi-Perron [Broise-Guivarc'h '99]

Lyapunov exponents

Let $\mathcal{T}$ be a finite set of substitutions and (D, S, ν) with $D \subset \mathcal{T}^{\mathbb{N}}$ be an ergodic shift equipped with a probability measure ν

The Lyapunov exponents $\theta_1, \theta_2, \ldots, \theta_d$ of (D, S, ν) are recursively defined by

$$\theta_1 + \theta_2 + \cdots + \theta_k = \lim_{n \to \infty} \frac{1}{n} \int_D \log \| \wedge^k (M_0 \cdots M_{n-1}) \| \, d\nu$$

where $\wedge^k$ denotes the k-fold wedge product

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The S-adic system (D, Σ, ν) satisfies the Pisot condition if

$$\theta_1 > 0 > \theta_2 \geq \theta_3 \geq \cdots \geq \theta_d$$

Theorem [Avila-Delecroix] The Arnoux-Rauzy S-adic system is Pisot

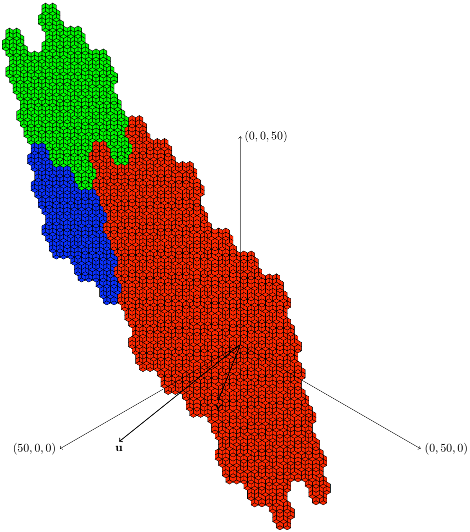
S-adic Pisot conjecture

Theorem [B., Steiner, Thuswaldner]

- For almost every $(\alpha, \beta) \in [0, 1]^2$, the S-adic system associated with the Brun multidimensional continued fraction algorithm of (α, β) is measurably conjugate to the translation by (α, β) on the torus $\mathbb{T}^2$
- For almost every Arnoux-Rauzy word, the associated S-adic system has discrete spectrum

Conjecture Every S-adic Pisot system has discrete spectrum

An example



- $S = \{\sigma_1, \sigma_2, \sigma_3\}$ (Arnoux-Rauzy)
- $\varphi : 1 \mapsto 1123,$
  $2 \mapsto 23,$
  $3 \mapsto 123$
  (Chacon substitution)
- $s = \lim_{k\to\infty} \varphi^k(1)$
- $\omega = \lim_{n\to\infty} \sigma_{s_0}\sigma_{s_1}\cdots\sigma_{s_n}(1)$
- 2-balancedness [B.,Cassaigne, Steiner]

Theorem The symbolic dynamical system generated by ω is measure-theoretically isomorphic to a rotation on $\mathbb{T}^2$