The Third Moment of Quadratic L-functions

Moments of L-functions Workshop

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Quadratic L-functions

For $d, m \in \mathbb{Z}$, $\chi_d(m) = \left(\frac{d}{m}\right)$ Kronecker symbol,

$$L(s, \chi_d) = \sum_{m=1}^{\infty} \left(\frac{d}{m}\right) m^{-s} = \prod_{p \text{ prime}} \left(1 - \left(\frac{d}{p}\right) p^{-s}\right)^{-1}$$

Convergent for $\Re(s) > 1$, meromorphic continuation to $\mathbb{C}$.
Functional equation: for d squarefree, $a = \frac{1-\chi_d(-1)}{2}$,

$$(\pi/d)^{-(s+a)/2}\Gamma((s+a)/2)L(s, \chi_d)$$

is symmetric under $s \mapsto 1 - s$.
Question: what are the probabilistic moments of $L(\frac{1}{2}, \chi_d)$ as d varies?

The Third Moment

Theorem (Diaconu, W.)

For $W : \mathbb{R} \to \mathbb{R}$ smooth, compactly supported, with bounded derivatives,

$$\sum_{\substack{d \text{ odd} \\ \text{squarefree}}} L(\frac{1}{2}, \chi_{2d})^3 W(d/X) = XQ_W(\log X) + R\hat{W}(3/4)X^{3/4} + O(X^{2/3+\epsilon})$$

Previous work on the third moment: Soundararajan (2000), Diaconu, Goldfeld, Hoffstein (2003), Young (2013).
Young's result has $O(X^{3/4+\epsilon})$. Our secondary term of size $X^{3/4}$ was conjectured by Q. Zhang (2005). Random matrix models don't seem to predict it.
Later I'll discuss what multiple Dirichlet series predict for higher moments.

The Multiple Dirichlet Series Approach

Goal: meromorphically continue to $s_1 = s_2 = s_3 = 1$, $s_4 = \frac{2}{3} + \epsilon$

$$
\begin{aligned}
Z(s_1, s_2, s_3, s_4) &= \sum_{d=1}^{\infty} L(s_1, \chi_d) L(s_2, \chi_d), L(s_3, \chi_d) d^{-s_4} \\
&= \sum_{d, m_1, m_2, m_3} \left(\frac{d}{m_1 m_2 m_3} \right) m_1^{-s_1} m_2^{-s_2} m_3^{-s_3} d^{-s_4} \\
&= \sum_{m_1, m_2, m_3} L(s_4, \tilde{\chi}_{m_1 m_2 m_3}) m_1^{-s_1} m_2^{-s_2} m_3^{-s_3}
\end{aligned}
$$

Advantage: Z inherits many functional equations from the L-functions.
Difficulties: Issues with the prime 2, χ versus $\tilde{\chi}$.
Terms where d and $m_1 m_2 m_3$ are not squarefree.

Multiple Dirichlet Series Background

Goldfeld, Hoffstein (1985), Diaconu, Goldfeld, Hoffstein (2003), Chinta, Gunnells (2010) Brubaker, Bump, Friedberg (2011).
I'll describe the first moment case and skim over issues with the prime 2.

$$Z(s_1, s_2) = \sum_{d=1}^{\infty} L(s_1, \chi_d) d^{-s_2} = \sum_{d,m} \left(\frac{d}{m}\right) m^{-s_1} d^{-s_2} = \sum_{m=1}^{\infty} L(s_2, \tilde{\chi}_m) m^{-s_1}$$

I'll describe the functional equations and meromorphic continuation.

Multiple Dirichlet Series Background

$\sum_{d=1}^{\infty} L(s_1, \chi_d) d^{-s_2}$ should have symmetry in $s_1 \mapsto 1 - s_1$, $s_2 \mapsto s_2 + s_1 - \frac{1}{2}$, inherited from the symmetry of $L(s_1, \chi_d)$.

Problem: non-squarefree d's. Solution: redefine the residue symbol.

Let $\chi'_d(m) = \left(\frac{d}{m}\right)' := \begin{cases} \sqrt{\gcd(d,m)} \left(\frac{d/\gcd(d,m)}{m/\gcd(d,m)}\right) & \text{if gcd is square} \\ 0 & \text{if not} \end{cases}$

- Matches the standard residue symbol whenever d or m is squarefree.
- $L(s, \chi'_d)$ and $L(s, \chi_d)$ only differ at finitely many primes.
- $(\pi/d)^{-(s+a)/2} \Gamma((s+a)/2) L(s, \chi'_d)$ is symmetric in $s \mapsto 1 - s$ for all d.

Functional Equations

$$Z(s_1, s_2) = \sum_{d=1}^{\infty} L(s_1, \chi'_d) d^{-s_2} = \sum_{d,m} \left(\frac{d}{m}\right)' m^{-s_1} d^{-s_2} = \sum_{m=1}^{\infty} L(s_2, \tilde{\chi}'_m) m^{-s_1}$$

has functional equations in

$$s_1 \mapsto 1 - s_1, \quad s_2 \mapsto s_2 + s_1 - \frac{1}{2}$$

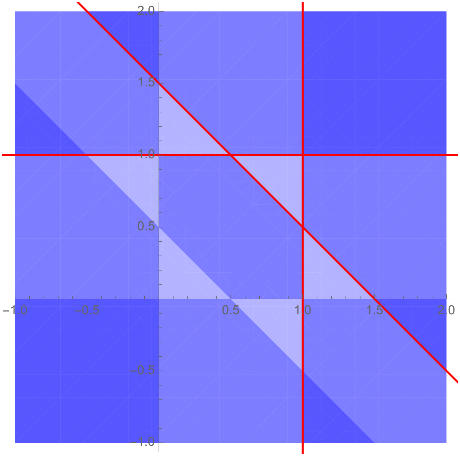
and

$$s_1 \mapsto s_1 + s_2 - \frac{1}{2}, \quad s_2 \mapsto 1 - s_2.$$

These generate a group of order 6 isomorphic to S_3.

Meromorphic Continuation

- Initial domain of meromorphicity $\Re(s_1 + s_2) > 3/2$ and poles
- Domain and poles after applying the functional equations
- Domain and poles after applying Bochner's convexity principle



Meromorphic continuation to $\mathbb{C}^2$ yields an asymptotic for the first moment.

Back to the Third Moment

$$Z(s_1, s_2, s_3, s_4) = \sum_{d=1}^{\infty} L(s_1, \chi_d) L(s_2, \chi_d), L(s_3, \chi_d) d^{-s_4}$$
$$= \sum_{d, m_1, m_2, m_3} \left(\frac{d}{m_1 m_2 m_3} \right) m_1^{-s_1} m_2^{-s_2} m_3^{-s_3} d^{-s_4}$$
$$= \sum_{m_1, m_2, m_3} L(s_4, \tilde{\chi}_{m_1 m_2 m_3}) m_1^{-s_1} m_2^{-s_2} m_3^{-s_3}$$

Replace $\left(\frac{d}{m_1 m_2 m_3} \right)$ with a function $H(m_1, m_2, m_3, d)$ which:

- Matches the standard residue symbol whenever d or m is squarefree.
- Only affects the L-functions $L(s_1, \chi_d) L(s_2, \chi_d) L(s_3, \chi_d)$ and $L(s_4, \tilde{\chi}_{m_1 m_2 m_3})$ at finitely many primes.
- Makes the functional equations uniform in d, $m_1 m_2 m_3$, when these are not squarefree.

Back to the Third Moment

Properties of the multiple Dirichlet series

$$Z(s_1,s_2,s_3,s_4) = \sum_{d,m_1,m_2,m_3} H(m_1,m_2,m_3,d)m_1^{-s_1}m_2^{-s_2}m_3^{-s_3}d^{-s_4}$$

- Group of 192 functional equations, generated by

$$\sigma_i : \begin{cases} s_i \mapsto 1-s_i \\ s_4 \mapsto s_4+s_i-\frac{1}{2} \end{cases} \quad \text{for } i=1,2,3, \quad \sigma_4 : \begin{cases} s_4 \mapsto 1-s_4 \\ s_i \mapsto s_i+s_4-\frac{1}{2} \end{cases}$$

- Meromorphic continuation to $\mathbb{C}^4$
- 12 poles: $s_1=1$, $s_2=1$, $s_3=1$, $s_4=1$, $s_1+s_4=\frac{3}{2}$, $s_2+s_4=\frac{3}{2}$, $s_3+s_4=\frac{3}{2}$, $s_1+s_2+s_4=2$, $s_1+s_3+s_4=2$, $s_2+s_3+s_4=2$, $s_1+s_2+s_3+s_4=\frac{5}{2}$, $s_1+s_2+s_3+2s_4=3$.
- Specializing $s_1=s_2=s_3=\frac{1}{2}$:
pole of order 8 at $s_4=1$, simple pole at $s_4=\frac{3}{4}$.

Our Work

Meromorphic continuation, poles, and residues of Z
$\Rightarrow$ asymptotic for the third moment of $L(\frac{1}{2}, \chi_d)$.
However, this asymptotic includes non-squarefree d, and we changed the definition of the L-function for these d values.
Need to sieve for squarefree d values (this was the new part of our paper).

The Sieving Step

$\sum_{d \text{ squarefree}} L(\frac{1}{2}, \chi_d)^3 d^{-s}$ can be expressed in terms of:

$$\frac{(s-1)^7(s-\frac{3}{4})}{(s+1)^8} \sum_{\substack{d, m_1, m_2, m_3 \\ \gcd(dm_1m_2m_3, c)=1}} \frac{H(m_1, m_2, m_3, d)\chi_{c_1}(m_1m_2m_3)\chi_{c_2}(d)}{(m_1m_2m_3)^{1/2}d^s}$$

for $c = c_1c_2c_3$ squarefree. Call this expression $\tilde{Z}^{(c)}(s, \chi_{c_1}, \chi_{c_2})$.
Need an upper bound for $\tilde{Z}^{(c)}(s, \chi_{c_1}, \chi_{c_2})$, $\Re(s) > \frac{2}{3}$, in terms of c_1, c_2, c_3.

$$\sigma_4\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_2\sigma_1\sigma_4 : (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, s) \mapsto (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1-s)$$

gives a functional equation, implying convexity bound :

$$\tilde{Z}^{(c)}(s, \chi_{c_1}, \chi_{c_2}) \ll c_1^{3(1-\Re(s))+\delta} c_2^{\frac{5}{2}(1-\Re(s))+\delta} c_3^{3(1-\Re(s))+\delta}$$

A Stronger Bound for $\tilde{Z}^{(c)}(s, \chi_{c_1}, \chi_{c_2})$

$$\tilde{Z}^{(c)}(s, \chi_{c_1}, \chi_{c_2}) \ll c_1^{3(1-\Re(s))+\delta} c_2^{\frac{5}{2}(1-\Re(s))+\delta} c_3^{3(1-\Re(s))+\delta}$$

The powers of c_1, c_2 are small enough to imply convergence of the sieving sum, but the power of c_3 is not. We'll lower it to $2 - \frac{5}{2}\Re(s) + \delta$.
Induction on the number of prime factors of c_3.
We'll use the local generating functions

$$f_{\epsilon_1,\epsilon_2}(s,p) := \sum_{\substack{a,b_1,b_2,b_3 \geq 0 \\ a \equiv \epsilon_1 \mod 2 \\ b_1+b_2+b_3 \equiv \epsilon_2 \mod 2}} H(p^{b_1}, p^{b_2}, p^{b_3}, p^a) p^{-as-\frac{1}{2}b_1-\frac{1}{2}b_2-\frac{1}{2}b_3}$$

$$|f_{0,0}(s,p)| = |1 + 3p^{-1} + p^{-2s} + \cdots| > A$$
$$|f_{1,0}(s,p)| = |p^{-s} + p^{-3s} + \cdots| < Bp^{-\Re(s)}$$
$$|f_{0,1}(s,p)| = |3p^{-1/2} + p^{-3/2} + \cdots| < Cp^{-1/2}$$
$$f_{1,1}(s,p) = 0$$

Recursive Refinement of Bounds

For the inductive step, we introduce one more prime factor p into c.

$$\begin{aligned}\tilde{Z}^{(c)}(s,\chi_{c_1},\chi_{c_2}) &= \tilde{Z}^{(pc)}(s,\chi_{pc_1},\chi_{c_2})\chi_{c_2}(p)f_{1,0}(s,p)\\ &+ \tilde{Z}^{(pc)}(s,\chi_{c_1},\chi_{pc_2})\chi_{c_1}(p)f_{0,1}(s,p)\\ &+ \tilde{Z}^{(pc)}(s,\chi_{c_1},\chi_{c_2})f_{0,0}(s,p)\end{aligned}$$

$$\begin{aligned}\tilde{Z}^{(c)}(s,\chi_{c_1},\chi_{c_2}) &= \tilde{Z}^{(pc)}(s,\chi_{pc_1},\chi_{c_2})\chi_{c_2}(p)f_{1,0}(s,p)\\ &+ \tilde{Z}^{(pc)}(s,\chi_{c_1},\chi_{pc_2})\chi_{c_1}(p)f_{0,1}(s,p)\\ &+ \tilde{Z}^{(pc)}(s,\chi_{c_1},\chi_{c_2})f_{0,0}(s,p)\end{aligned}$$

By the inductive hypothesis and the estimates for local generating functions, the blue terms are all

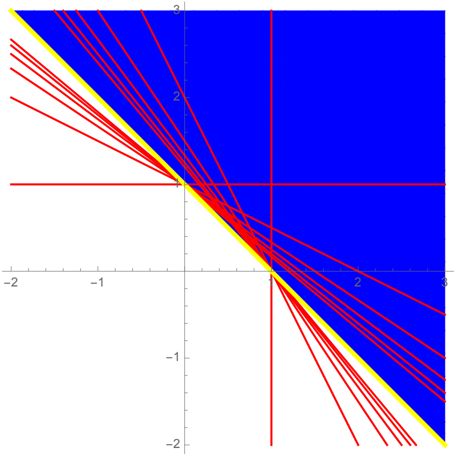
$$\ll c_1^{3(1-\Re(s))+\delta} c_2^{\frac{5}{2}(1-\Re(s))+\delta} (pc_3)^{2-\frac{5}{2}\Re(s)+\delta}$$

and thus the red term is as well.

Higher Moments

For the fourth and higher moments, the multiple Dirichlet series will have:

- Infinitely many functional equations, infinitely many poles.
- Poles accumulate at natural boundary of meromorphic continuation.
- The ray $(\frac{1}{2}, \ldots \frac{1}{2}, s)$, $s > \frac{1}{2}$ is in the domain, but not accessible by functional equations.
- This ray intersects infinitely many poles $\Rightarrow$ infinitely many secondary terms in asymptotics.



Higher Moments

Problems:

- Define the modified residue symbol for the fourth and higher moments. W. (2014), Patnaik, Puskas (2019), Diaconu, Pasol (forthcoming)
- Meromorphically continue to $(\frac{1}{2}, \ldots \frac{1}{2})$.

Thank you!