# Twisted first moment of $\mathrm{GL}(3) \times \mathrm{GL}(2)$ $L$-functions

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# GL-what‽ $L$-functions?

We all know and love Maass forms: $f\colon \mathbb{H} \to \mathbb{C}$ that are

- Laplace eigenfunctions, $\Delta f = (1/4 + t_f^2) f$;
- automorphic,

$$f\Bigg(\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in \mathrm{SL}(2,\mathbb{Z})} z\Bigg) = f(z);$$

- of polynomial growth as $z \to i\infty$.

$$L\text{-functions: } L(s, f) = \sum_{n \geq 1} \frac{n\text{th Fourier coefficient}}{n^s}$$

This is clearly a GL(2) thing!

A cute fact (Iwasawa decomposition): $\gamma \in \mathrm{GL}(2,\mathbb{R})$ can be decomposed as

$$\gamma = \begin{pmatrix} y & x \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \delta & \\ & \delta \end{pmatrix}$$

where

$$x \in \mathbb{R}, \quad y > 0, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ orthogonal}, \quad \delta \neq 0$$

meaning

$$\mathbb{H} \cong \mathrm{GL}(2,\mathbb{R})/(\mathrm{O}(2,\mathbb{R}) \cdot \mathbb{R}^{\times}).$$

$\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{H}$ by multiplication of the coset representatives!

In summary, $L$-functions:

- $u_j$ an orthonormal basis of GL(2) Maass cusp forms with Hecke eigenvalues $\lambda_j(n)$:

$$L(s, u_j) = \sum_{n \geq 1} \frac{\lambda_j(n)}{n^s}$$

- $f$ a GL(3) Maass form with "Fourier" coefficients $A(n, m)$:

$$L(s, f) = \sum_{m \geq 1} \frac{A(m, 1)}{m^s} \quad \text{or} \quad L(s, \tilde{f}) = \sum_{m \geq 1} \frac{A(1, m)}{m^s}.$$

- We can even combine them into a GL(3) $\times$ GL(2) thing!

$$L(s, f \times u_j) = \sum_{m \geq 1} \sum_{n \geq 1} \frac{\lambda_j(n) A(n, m)}{(m^2 n)^s}.$$

## Theorem (S. (2021))

*Let $T$ be large and $M$ slightly smaller than $T$ (but not too tiny), and let $p$ be a prime less than $T$. Then*

$$\sum_{T-M\leq t_j\leq T+M} \omega_j\lambda_j(p)L\Big(\frac{1}{2}, f\times u_j\Big) + (\textit{continuous spectrum term})$$
$$= C_{f,p}TM + (\textit{power saving error})$$

*where*

$$C_{f,p} = \frac{L(1,\tilde{f})(A(p,1)p-1) + L(1,f)(A(1,p)p-1)}{p^{3/2}\pi}.$$

Why is this interesting/useful, and how does one prove it?

# Why this is interesting!

$$\sum_{T-M\leq t_j\leq T+M} \omega_j\lambda_j(p)L\Big(\frac{1}{2}, f\times u_j\Big) \quad \text{grows like} \quad C_{f,p}TM$$

In increasing level of difficulty:

1. $L(1/2, f\times u_j)\neq 0$ for infinitely many $u_j$
2. $f$ is uniquely determined by $L(1/2, f\times u_j)$
3. If you have also have a second moment, proportion of nonvanishing!

Philosophically: individual Maass (cusp) forms are hard to get your hands on, but their moments can reveal useful information!

# How one proves this...

Hecke eigenvalues $\lambda_j(n)$ have a kind of "orthogonality" relation:

$$\sum_{j\geq 1} h(t_j)\omega_j \lambda_j(m)\lambda_j(n) + (\text{continuous specturm term})$$

$$= \delta(m,n)H + \sum_{c\geq 1} \frac{S(m,n;c)}{c} H^{\pm}\Big(\frac{4\pi\sqrt{|mn|}}{c}\Big)$$

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Take $k(t) = e^{-(t-T)^2/M^2} + e^{-(t+T)^2/M^2}\dots$

The diagonal term is:

$$\mathcal{D} = \sum_{m\geq 1}\sum_{n\geq 1} \frac{A(n,m)}{(m^2n)^{1/2}}\delta(n,p)H_{m,n}.$$

We rearrange:

$$\begin{aligned}\mathcal{D} &= \sum_{(m,p)=1} \frac{A(p,m)}{mp^{1/2}}H_{m,p} + \sum_{(m,p)>1} \frac{A(p,m)}{mp^{1/2}}H_{m,p} \\ &= \sum_{(m,p)=1} \frac{A(p,m)}{mp^{1/2}}H_{m,p} + \sum_{m\geq 1} \frac{A(p,mp)}{(mp)p^{1/2}}H_{mp,p}.\end{aligned}$$

By multiplicativity, factor $A(p,m) = A(p,1)A(1,m)$ and

$$\begin{aligned}A(p,mp) &= \sum_{d|(p,mp)} \mu(d)A\Big(\frac{p}{d},1\Big)A\Big(1,\frac{pm}{d}\Big) \\ &= A(p,1)A(1,pm) - A(1,1)A(1,m).\end{aligned}$$

$$\mathcal{D} = \frac{A(p,1)}{p^{1/2}} \sum_{m\geq 1} \frac{A(1,m)}{m} H_{m,p} - \frac{1}{p^{3/2}} \sum_{m\geq 1} \frac{A(1,m)}{m} H_{mp,p}$$

Here

$$H_{m,p} = \frac{2}{\pi} \int_0^\infty k(t) V(m^2 p, t) \tanh(\pi t) t \, dt$$

and

$$V(m^2 p, t) = \frac{1}{2\pi i} \int_{(1000)} (m^2 p)^{-u} F(u) \frac{\gamma(1/2 + u, t)}{\gamma(1/2, t)} \frac{du}{u}.$$

Bring the $m$-sum all the way in:

$$\frac{2}{\pi} \int_0^\infty k(t) \tanh(\pi t) t \left( \frac{1}{2\pi i} \int_{(1000)} p^{-u} \underbrace{\Big( \sum_{m\geq 1} \frac{A(1,m)}{m^{1+2u}} \Big)}_{=L(1+2u,\tilde{f})} F(u) \frac{\gamma(\frac{1}{2}+u,t)}{\gamma(\frac{1}{2},t)} \frac{du}{u} \right) dt.$$

*Fin*