The eighth moment of the Riemann zeta function

Quanli Shen
(joint work with Nathan Ng and Peng-Jie Wong)

Workshop on Moments of L-functions
UNBC
July 29, 2022

Table of Contents

Table of Contents

The $2k$-th moments of $|\zeta(\frac{1}{2}+it)|$

- $2k$-th moments:

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt.$$

The $2k$-th moments of $|\zeta(\frac{1}{2}+it)|$

- $2k$-th moments:

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt.$$

- Lindelöf hypothesis (LH):

$$|\zeta(\tfrac{1}{2}+it)| \ll_\varepsilon (1+|t|)^\varepsilon \text{ for all } \varepsilon > 0.$$

The $2k$-th moments of $|\zeta(\frac{1}{2}+it)|$

- $2k$-th moments:

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt.$$

- Lindelöf hypothesis (LH):

$$|\zeta(\tfrac{1}{2}+it)| \ll_\varepsilon (1+|t|)^\varepsilon \text{ for all } \varepsilon > 0.$$

- The Lindelöf hypothesis is true if and only if

$$I_k(T) \ll_{k,\varepsilon} T^{1+\varepsilon} \text{ for all } k \in \mathbb{N} \text{ and all } \varepsilon > 0.$$

Bounds for $I_k(T)$

Theorem (Soundararajan, 2009)

The RH implies for any $k > 0$

$$I_k(T) \ll_k T(\log T)^{k^2+\varepsilon}.$$

Theorem (Harper, 2013)

The RH implies for any $k > 0$

$$I_k(T) \ll_k T(\log T)^{k^2}.$$

Bounds for $I_k(T)$

Theorem (Soundararajan, 2009)

The RH implies for any $k > 0$

$$I_k(T) \ll_k T(\log T)^{k^2+\varepsilon}.$$

Theorem (Harper, 2013)

The RH implies for any $k > 0$

$$I_k(T) \ll_k T(\log T)^{k^2}.$$

Theorem (Radziwill-Soundararajan, 2013, Heap-Soundararajan, 2020)

For any $k > 0$,

$$I_k(T) \gg T(\log T)^{k^2}.$$

Asymtotics for $I_k(T)$

Theorem (Hardy-Littlewood, 1918)

$$I_1(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^2 \, dt \sim T \log T.$$

Asymtotics for $I_k(T)$

Theorem (Hardy-Littlewood, 1918)

$$I_1(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^2 \, dt \sim T \log T.$$

Theorem (Ingham, 1926)

$$I_2(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^4 \, dt \sim \frac{T}{2\pi^2} (\log T)^4.$$

Higher moments ($2k = 6, 8$)

Conjecture (Folklore)

For $k > 0$,

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} \, dt \sim \frac{g_k}{(k^2)!} \cdot a_k \cdot T(\log T)^{k^2},$$

- $a_k = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m}$.
- Difficult to predict g_k.

Higher moments ($2k = 6, 8$)

Conjecture (Folklore)

For $k > 0$,

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k}\, dt \sim \frac{g_k}{(k^2)!} \cdot a_k \cdot T(\log T)^{k^2},$$

- $a_k = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m}$.
- Difficult to predict g_k.

Conjecture (Conrey and Ghosh, 1996)

$$I_3(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^6\, dt \sim \frac{42}{9!} \cdot a_3 \cdot T(\log T)^9.$$

Conjecture (Conrey and Gonek, 1998)

$$I_4(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^8\, dt \sim \frac{24024}{16!} \cdot a_4 \cdot T(\log T)^{16}.$$

Higher moments ($k \in \mathbb{N}$)

Conjecture (Keating and Snaith, 1998)

For $k \in \mathbb{N}$,

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} \, dt \sim \frac{g_k}{(k^2)!} \cdot a_k \cdot T(\log T)^{k^2}$$

where

$$g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

Higher moments ($k \in \mathbb{N}$)

Conjecture (Keating and Snaith, 1998)

For $k \in \mathbb{N}$,

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} \, dt \sim \frac{g_k}{(k^2)!} \cdot a_k \cdot T(\log T)^{k^2}$$

where

$$g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

- Method: Random Matrices Theory.

Higher moments ($k \in \mathbb{N}$)

Conjecture (Keating and Snaith, 1998)

For $k \in \mathbb{N}$,

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} \, dt \sim \frac{g_k}{(k^2)!} \cdot a_k \cdot T(\log T)^{k^2}$$

where

$$g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

- Method: Random Matrices Theory.
- Conrey, Farmer, Keating, Rubinstein and Snaith (2005) obtained the conjecture with full main terms. "Recipe" method.
- Diaconu, Goldfeld and Hoffstein (2003) also made the conjecture. Multiple Dirichlet Series.

Table of Contents

Additive divisor sum

Defination

The k-th **divisor function** *is defined by*

$$d_k(n) := \#\{(m_1, \dots, m_k) \in \mathbb{N}^k \mid m_1 \cdots m_k = n\}.$$

- $\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}, \quad \mathfrak{Re}(s) > 1.$

Additive divisor sum

Defination

The k*-th* **divisor function** *is defined by*

$$d_k(n) := \#\{(m_1, \dots, m_k) \in \mathbb{N}^k \mid m_1 \cdots m_k = n\}.$$

- $\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}, \quad \mathfrak{Re}(s) > 1.$

Defination

The **additive divisor sum** *is defined by*

$$D_{k,\ell}(x, r) := \sum_{n \leq x} d_k(n) d_\ell(n + r) \text{ for } r \in \mathbb{Z}\backslash\{0\}.$$

Additive divisor sum

Defination

The k*-th* **divisor function** *is defined by*

$$\color{red}{d_k(n)} := \#\{(m_1, \dots, m_k) \in \mathbb{N}^k \mid m_1 \cdots m_k = n\}.$$

- $\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}, \quad \mathfrak{Re}(s) > 1.$

Defination

The **additive divisor sum** *is defined by*

$$D_{k,\ell}(x, r) := \sum_{n \le x} d_k(n) d_\ell(n + r) \text{ for } r \in \mathbb{Z} \backslash \{0\}.$$

- The cases of $k \ge 2$, $\ell = 2$ have been proved by Drappeau, Motohashi, Topacogullari, many others.

Additive divisor sum

Defination

The k**-th divisor function** *is defined by*

$$d_k(n) := \#\{(m_1, \ldots, m_k) \in \mathbb{N}^k \mid m_1 \cdots m_k = n\}.$$

- $\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}, \quad \mathfrak{Re}(s) > 1.$

Defination

The **additive divisor sum** *is defined by*

$$D_{k,\ell}(x, r) := \sum_{n \leq x} d_k(n) d_\ell(n + r) \text{ for } r \in \mathbb{Z}\backslash\{0\}.$$

- The cases of $k \geq 2$, $\ell = 2$ have been proved by Drappeau, Motohashi, Topacogullari, many others.

Theorem (Motohashi, 1994)

$$D_{2,2}(x, r) = x(c_0(r)(\log x)^2 + c_1(r)(\log x) + c_2(r)) + O(x^{\frac{2}{3}+\varepsilon})$$

uniformly for $|r| \leq x^{\frac{20}{27}}$, *where* $c_0(r) = \frac{6}{\pi^2} \sum_{d|r} d^{-1}$.

Additive Divisor Conjecture (Vinogradov-Ivic-Conrey-Gonek, 1989-1998)

For every $\varepsilon > 0$ *and for every* $\varepsilon' \in (0,1)$, *we have*

$$D_{k,\ell}(x,r) = \sum_{n \le x} d_k(n) d_\ell(n+r) = main\ \ term + \mathcal{O}(x^{\frac{1}{2}+\varepsilon}),$$

uniformly for $1 \le |r| \le x^{1-\varepsilon'}$.

Additive Divisor Conjecture (Vinogradov-Ivic-Conrey-Gonek, 1989-1998)

For every $\varepsilon > 0$ *and for every* $\varepsilon' \in (0,1)$, *we have*

$$D_{k,\ell}(x,r) = \sum_{n \le x} d_k(n) d_\ell(n+r) = main\ \ term + \mathcal{O}(x^{\frac{1}{2}+\varepsilon}),$$

uniformly for $1 \le |r| \le x^{1-\varepsilon'}$.

- The main term can be expressed explicitly, and a simple form of it was obtained by Ng-Thom (2016) and Tao (blogpost, 2016).

Additive Divisor Conjecture (Vinogradov-Ivic-Conrey-Gonek, 1989-1998)

For every $\varepsilon > 0$ *and for every* $\varepsilon' \in (0,1)$, *we have*

$$D_{k,\ell}(x,r) = \sum_{n \le x} d_k(n) d_\ell(n+r) = main \ \ term + \mathcal{O}(x^{\frac{1}{2}+\varepsilon}),$$

uniformly for $1 \le |r| \le x^{1-\varepsilon'}$.

- The main term can be expressed explicitly, and a simple form of it was obtained by Ng-Thom (2016) and Tao (blogpost, 2016).

Smoothed Additive Divisor Conjecture (Non-precise version)

Let $f : [x, 2x] \times [y, 2y] \to \mathbb{R}$ *be smooth. Assume* $x \asymp y$. *For every* $\varepsilon > 0$ *and for every* $\varepsilon' \in (0,1)$, *we have*

$$\sum_{m-n=r} d_k(m) d_\ell(n) f(m,n) = main \ \ term + \mathcal{O}(x^{\frac{1}{2}+\varepsilon}),$$

uniformly for $1 \le |r| \le x^{1-\varepsilon'}$.

- Results for the case of $k = \ell = 2$ have been obtained by Duke-Friedlander-Iwaniec 1994 and Aryan 2017.

Table of Contents

The sixth moment $I_3(T)$ and the eight moment $I_4(T)$

The ternary additive divisor sum ($k = \ell = 3$): $\sum_{n \le x} d_3(n) d_3(n+r)$.

Theorem (Ng, 2021)

The smoothed ternary Additive Divisor Conjecture implies

$$I_3(T) \sim \frac{42}{9!} \cdot a_3 \cdot T(\log T)^9.$$

The sixth moment $I_3(T)$ and the eight moment $I_4(T)$

The ternary additive divisor sum ($k = \ell = 3$): $\sum_{n \le x} d_3(n) d_3(n+r)$.

Theorem (Ng, 2021)

The smoothed ternary Additive Divisor Conjecture implies

$$I_3(T) \sim \frac{42}{9!} \cdot a_3 \cdot T(\log T)^9.$$

- Ng actually proved the full main term with a power savings error.
- Conrey and Gonek (1998) previously provided a heuristic argument for this.
- Ivic (1996) showed an averaged form of the ternary Additive Divisor Conjecture implies $I_3(T) \ll T^{1+\varepsilon}$.

The sixth moment $I_3(T)$ and the eight moment $I_4(T)$

The ternary additive divisor sum ($k = \ell = 3$): $\sum_{n \leq x} d_3(n) d_3(n+r)$.

Theorem (Ng, 2021)

The smoothed ternary Additive Divisor Conjecture implies

$$I_3(T) \sim \frac{42}{9!} \cdot a_3 \cdot T(\log T)^9.$$

- Ng actually proved the full main term with a power savings error.
- Conrey and Gonek (1998) previously provided a heuristic argument for this.
- Ivic (1996) showed an averaged form of the ternary Additive Divisor Conjecture implies $I_3(T) \ll T^{1+\varepsilon}$.

The quaternary additive divisor sum ($k = \ell = 4$): $\sum_{n \leq x} d_4(n) d_4(n+r)$.

Theorem (Ng-S.-Wong, 2022+, arXiv)

The RH and the smooth quaternary Additive Divisor Conjecture implies

$$I_4(T) \sim \frac{24024}{16!} \cdot a_4 \cdot T(\log T)^{16}.$$

- Proof based on works of Ng, Conrey-Gonek and Ivic. A key new input is the application of the sharp upper bound of shifted moments of zeta.

Table of Contents

Shifted moments of zeta

Conjecture (Chandee, 2010)

Let k be an even number. Let α_1, α_2 be real and satisfy certain conditions. Then

$$\int_0^T |\zeta(\tfrac{1}{2} + \mathrm{i}(t + \alpha_1))|^k \; |\zeta(\tfrac{1}{2} + \mathrm{i}(t + \alpha_2))|^k dt$$

$$\asymp_k \begin{cases} T(\log T)^{k^2} & \text{if } \lim_{T\to\infty} |\alpha_1 - \alpha_2| \log T = 0, \\ T(\log T)^{k^2} & \text{if } \lim_{T\to\infty} |\alpha_1 - \alpha_2| \log T = c \neq 0, \\ T\left(\frac{\log T}{|\alpha_1 - \alpha_2|}\right)^{\frac{k^2}{2}} & \text{if } \lim_{T\to\infty} |\alpha_1 - \alpha_2| \log T = \infty. \end{cases}$$

- When $\alpha_1 = \alpha_2 = 0$, it is $\int_0^T |\zeta(\frac{1}{2} + \mathrm{i}t)|^{2k} dt$.

Shifted moments of zeta

Conjecture (Chandee, 2010)

Let k be an even number. Let α_1, α_2 be real and satisfy certain conditions. Then

$$\int_0^T |\zeta(\tfrac{1}{2} + \mathrm{i}(t + \alpha_1))|^k \ |\zeta(\tfrac{1}{2} + \mathrm{i}(t + \alpha_2))|^k dt$$

$$\asymp_k \begin{cases} T(\log T)^{k^2} & \text{if } \lim_{T\to\infty} |\alpha_1 - \alpha_2| \log T = 0, \\ T(\log T)^{k^2} & \text{if } \lim_{T\to\infty} |\alpha_1 - \alpha_2| \log T = c \neq 0, \\ T\left(\frac{\log T}{|\alpha_1 - \alpha_2|}\right)^{\frac{k^2}{2}} & \text{if } \lim_{T\to\infty} |\alpha_1 - \alpha_2| \log T = \infty. \end{cases}$$

- When $\alpha_1 = \alpha_2 = 0$, it is $\int_0^T |\zeta(\frac{1}{2} + \mathrm{i}t)|^{2k} dt$.
- It shows the correlation between zeta functions.

Shifted moments of zeta

Conjecture (Chandee, 2010)

Let k be an even number. Let α_1, α_2 be real and satisfy certain conditions. Then

$$\int_0^T |\zeta(\tfrac{1}{2} + \mathrm{i}(t + \alpha_1))|^k \, |\zeta(\tfrac{1}{2} + \mathrm{i}(t + \alpha_2))|^k dt$$

$$\asymp_k \begin{cases} T(\log T)^{k^2} & \text{if } \lim_{T\to\infty} |\alpha_1 - \alpha_2| \log T = 0, \\ T(\log T)^{k^2} & \text{if } \lim_{T\to\infty} |\alpha_1 - \alpha_2| \log T = c \neq 0, \\ T\left(\frac{\log T}{|\alpha_1 - \alpha_2|}\right)^{\frac{k^2}{2}} & \text{if } \lim_{T\to\infty} |\alpha_1 - \alpha_2| \log T = \infty. \end{cases}$$

- When $\alpha_1 = \alpha_2 = 0$, it is $\int_0^T |\zeta(\frac{1}{2} + \mathrm{i}t)|^{2k} dt$.
- It shows the correlation between zeta functions.
- Chandee proved the lower bound, and also an upper bound off by $(\log X)^{\varepsilon}$.

Theorem (Chandee, 2010)

Let $k_i > 0$. Let α_i be real and satisfy certain conditions. Then RH implies

$$\int_0^T |\zeta(\tfrac{1}{2} + \mathrm{i}(t + \alpha_1))|^{2k_1} \cdots |\zeta(\tfrac{1}{2} + \mathrm{i}(t + \alpha_m))|^{2k_m} dt$$

$$\ll_{\mathbf{k},\varepsilon} T(\log T)^{k_1^2 + \cdots k_m^2 + \varepsilon} \prod_{i<j} \left(\min\left\{ \frac{1}{|\alpha_i - \alpha_j|}, \log T \right\} \right)^{2k_i k_j}.$$

Theorem (Chandee, 2010)

Let $k_i > 0$. Let α_i be real and satisfy certain conditions. Then RH implies

$$\int_0^T |\zeta(\tfrac{1}{2} + \mathrm{i}(t + \alpha_1))|^{2k_1} \cdots |\zeta(\tfrac{1}{2} + \mathrm{i}(t + \alpha_m))|^{2k_m} dt$$

$$\ll_{\mathbf{k},\varepsilon} T(\log T)^{k_1^2 + \cdots k_m^2 + \varepsilon} \prod_{i<j} \left(\min \left\{ \frac{1}{|\alpha_i - \alpha_j|}, \log T \right\} \right)^{2k_i k_j}.$$

- Let $\alpha_1 = \alpha_2 = \cdots \alpha_m = 0$. It is

  $$\int_0^T |\zeta(\tfrac{1}{2} + \mathrm{i}t)|^{2k} dt \ll T(\log T)^{k^2+\varepsilon}.$$

  (Soundararajan, 2009)
- Application: Upper bounds of shifted moments can be applied to certain problems on the edge of what can be solved.
- When $|\alpha_i - \alpha_j| = 0$ for all $i < j$, the shifted moment $\ll T(\log T)^{(k_1+\cdots k_m)^2+\varepsilon}$.
  When $|\alpha_i - \alpha_j| \gg 1$ for all $i < j$, the shifted moment $\ll T(\log T)^{k_1^2+\cdots+k_m^2+\varepsilon}$.
  There is a gap between $(k_1 + \cdots k_m)^2$ and $k_1^2 + \cdots k_m^2$, which one can empoly.

Applications of shifted moments

Let f be a cusp form. Let $\chi_d(n) := \left(\frac{d}{n}\right)$ denote the Kronecker symbol.

Theorem (Soundararajan-Young, 2010)

Assume GRH. Let α_1, α_2 be real and $|\alpha_1|, |\alpha_2| \leq X$ and let $0 \leq \sigma \leq 1/\log X$. Then

$$\sum_{\substack{|d| \leq X \\ d \text{ fund. discr.}}} |L(\tfrac{1}{2} + \sigma + i\alpha_1, f \otimes \chi_d)||L(\tfrac{1}{2} + \sigma + i\alpha_2, f \otimes \chi_d)|$$

$$\ll X(\log X)^{1/2+\varepsilon} \left(1 + \min\left\{(\log X)^{1/2}, \frac{1}{|\alpha_1 - \alpha_2|^{1/2}}\right\}\right).$$

The upper bound for shifted moments leads to

Theorem (Soundararajan-Young, 2010)

Under GRH,

$$\sum_{\substack{0<d<X \\ d \text{ fund. discr.}}} L(\tfrac{1}{2}, f \otimes \chi_d)^2 \sim c \cdot X \log X,$$

where c is an explicit constant.

The upper bound for shifted moments leads to

Theorem (Soundararajan-Young, 2010)

Under GRH,

$$\sum_{\substack{0<d<X \\ d \text{ fund. discr.}}} L(\tfrac{1}{2}, f \otimes \chi_d)^2 \sim c \cdot X \log X,$$

where c is an explicit constant.

- $$L(\tfrac{1}{2}, f \otimes \chi_d) \approx \sum_{n \le d^{1+\varepsilon} \ll X^{1+\varepsilon}} \frac{\lambda_f(n) d(n)}{n^{\frac{1}{2}}}.$$

  is of length $X^{1+\varepsilon}$.

The upper bound for shifted moments leads to

Theorem (Soundararajan-Young, 2010)

Under GRH,

$$\sum_{\substack{0<d<X \\ d \text{ fund. discr.}}} L(\tfrac{1}{2}, f \otimes \chi_d)^2 \sim c \cdot X \log X,$$

where c is an explicit constant.

- $$L(\tfrac{1}{2}, f \otimes \chi_d) \approx \sum_{n \leq d^{1+\varepsilon} \ll X^{1+\varepsilon}} \frac{\lambda_f(n) d(n)}{n^{\frac{1}{2}}}.$$

  is of length $X^{1+\varepsilon}$. Hard to handle this length.

The upper bound for shifted moments leads to

Theorem (Soundararajan-Young, 2010)

Under GRH,

$$\sum_{\substack{0<d<X \\ d \text{ fund. discr.}}} L(\tfrac{1}{2}, f \otimes \chi_d)^2 \sim c \cdot X \log X,$$

where c is an explicit constant.

- $$L(\tfrac{1}{2}, f \otimes \chi_d) \approx \sum_{n \le d^{1+\varepsilon} \ll X^{1+\varepsilon}} \frac{\lambda_f(n) d(n)}{n^{\frac{1}{2}}}.$$

  is of length $X^{1+\varepsilon}$. Hard to handle this length.
- Idea:

  Length of Poly. $X^{1+\varepsilon}$ (Hard)

The upper bound for shifted moments leads to

Theorem (Soundararajan-Young, 2010)

Under GRH,

$$\sum_{\substack{0<d<X \\ d \text{ fund. discr.}}} L(\tfrac{1}{2}, f \otimes \chi_d)^2 \sim c \cdot X \log X,$$

where c is an explicit constant.

- $$L(\tfrac{1}{2}, f \otimes \chi_d) \approx \sum_{n \leq d^{1+\varepsilon} \ll X^{1+\varepsilon}} \frac{\lambda_f(n) d(n)}{n^{\frac{1}{2}}}.$$

  is of length $X^{1+\varepsilon}$. Hard to handle this length.
- Idea:

$$\text{Length of Poly. } X^{1+\varepsilon} \text{ (Hard)} \xrightarrow[\text{Tool: Upper bound of shifted moments}]{\text{is reduced to}} \frac{X}{(\log X)^{100}} \text{ (Solvable)}$$

The upper bound for shifted moments leads to

Theorem (Soundararajan-Young, 2010)

Under GRH,

$$\sum_{\substack{0<d<X \\ d \text{ fund. discr.}}} L(\tfrac{1}{2}, f \otimes \chi_d)^2 \sim c \cdot X \log X,$$

where c is an explicit constant.

- $$L(\tfrac{1}{2}, f \otimes \chi_d) \approx \sum_{n \leq d^{1+\varepsilon} \ll X^{1+\varepsilon}} \frac{\lambda_f(n) d(n)}{n^{\frac{1}{2}}}.$$
  is of length $X^{1+\varepsilon}$. Hard to handle this length.
- Idea:

$$\text{Length of Poly. } X^{1+\varepsilon} \text{ (Hard)} \xrightarrow[\text{Tool: Upper bound of shifted moments}]{\text{is reduced to}} \frac{X}{(\log X)^{100}} \text{ (Solvable)}$$

- $\sum_{\substack{0<d<X \\ d \text{ fund. discr.}}} L(\tfrac{1}{2}, \chi_d)^4$. Florea, 2017 (function field), and S., 2020.
  $\sum_{q \leq Q} \sum_{\substack{\chi \mod q \\ \chi \text{ even, primitive}}} \int_{-\infty}^{\infty} |\Lambda(\tfrac{1}{2} + iy, \chi)|^8 dy$, where Λ is the completed L-function of the Dirichlet L-function. Chandee-Li, 2014.

- In our consideration, we face polynomials of length $T^{2+\varepsilon}$.

- In our consideration, we face polynomials of length $T^{2+\varepsilon}$.
-

$$\text{Length of Poly. } T^{2+\varepsilon} \text{ (Hard)} \xrightarrow[\text{Tool: Upper bound of shifted moments}]{\text{is reduced to}} \frac{T^2}{(\log T)^{100}} \text{ (Still hard)}$$

- In our consideration, we face polynomials of length $T^{2+\varepsilon}$.
-

$$\text{Length of Poly. } T^{2+\varepsilon} \text{ (Hard)} \xrightarrow[\text{Tool: Upper bound of shifted moments}]{\text{is reduced to}} \frac{T^2}{(\log T)^{100}} \text{ (Still hard)}$$

- Idea:

$$\text{Length of Poly. } T^{2+\varepsilon} \text{ (Hard)} \xrightarrow[\text{Tool: Sharp upper bound of shifted moments}]{\text{is reduced to}} T^{2-\varepsilon} \text{ (Solvable)}$$

- In our consideration, we face polynomials of length $T^{2+\varepsilon}$.
- $$\text{Length of Poly. } T^{2+\varepsilon} \text{ (Hard)} \xrightarrow[\text{Tool: Upper bound of shifted moments}]{\text{is reduced to}} \frac{T^2}{(\log T)^{100}} \text{ (Still hard)}$$
- Idea:

$$\text{Length of Poly. } T^{2+\varepsilon} \text{ (Hard)} \xrightarrow[\text{Tool: Sharp upper bound of shifted moments}]{\text{is reduced to}} T^{2-\varepsilon} \text{ (Solvable)}$$

Sharp upper bound of shifted moments:

Theorem (Ng-S.-Wong, 2022+, arXiv)

Let $k \geq 1$. Let α_1, α_2 be real and satisfy certain conditions. Then RH implies

$$\int_0^T |\zeta(\tfrac{1}{2} + i(t + \alpha_1))|^k \, |\zeta(\tfrac{1}{2} + i(t + \alpha_2))|^k dt \ll_k T(\log T)^{\frac{k^2}{2}} \mathcal{F}(T, \alpha_1, \alpha_2)^{\frac{k^2}{2}},$$

where

$$\mathcal{F}(T, \alpha_1, \alpha_2) = \begin{cases} \min\left\{\frac{1}{|\alpha_1 - \alpha_2|}, \log T\right\} & \text{if } |\alpha_1 - \alpha_2| \leq \frac{1}{100}; \\ \log(2 + |\alpha_1 - \alpha_2|) & \text{if } |\alpha_1 - \alpha_2| > \frac{1}{100}. \end{cases}$$

- It is the version of Chandee's upper bound without $(\log X)^{\varepsilon}$.

- Let $\alpha_1 = \alpha_2 = 0$. It is

$$\int_0^T |\zeta(\tfrac{1}{2} + \mathrm{i}t)|^{2k} dt \ll T(\log T)^{k^2}.$$

(Harper, 2013)

- Together with the lower bound given by Chandee, it completes the proof of Chandee's conjecture.
- The idea has a cost of not getting an error term in the asymptotic for $I_4(T)$:

$$I_4(T) \sim \frac{24024}{16!} \cdot a_4 \cdot T(\log T)^{16}.$$

Table of Contents

Sketch of the proof

- Approximate functional equation:

$$|\zeta(\tfrac{1}{2}+\mathrm{i}t)|^8 \approx \sum_{\substack{m,n=1 \\ m,n \le T^{2+\varepsilon}}}^{\infty} \frac{d_4(m)d_4(n)}{m^{\frac{1}{2}}n^{\frac{1}{2}}} \left(\frac{m}{n}\right)^{-it}.$$

Sketch of the proof

- Approximate functional equation:

$$|\zeta(\tfrac{1}{2}+\mathrm{i}t)|^8 \approx \sum_{\substack{m,n=1 \\ m,n\leq T^{2+\varepsilon}}}^{\infty} \frac{d_4(m)d_4(n)}{m^{\frac{1}{2}}n^{\frac{1}{2}}}\left(\frac{m}{n}\right)^{-it}.$$

- $I_4(T)$ is like

$$\int_T^{2T} \sum_{\substack{m,n=1 \\ m,n\leq T^{2+\varepsilon}}}^{\infty} \frac{d_4(m)d_4(n)}{m^{\frac{1}{2}}n^{\frac{1}{2}}}\left(\frac{m}{n}\right)^{-it} dt$$

Sketch of the proof

- Approximate functional equation:

$$|\zeta(\tfrac{1}{2}+\mathrm{i}t)|^8 \approx \sum_{\substack{m,n=1 \\ m,n\le T^{2+\varepsilon}}}^{\infty} \frac{d_4(m)d_4(n)}{m^{\frac{1}{2}}n^{\frac{1}{2}}}\left(\frac{m}{n}\right)^{-it}.$$

- $I_4(T)$ is like

$$\int_T^{2T} \sum_{\substack{m,n=1 \\ m,n\le T^{2+\varepsilon}}}^{\infty} \frac{d_4(m)d_4(n)}{m^{\frac{1}{2}}n^{\frac{1}{2}}}\left(\frac{m}{n}\right)^{-it} dt$$

- Use the sharp upper bound for shifted moments of zeta to obtain

$$\int_T^{2T} \sum_{\substack{m,n=1 \\ m,n\le T^{2-\varepsilon}}}^{\infty} \frac{d_4(m)d_4(n)}{m^{\frac{1}{2}}n^{\frac{1}{2}}}\left(\frac{m}{n}\right)^{-it} dt$$

Sketch of the proof

- Approximate functional equation:

$$|\zeta(\tfrac{1}{2}+\mathrm{i}t)|^8 \approx \sum_{\substack{m,n=1 \\ m,n\le T^{2+\varepsilon}}}^{\infty} \frac{d_4(m)d_4(n)}{m^{\frac{1}{2}}n^{\frac{1}{2}}}\left(\frac{m}{n}\right)^{-it}.$$

- $I_4(T)$ is like

$$\int_T^{2T} \sum_{\substack{m,n=1 \\ m,n\le T^{2+\varepsilon}}}^{\infty} \frac{d_4(m)d_4(n)}{m^{\frac{1}{2}}n^{\frac{1}{2}}}\left(\frac{m}{n}\right)^{-it} dt$$

- Use the sharp upper bound for shifted moments of zeta to obtain

$$\int_T^{2T} \sum_{\substack{m,n=1 \\ m,n\le T^{2-\varepsilon}}}^{\infty} \frac{d_4(m)d_4(n)}{m^{\frac{1}{2}}n^{\frac{1}{2}}}\left(\frac{m}{n}\right)^{-it} dt$$

- The case $m=n$ is the diagonal term, which gives one main term.
- The case $m \neq n$ is the off-diagonal terms, which is much more complicated.

- For off-diagonal terms ($m \neq n$), use Smooth Partition of Unity to put variables in dyadic boxes:

$$\sum_{\substack{M,N \\ MN \leq T^{4-\varepsilon}, M \asymp N}} \frac{2T}{\sqrt{MN}} \sum_{1 \leq |r| \ll \frac{M}{T_0} T^{\varepsilon}} \widetilde{D}_{2,2}(f_r, r).$$

- Apply the Additive Divisor Conjecture in $\sum_{1 \leq |r| \ll \frac{M}{T_0} T^{\varepsilon}} \widetilde{D}_{2,2}(f_r, r)$.

- For off-diagonal terms ($m \neq n$), use Smooth Partition of Unity to put variables in dyadic boxes:

$$\sum_{\substack{M,N \\ MN \leq T^{4-\varepsilon}, M \asymp N}} \frac{2T}{\sqrt{MN}} \sum_{1 \leq |r| \ll \frac{M}{T_0} T^{\varepsilon}} \widetilde{D}_{2,2}(f_r, r).$$

- Apply the Additive Divisor Conjecture in $\sum_{1 \leq |r| \ll \frac{M}{T_0} T^{\varepsilon}} \widetilde{D}_{2,2}(f_r, r)$.
- Evaluation of certain contour integrals gives the other main term.

Thank you for listening!