Upper bounds for negative moments of the Riemann zeta- function

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Introduction

For $\Re s > 1$,

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left(1 - p^{-s}\right)^{-1}.$$

- It has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$.
- It satisfies a functional equation $\zeta(s) \leftrightarrow \zeta(1 - s)$
- Trivial zeros at $s = -2m, m \geq 1$.
- (RH) The non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$.
- Lindelöf hypothesis $|\zeta(1/2 + it)| \ll t^{\epsilon}, \forall \epsilon > 0$.

Moments of $\zeta(s)$

- (RH) The non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$.
- (Lindelöf hypothesis) $|\zeta(1/2 + it)| = O(t^{\epsilon}), \forall \epsilon > 0$.
- Hardy and Littlewood (1916): moments of $\zeta(s)$

$$I_k(T) = \int_0^T |\zeta(1/2 + it)|^{2k} \, dt$$

Lindelöf hypothesis $\iff I_k(T) \ll T^{1+\epsilon}, k = 1, 2, \ldots.$

Moments of $\zeta(s)$

Conjecture (Keating, Snaith)

For $k > 0$,

$$\int_0^T |\zeta(1/2+it)|^{2k}\,dt \sim a_k g_k (k^2)!\, T(\log T)^{k^2},$$

$$a_k = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m},$$

$$g_k = ???.$$

- Heuristic idea:

$$\int_0^T |\zeta(1/2+it)|^{2k}\,dt \sim \int_0^T \sum_{m,n} \frac{d_k(m)d_k(n)}{\sqrt{mn}} \left(\frac{m}{n}\right)^{it} dt.$$

Moments of $\zeta(s)$

- Keep $m = n$.

$$\sum_{n \le T} \frac{d_k(n)^2}{n} \sim a_k (\log T)^{k^2}.$$

$$g_k = ???.$$

- $g_1 = 1$ (Hardy, Littlewood)
- $g_2 = \frac{1}{12}$ (Ingham)
- $g_3 = \frac{42}{9!}$ (conjecture Conrey, Gosh)
- $g_4 = \frac{24024}{16!}$ (conjecture Conrey, Gonek)

Conjecture (Keating, Snaith)

$$\textit{RMT:}\ g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

Hybrid Model

- Assume RH. Let $\rho_n = 1/2 + \gamma_n$ denote the non-trivial zeros of $\zeta(s)$.

$$\zeta(1/2+it) \approx \prod_{p \le x} (1 - p^{-1/2-it})^{-1} \prod_{\substack{\gamma_n \\ |\gamma_n - t| < \frac{1}{\log x}}} \left(1 - \frac{1/2+it}{\rho_n}\right).$$

Assume

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2k}\,dt \sim \left(\frac{1}{T}\int_0^T \Big|\prod_{p\le x}(1-p^{-1/2-it})\Big|^{-2k} dt\right)$$
$$\times \left(\frac{1}{T}\int_0^T \Big|\prod_{\substack{\gamma_n \\ |\gamma_n - t| < \frac{1}{\log x}}} \left(1 - \frac{1/2+it}{\rho_n}\right)\Big|^{-2k} dt\right)$$

Conjecture (Gonek, Hughes, Keating)

$$\int_0^T |\zeta(1/2+it)|^{2k}\,dt \sim a_k g_k (k^2)!\, T(\log T)^{k^2}.$$

Moments of $\zeta(s)$

Conjecture (Conrey, Farmer, Keating, Rubinstein, Snaith)

$$\int_0^T |\zeta(1/2+it)|^{2k}\,dt = TP_k(\log T) + O(T^{1-\delta}),$$

where P_k is a polynomial of degree k^2 and $\delta > 0$.

The same conjecture can be obtained by work of Conrey-Keating using long Dirichlet polynomials and results on divisor correlations.

- $k = 1$: Hardy, Littlewood (1916)
- $k = 2$: Ingham (1932), Heath-Brown (1979)
- $k = 3$: Ng (2016), conditional on conjectures about ternary additive divisor sums
- $k = 4$: Ng, Shen, Wong (2022), conditional on conjectures about quaternary additive divisor sums

Moments of $\zeta(s)$

- Under RH, lower bounds of the right order of magnitude for all $k > 0$, due to Ramachandra and Heath-Brown
- Unconditional sharp lower bounds for $k \geq 1$, due to Radziwill-Soundararajan
- Unconditional sharp lower bounds for $0 < k < 1$, due to Heap-Soundararajan
- Under RH, upper bounds of the right order of magnitude for $0 \leq k \leq 2$ due to Ramachandra and Heath-Brown
- Sharp upper bounds for $k = 1/n$ due to Heath-Brown, and for $k = 1 + 1/n$, due to Bettin-Chandee-Radziwill
- Unconditional sharp upper bounds for $0 \leq k \leq 2$, due to Heap-Radziwill-Soundararajan

Moments of $\zeta(s)$

Theorem (Soundararajan)

Assume the RH. Then for all positive real k and any $\epsilon > 0$,

$$\int_0^T |\zeta(1/2+it)|^{2k}\,dt \ll T(\log T)^{k^2+\epsilon}.$$

Theorem (Harper)

Assume the RH. Then for all positive real k,

$$\int_0^T |\zeta(1/2+it)|^{2k}\,dt \ll T(\log T)^{k^2}.$$

Negative moments of $\zeta(s)$

Conjecture (Gonek, 1989)

Let $k > 0$ be fixed. Uniformly for $\frac{1}{\log T} \leq \delta \leq 1$, we have

$$I_{-k}(\delta, T) = \frac{1}{T}\int_1^T \left|\zeta\left(\frac{1}{2} + \delta + it\right)\right|^{-2k} dt \asymp \left(\frac{1}{\delta}\right)^{k^2},$$

and uniformly for $0 < \delta \leq \frac{1}{\log T}$, we have

$$I_{-k}(\delta, T) \asymp \begin{cases} (\log T)^{k^2} & \text{if } k < 1/2 \\ \log(e/(\delta \log T))(\log T)^{k^2} & \text{if } k = 1/2 \\ (\delta \log T)^{1-2k}(\log T)^{k^2} & \text{if } k > 1/2. \end{cases}$$

- Random matrix theory computations (Berry-Keating; Forrester-Keating) suggest transition regimes when $k = (2n+1)/2$, for n a positive integer

"Conjectures"

Conjecture

For $\frac{1}{\log T} \le \delta \le 1$, *we have*

$$I_{-k}(\delta, T) \sim a_k \left(\frac{1}{\delta}\right)^{k^2},$$

and for $0 < \delta \le \frac{1}{\log T}$, *we have*

$$I_{-k}(\delta, T) \sim$$

$$\begin{cases} a_k (\log T)^{k^2} (\delta \log T)^{-j(2k-j)} & \text{if } j - \frac{1}{2} < k < j + \frac{1}{2} \\ a_k \log\left(\frac{e}{\delta \log T}\right) (\log T)^{k^2} (\delta \log T)^{-j(j-1)} & \text{if } k = j - \frac{1}{2} \ (j \ge 1), \end{cases}$$

and

$$a_k = \prod_p \left(1 - \frac{1}{p^{1+2\delta}}\right)^{k^2} \left(1 + \sum_{j=1}^{\infty} \frac{\mu_k(p^j)^2}{p^{(1+2\delta)j}}\right).$$

Negative moments of $\zeta(s)$

- Gonek obtained lower bounds consistent with the conjecture for all $k > 0$ in the range $\frac{1}{\log T} \leq \delta \leq 1$ and for $k < 1/2$ in the range $0 < \delta \leq \frac{1}{\log T}$.
- $$J_{-k}(T) = \sum_{0<\Im\rho\leq T} |\zeta'(\rho)|^{-2k}.$$
- Under RH and the assumption that all zeros are simple, Gonek showed that
$$J_{-1}(T) \gg T.$$
- Upper bounds?

Upper bounds for negative moments

Theorem (Bui-F., 2022)

Let $k > 0$ and $\alpha > 0$ such that $u = \frac{\log \frac{1}{\alpha}}{\log\log T} \ll 1.$

$$I_{-k}(\delta, T) \ll \begin{cases} (\log\log T)^k \left(\frac{\log(\alpha \log T)}{\alpha}\right)^{k^2} & \text{if } \alpha \log T \to \infty,\ k < 1/2, \\ (\log\log T)^k (\log T)^{ck^2} & \text{if } \alpha \asymp \frac{1}{\log T},\ k < 1/2 \\ (\log\log T)^k \left(\frac{\log(\alpha \log T)}{\alpha}\right)^{\frac{k}{2}} & \text{if } \frac{(\log\log T)^{\frac{4}{k}+\varepsilon}}{(\log T)^{\frac{1}{2k}}} \ll \alpha = o\left(\frac{1}{\log T}\right), \\ & \quad k < 1/2, \\ (\log\log T)^k \left(\frac{\log(\alpha \log T)}{\alpha}\right)^{k^2} & \text{if } \alpha \gg \frac{(\log\log T)^{\frac{4}{k}+\varepsilon}}{(\log T)^{\frac{1}{2k}}},\ k \geq 1/2, \\ T^{(1+\varepsilon)(ku - 1/2 + 2k\varepsilon)} & \text{if } \alpha = o\left(\frac{(\log\log T)^{\frac{4}{k}+\varepsilon}}{(\log T)^{\frac{1}{2k}}}\right). \end{cases}$$

Asymptotic formulas for negative moments

Corollary

Assume RH. Let $k > 0$, $C > 0$ and
$\alpha \geq \max\left\{ C\frac{(\log\log T)^{\frac{4}{k}+\varepsilon}}{(\log T)^{\frac{1}{2k}}}, \frac{(1+\varepsilon)(2k+1)\log\log T}{2\log T} \right\}$. *Then we have*

$$I_{-k}(\delta, T)\,dt = \big(1+o(1)\big)\zeta(1+2\alpha)^{k^2}\prod_p\left(1-\frac{1}{p^{1+2\alpha}}\right)^{k^2}\left(1+\sum_{j=1}^{\infty}\frac{\mu_k(p^j)^2}{p^{(1+2\alpha)j}}\right),$$

where $\mu_k(n)$ denotes the Dirichlet coefficients of $\zeta(s)^{-k}$.

The Ratios Conjecture for $\zeta(s)$

Conjecture (Farmer, 1993)

For $s = 1/2 + it$ *and complex numbers* $\alpha, \beta, \gamma, \delta$ *of size* $c/\log T$, *such that* $\Re\alpha, \Re\beta, \Re\gamma, \Re\delta > 0$ *we have*

$$\frac{1}{T}\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\,dt \sim 1 + (1 - T^{-\alpha-\beta})\frac{(\alpha-\gamma)(\beta-\delta)}{(\alpha+\beta)(\gamma+\delta)}.$$

- The conjecture implies many interesting results about zeros of $\zeta(s)$, such as the pair correlation conjecture of Montgomery.
- By adapting the "recipe" used by Conrey, Farmer, Keating, Rubinstein and Snaith to conjecture asymptotic formulas for moments of L–functions, one can make the following conjecture.

Conjecture (Conrey, Farmer, Zirnbauer, 2007)

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\,dt$$
$$= \int_0^T \Big(\frac{\zeta(1+\alpha+\beta)\zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)}A(\alpha,\beta,\gamma,\delta)$$
$$+ \big(\frac{t}{2\pi}\big)^{-\alpha-\beta}\frac{\zeta(1-\alpha-\beta)\zeta(1+\gamma+\delta)}{\zeta(1-\beta+\delta)\zeta(1-\alpha+\gamma)}A(-\beta,-\alpha,\gamma,\delta)\Big)\,dt + O\Big(T^{1/2+\epsilon}\Big),$$

where

$$A(\alpha,\beta,\gamma,\delta) = \prod_p \frac{\big(1-\frac{1}{p^{1+\gamma+\delta}}\big)\big(1-\frac{1}{p^{1+\beta+\gamma}}-\frac{1}{p^{1+\alpha+\delta}}+\frac{1}{p^{1+\gamma+\delta}}\big)}{\big(1-\frac{1}{p^{1+\beta+\gamma}}\big)\big(1-\frac{1}{p^{1+\alpha+\delta}}\big)}$$

for $|\Re\alpha|, |\Re\beta| < 1/4$,

$$\frac{1}{\log T} \ll \Re\gamma, \Re\delta < 1/4,\ \ \Im\alpha, \Im\beta, \Im\gamma, \Im\delta \ll T^{1-\epsilon}.$$

Applications of the Ratios Conjecture

- Almost all integers can be written as the sum of three cubes (V. Wang, 2022)
- Compute the lower order terms for the pair correlation of the zeros of $\zeta(s)$, which were previously heuristically computed by Bogomolny and Keating.
- Compute mollified moments of $\zeta(s)$ or other L–functions
- Obtain conjectures for moments of $|\zeta'(\rho)|$
- Compute the one-level density of zeros in families of L–functions, for test functions whose Fourier transforms have any support.

Conjecture (Chowla's conjecture)

$L(1/2, \chi) \neq 0$ *for any* χ *a Dirichlet character.*

- Soundararajan: $\geq 87.5\%$ of $L(1/2, \chi_d) \neq 0$
- Ozluk-Snyder: $\geq 93.75\%$ of $L(1/2, \chi_d) \neq 0$ by computing the one-level density of zeros with support $(-2, 2)$ (GRH)
- The Ratios Conjecture $\Rightarrow 100\%$ of $L(1/2, \chi_d) \neq 0$

The Ratios Conjecture in Random Matrix Theory

One can compute ratios of characteristic polynomials in matrix ensembles:

- Conrey-Farmer-Zirnbauer
- Borodin-Strahov
- Conrey-Forrester-Snaith
- Bump-Gamburd
- Huckleberry-Puttmann-Zirnbauer

Theorem (Conrey-Farmer-Zirnbauer)

For $\Re\gamma_k > 0$, *we have*

$$\int_{Usp(2N)} \frac{\prod_{k=1}^{K} \Lambda_A(e^{-\alpha_k})}{\prod_{k=1}^{K} \Lambda_A(e^{-\gamma_k})}\, dA$$

$$= \sum_{\epsilon\in\{-1,1\}^K} e^{N\sum_{k=1}^{K}(\epsilon_k\alpha_k-\alpha_k)} \frac{\prod_{j\le k\le K} z(\epsilon_j\alpha_j+\epsilon_k\alpha_k) \prod_{q<r\le K} z(\gamma_q+\gamma_r)}{\prod_{k=1}^{K}\prod_{q=1}^{K} z(\epsilon_k\alpha_k+\gamma_q)},$$

where $z(x) = (1-e^{-x})^{-1}$.

Ideas of proof

- Key Lemma (adapted from work of Carneiro and Chandee)

$$\log \frac{1}{|\zeta(1/2+\delta+it)|} \le \Re \sum_{p\le x} \frac{a(p)}{p^{\frac{1}{2}+\delta+it}} + \frac{\log T}{\log x} \log \frac{1}{1-x^{-\delta}} + O(1).$$

- $|\zeta(1/2+\delta+it)|^{-2k} \ll \left(\frac{1}{1-x^{-\delta}}\right)^{\frac{2k\log T}{\log x}} \exp\left(2k\Re\sum_{p\le x}\frac{a(p)}{p^{\frac{1}{2}+\delta+it}}\right).$
- Let $\mathcal{P}_j(t) = \sum_{T^{\beta_{j-1}}\le p\le T^{\beta_j}} \frac{a(p)}{p^{1/2+\delta+it}},$
- $1 = T^{\beta_0}, T^{\beta_1}, \dots, T^{\beta_K}$

$$\beta_1 \asymp \frac{\log\log T}{\log T}, \beta_j = r^{j-1}\beta_1, r > 1$$

Stop when $\beta_K =$ small constant.

Ideas of proof

- Assume $\delta \gg \frac{1}{(\log T)^{\frac{1}{2k}-\epsilon}}$.
- There are 3 possibilities for $t \in [0, T]$:

1. $t \in \mathcal{S}_0$ "Exceptional Set"

$$|\Re \mathcal{P}_1(t)| > \beta_1^{-d},\ d = 1 - \epsilon.$$

$$\int_{\mathcal{S}_0} |\zeta(1/2+\delta+it)|^{-2k} \leq \int_0^T |\zeta(1/2+\delta+it)|^{-2k} (\beta_1^d |\Re \mathcal{P}_1(t)|)^{s_0}\, dt$$

$$\leq \beta_1^{ds_0} \left(\int_0^T |\zeta(1/2+\delta+it)|^{-4k}\, dt \right)^{1/2} \left(\int_0^T |\Re \mathcal{P}_1(t)|)^{2s_0}\, dt \right)^{1/2}$$

Ideas of proof

$$\int_{\mathcal{S}_0} |\zeta(1/2+\delta+it)|^{-2k} \leq$$

$$\beta_1^{ds_0} \Big(\int_0^T |\zeta(1/2+\delta+it)|^{-4k} \, dt \Big)^{1/2} \Big(\int_0^T (|\Re \mathcal{P}_1(t)|)^{2s_0} \, dt \Big)^{1/2}$$

- Use a priori bound for the first term; use pointwise bound

$$\frac{1}{|\zeta(1/2+\delta+it)|} \ll \Big(\frac{1}{1-(\log T)^{-\delta}} \Big)^{\frac{\log T}{2 \log \log T}}.$$

- Compute moments of the sum over the primes. Need $s_0\beta_0 \leq 1$.
- Contribution from exceptional set is $o(T)$.

Ideas of proof

2. t is such that $|\Re \mathcal{P}_h(t)| \le \beta_h^{-d}$, $h \le j$, but

$$|\Re \mathcal{P}_{j+1}(t)| > \beta_{j+1}^{-d}.$$

Call this set $\mathcal{T}_j$. Let

$$E_\ell(t) = \sum_{s \le \ell} \frac{t^s}{s!}.$$

If $t \le \ell/e^2$, then

$$e^t \le (1 + e^{-\ell/2}) E_\ell(t).$$

- Since $|\Re \mathcal{P}_h(t)| \le \beta_h^{-d}$, we can approximate

$$\exp(2k\Re \mathcal{P}_h(t)) \ll E_{(\beta_h)^{-d}}(2k\Re \mathcal{P}_h(t)).$$

Ideas of proof

- Use Key Lemma with $x = T^{\beta_j}$. We get that

$$\int_{\mathcal{T}_j} |\zeta(1/2+\delta+it)|^{-2k} \le \exp\left(\frac{2k}{\beta_j}\log\frac{1}{1-T^{-\beta_j\delta}}\right)$$

$$\times \int_{\mathcal{T}_j} \exp\left(\sum_{p\le T^{\beta_j}} \frac{a(p)}{p^{1/2+\delta+it}}\right) dt$$

$$\ll \exp\left(\frac{2k}{\beta_j}\log\frac{1}{1-T^{-\beta_j\delta}}\right) \int_0^T \prod_{h\le j} E_{\beta_h^{-d}}(2k\Re\mathcal{P}_h(t))$$

$$\times \left(\beta_{j+1}^d |\Re\mathcal{P}_{j+1}(t)|\right)^{s_{j+1}}\ dt$$

- Can compute moments as long as

$$\sum_{h=0}^{j} \beta_h^{1-d} + s_{j+1}\beta_{j+1}/2 < 1.$$

- Contribution is small.

Ideas of proof

3. t is such that $|\Re \mathcal{P}_h(t)| \leq \beta_h^{-d}, h \leq K$. Call this set $\mathcal{T}_K$.
   - Use Key Lemma with $x = T^{\beta_K}$. We get that

$$\int_{\mathcal{T}_K} |\zeta(1/2+\delta+it)|^{-2k} \leq \exp\left(\frac{2k}{\beta_K} \log \frac{1}{1-T^{-\beta_K \delta}}\right) \times \int_0^T \prod_{h \leq K} E_{\beta_h^{-d}}(2k\Re \mathcal{P}_h(t))\, dt$$

   - Can compute moments as long as

$$\sum_{h=0}^{K} \beta_h^{1-d} < 1.$$

   - $$\frac{1}{T} \int_{\mathcal{T}_K} |\zeta(1/2+\delta+it)|^{-2k}\, dt \ll (\log\log T/\delta)^{k^2}.$$

Improved bounds

$$\alpha = \frac{(\log\log T)^b}{(\log T)^{\frac{1}{2k}}}, b > \frac{4}{k}.$$

- Repeat previous argument, with different choices of parameters. Obtain
$$I_{-k}(\delta, T) \ll \exp\left((\log T)^{\frac{3}{kb-1}}\right)(\log T)^{k^2}.$$
- Iterate
$$I_{-k}(\delta, T) \ll \exp\left((\log T)^{\left(\frac{3}{kb-1}\right)^m}\right)(\log T)^{k^2}.$$
- $$\dots$$
- $$I_{-k}(\delta, T) \ll (\log T)^{k^2}.$$

Thank you!