Timed automata

models, languages, dynamics

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PIMS/EQINOCS Workshop on Automata Theory and Symbolic Dynamics

Context

Timed automata

- A model for verification of real-time systems
- Invented by Alur and Dill in early 1990s
- Precursors: time Petri nets (Bethomieu)
- Now: an efficient model for verification, supported by tools (UPPAAL)
- A popular researh topic (¿8000 citation for papers by Alur and Dill)
  - modeling and verification
  - decidability and algorithmics
  - automata and language theory
  - very recent: dynamics
- Inspired by TA: hybrid automata, data automata, automata on nominal sets

Outline

Before we begin: timed words and languages

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- For us it sits in $(\mathbb{R}_+ \times \Sigma)^*$ (words on an infinite alphabet), that is $w =$ $(0.8, a), (2.66, b), (1.5, b), (0, a), (3.14159, b), (2.71828, b)$.

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- Geometrically w is a point in several copies of $\mathbb{R}^n$:

$$w = (0.8, 2.66, 1.5, 0, 3.14159, 2.71828) \in \mathbb{R}^6_{abbabb}$$

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- A *timed language* is a set of timed words - examples below.

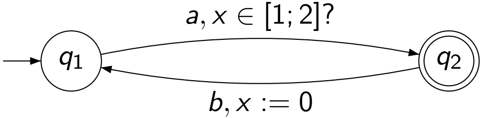
What are TA

Recipe: how to make a timed automaton

- take a finite automaton
- put it into continuous time
- add some variables $x_1, \ldots, x_n$, called clocks.
- make all them run: $\dot{x}_i = 1$ everywhere.
- add guards to some transitions (e.g. $x_3 < 7$)
- add resets to some transitions (e.g. $x_2 := 0$)
- serve and enjoy!

An example of a timed automaton

- Timed automaton (we forget to write $\dot{x} = 1$):



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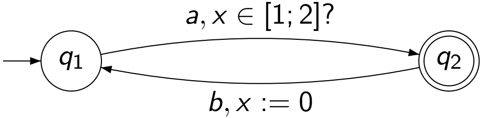

- Its run

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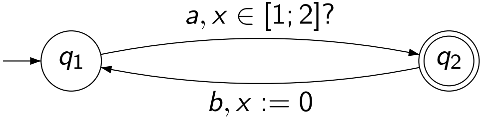

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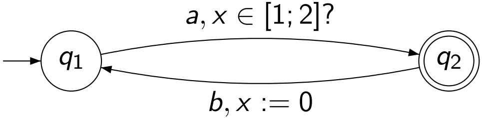

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- Its *trace* $1.83\,a\,4.1\,b\,1\,a$ a *timed word*
- Its *timed language*: set of all the traces starting in q_1, ending in q_2:

$$\{t_1\, a\, s_1\, b\, t_2\, a\, s_2\, b \ldots t_n\, a \mid \forall i . t_i \in [1;2]\}$$

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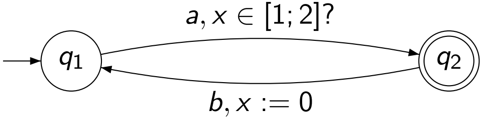

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Observation

Clock value of x: time since the last reset of x.

Some simple exercises

Draw timed automata for specifications:

- Request a arrives every 5 minutes.

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- Request a arrives every 5 to 7 minutes.
- a arrives every 5 to 7 minutes; and b arrives every 3 to 10 minutes.
- Request a is serviced within 2 minutes by c or rejected within 1 minute by r.
- The same, but a arrives every 5 to 7 minutes.

Modeling exercise 2

Scheduling

Schedule two jobs on one CPU and one printer with a total execution time up to 16 minutes.

- Job 1 : Compute (10 min); Print (5 min)
- Job 2 : Download (3 min); Compute (1 min); Print (2 min)

Try it :

1. without preemption;
2. with preemptible computing.

Main theorem

Theorem (Alur, Dill)

Reachability is decidable for timed automata.

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Classical formulation

Empty language problem is decidable for TA

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Classical formulation

Empty language problem is decidable for TA

Both are the same

Non-empty language $\Leftrightarrow$ Reach(Init,Fin)

Proof idea

- Split the state space $Q \times \mathbb{R}^n$ into regions s.t.
  - all the states in one region have the same behavior;
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- What does it mean: the same behavior?
- How to invent it?

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Two difficulties

- What does it mean: the same behavior? Bisimulation.
- How to invent it? A&D invented it using ideas of Berthomieu (Time Petri nets). In fact it is rather natural.

Region equivalence

Definition

Two states of a TA are region equivalent: $(q, \mathbf{x}) \approx (p, \mathbf{y})$ if

- Same location: $p = q$
- Same integer parts of clocks: $\forall i\, (\lfloor x_i \rfloor = \lfloor y_i \rfloor)$
- Same order of fractional parts of clocks
  $\forall i, j\, (\{x_i\} < \{x_j\} \Leftrightarrow \{y_i\} < \{y_j\})$

Look at the picture!

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Definition

Equivalence classes of $\approx$ are called regions.

Lemma (Region equivalence is a bisimulation)

Equivalent states can make the same transitions, and arrive to equivalent states.

Decision algorithm

- Build a region automaton RA
  - States are regions.
  - There is a transition $r_1 \xrightarrow{a} r_2$ if some (all) element of r_1 can go to some element of r_2 on a.
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- Check whether some final region in RA is reachable from initial region.

Outline

1. TA: the model
   - Decidability
2. Timed language theory
3. Timed symbolic dynamics
4. Conclusions

Closure properties

Definition

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Fact

Determinization impossible for timed automata.

Decidability properties

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Proof.

Immediate from Alur&Dill's theorem. □

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Proof.

Encoding of runs of Minsky Machine as a timed languages. □

Reminder: regular expressions

Definition

Regular expressions: $E ::= 0 \mid \varepsilon \mid a \mid E + E \mid E \cdot E \mid E^*$

Theorem (Kleene)

Finite automata and regular expression define the same class of languages.

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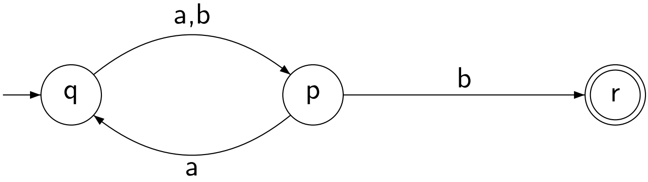
Example



$$((a+b)a)^*(a+b)b$$

Timed regular expressions

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Semantics:

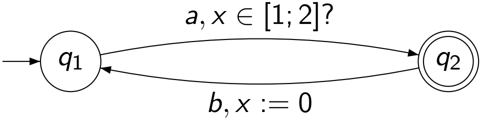
$$\|\underline{\mathbf{t}}\| = \mathbb{R}_{\geq 0} \quad \|a\| = \{a\} \qquad \|0\| = \emptyset \quad \|\varepsilon\| = \{\varepsilon\}$$

$$\|E_1 \cdot E_2\| = \|E_1\| \cdot \|E_2\| \qquad \|E_1 + E_2\| = \|E_1\| \cup \|E_2\|$$

$$\|\langle E\rangle\|_I = \{\sigma \in \|E\| \mid \ell(\sigma) \in I\} \qquad \|E^*\| = \|E\|^*$$

$$\|E_1 \wedge E_2\| = \|E_1\| \cap \|E_2\| \qquad \|[a \mapsto z]E\| = [a \mapsto z]\|E\|$$

A good example and a theorem



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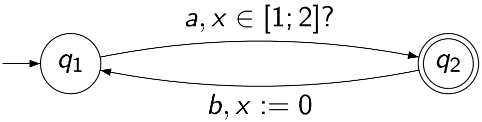


$$\{L = \{t_1\, a\, s_1\, b\, t_2\, a\, s_2\, b \ldots t_n\, a \mid \forall i . t_i \in [1;2]\}$$

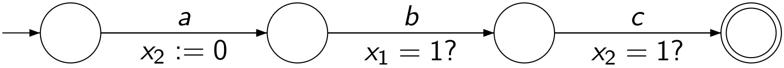
An expression for L : $\left(\langle \underline{\mathbf{t}} a \rangle_{[1;2]} \underline{\mathbf{t}} b\right)^*$

Theorem (A., Caspi, Maler,95)

Timed Automata and Timed regular expressions (with $\wedge$ and $[a \mapsto z]$) define the same class of timed languages

A nasty example

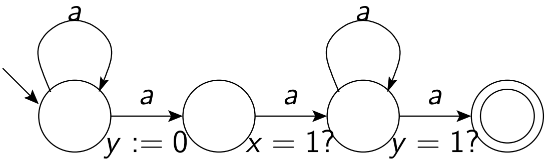
Intersection needed [ACM]



$$\{t_1 a t_2 b t_3 c \mid t_1 + t_2 = 1, t_2 + t_3 = 1\} = \underline{\mathbf{t}} a \langle \underline{\mathbf{t}} b \underline{\mathbf{t}} c \rangle_1 \wedge \langle \underline{\mathbf{t}} a \underline{\mathbf{t}} b \rangle_1 \underline{\mathbf{t}} c$$

Another nasty example

Renaming needed [Herrmann]



$$[b \mapsto a]\Big((\underline{\mathbf{t}}a)^*\langle \underline{\mathbf{t}}b(\underline{\mathbf{t}}a)^*\rangle_1 \wedge \langle(\underline{\mathbf{t}}a)^*\underline{\mathbf{t}}b\rangle_1(\underline{\mathbf{t}}a)^*\Big).$$

Outline

Preliminary considerations

- Aim: describe timed regular languages as dynamical systems
- Solution: from Nicolas Basset's MSc thesis (2010)
- Limitations: deterministic automata, bounded intervals between events

Shifts

Full timed shift

- Fix Σ an alphabet and $M \in \mathbb{R}$.
- Let $C = \Sigma \times [0; M]$ - a compact set (with a natural metrics).
- $S = C^{\mathbb{Z}}$: set of bi-infinite timed words (with a natural metrics). E.g. $\dots t_{-1}a_{-1}t_0a_0t_1a_1t_2\dots$
- Shift $\sigma : S \to S$ E.g. $\sigma(\{c_n\}) = \{c_{n+1}\}$.

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The simplest example: studied by Weiss and Lindenstrauss!

- $\Sigma = \{a\}; M = 1; C \cong [0; 1]$
- $S = [0; 1]^{\mathbb{Z}} = \{t_{-1}, t_0, t_1, \dots\}$

Timed subshifts

Definition

A subshift $X \subset S$ with X closed and σ-invariant.

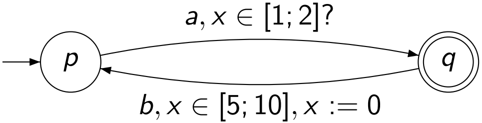
Two standard ways to define subshifts

- by an open set of forbidden patterns;
- by a closed set of allowed patterns for some lengths.

A timed automaton

- which is deterministic, w/o initial final states;
- with closed guards;
- corresponds to a timed subshift;
- we call it sofic!

Timed dynamics – an example



exercice

Compute forbidden patterns.

What about entropy?

We try to compute it for $[0;1]^{\mathbb{Z}}$

- Define $C(n, \epsilon)$: size of ϵ-net in $[0;1]^n$.
- Compute it: $C(n, \epsilon) = (1/\epsilon)^n$.

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- Growth rate of C:

$$h_\epsilon = \lim_n \frac{\log C(n,\epsilon)}{n} = \log \frac{1}{\epsilon}.$$

- Entropy $h = \lim_{\epsilon\to 0} h_\epsilon =$

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The same is true for almost all reasonable timed subshifts.

Entropy renormalized

We still try!

Given a timed subshift L

- Define L_n -set of timed words of length n (btw $L_n \subset \mathbb{R}^n \times \Sigma^n$)
- Define $C(n, \epsilon)$: size of ϵ-net in L_n.
- Compute it: $C(n, \epsilon) = (1/\epsilon)^n \operatorname{Vol}(L_n)$.

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- Compute it: $C(n,\epsilon) = (1/\epsilon)^n \operatorname{Vol}(L_n)$.
- Growth rate of C:

$$h_\epsilon = \lim_n \frac{\log C(n,\epsilon)}{n} = \log \frac{1}{\epsilon} + \lim_n \frac{\log \operatorname{Vol}(L_n)}{n}.$$

- We call the last term *volumic entropy*:

$$H(L) = \lim_n \frac{\log \operatorname{Vol}(L_n)}{n}$$

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- Compute it: $C(n,\epsilon) = (1/\epsilon)^n \operatorname{Vol}(L_n)$.
- Growth rate of C:

$$h_\epsilon = \lim_n \frac{\log C(n,\epsilon)}{n} = \log\frac{1}{\epsilon} + \lim_n \frac{\log \operatorname{Vol}(L_n)}{n}.$$

- We call the last term *volumic entropy*:

$$H(L) = \lim_n \frac{\log \operatorname{Vol}(L_n)}{n}$$

Information production in a timed subshift

Outline

It was in this talk

- Timed automata: a beautiful variant of automat coming from practice
- Infinite-state, but many questions decidable
- A non-trivial theory of languages
- Symbolic dynamics can be defined, many thing remain to study
- . . . in particular the flow

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- Next talk (Aldric): how to characterize and compute the volumes and the volumic entropy.
- After that (Nicolas): MME for timed automata
- Merci EQINOCS, more results will follow