Logic in $K - 12$:

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Logical Thinking, Mathematical Thinking, Computational Thinking: What is the Difference?
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Theme

Formal logic is not an appropriate topic for the general K-12 curriculum.

Teachers' understanding of basic logical principles are helpful/necessary in presenting many mathematical comments correctly and help students understand them.

The examples today are similar to those the 2013 course 'Logic across the Mathematics curriculum' that is described here
`http://homepages.math.uic.edu/~jbaldwin/pub/logicaug20.pdf`
[Bal13]

Overview

1. Symbols
2. Say what you are talking about!
3. Inference
4. Variables and Quantifiers
5. Engaging with real situations

Solving equations

What goes in the box?

$$8 + 4 = \square + 5$$

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How does success on the problem change as students move from 1st to 6th grade?

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TABLE 1

Percent of children offering various solutions to 8 + 4 = □ + 5

Grade	Answers Given: 7	12	17	12 and 17	Other	Number of Children
1	0	79	7	0	14	42
1 and 2	6	54	20	0	20	84
2	6	55	10	14	15	174
3	10	60	20	5	5	208
4	7	9	44	30	11	57
5	7	48	45	0	0	42
6	0	84	14	2	0	145

DECEMBER 1999

Why?

What does = mean?

Two Answers

1. CS: Evaluate
2. MATH: The two sides of the equation have the same value.

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Too much drill on meaning 1) drives out meaning 2. They should be distinguished early on.

Some grade 1-2 remedies:

1. Ask $\square = 2 + 5$ as often as $2 + 5 = \square$.
2. Is $8 = 8$?
3. Does $8 + 2 = 6 + 4$?
4. Does $7 + 4 = 15 - 4$?

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In pilot 2nd grade using these techniques 14/16 students answer 7 to opening problem.
[FLT99]
[MC01]

Say what you are talking about!

Primary School: Counting Numbers

The study is of the structure $(\mathbb{N}, 0, 1, +, \times)$.

Context matters

In this context the teacher is mathematically correct but educationally delinquent to say.

'Borrow because you can't subtract a bigger number from a smaller one'
or

'Multiplication is repeated addition.'

Middle School and High School

The study is of the structures

1. integers: $(\mathbb{Z}, 0, 1, +, \times)$,
2. rationals: $(\mathbb{Q}, 0, 1, +, \times)$,
3. reals: $(\Re, 0, 1, +, \times)$.

The two statements:

'Borrow because you can't subtract a bigger number from a smaller one'

'Multiplication is repeated addition.'

ARE FALSE

Questions and Metaphors in first grade and 7th

How many? vs Which way?
'take away' vs 'comparison

See the appendix to Bob Moses and Charles E. Cobb *Radical Equations: Math Literacy and Civil Rights*
[MC01]

Multiplication is not repeated addition

How is $\frac{3}{4} \times \frac{7}{8}$ repeated addition?

Inverse

Repeated addition motivates multiplication of a real number by a natural number; but it does not motivate multiplication of a natural number by an arbitrary real.

`https://denisegaskins.com/2008/07/01/if-it-aint-repeated-addition/#:~:text=But%20according%20to%20Devlin%3A,learn%20that%20it%20is%20not.`

Geometry is a better motivator

Freshmen in college told me, ‘I know the area is $2\ell + 2w$ or ℓw but I don’t know which.

Geometry motivates Algebra

1. Using multiplication to count the small squares in a rectangle yields distributive and commutative laws.
   And a picture to avoid my freshmen forgetting.
2. To motivate the inverse of a, ask what is the length b of the base of a rectangle with height a and area 1?
3. proportion and similarity

See Chval and Page:
`https://www.amazon.com/Books-Kathryn-Chval/s?rh=n%3A283155%2Cp_27%3AKathryn+Chval`

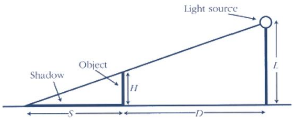
Similarity

A triangular clothes hangar and its reflection in a mirror.

Figure: Similarity Demonstrated

Foreshadowing Trigonometry

The picture below shows the variables that we will consider in these explorations.



Experiment 1 – Varying D.

Set up your light at a height that you choose. L = _________

Use your cubes to construct a tower to serve as H and measure its height. H = _______

What will be the independent variable you are controlling?

What will be the dependent variable you are measuring?

Figure: Shadow problem

Interactive Mathematics Program
`https://www.google.com/search?q=shadows+interactive+mathematics+program&rlz=1C1CHBF_enUS752US752&sxsrf=ALiCzsYIGo4VkT7hFF0YQGf4mRsgEeIcxQ%3A1652732021063&ei=dbCCYva8A5qpptQPm8-QmAc&ved=`

Inference

Basic Inference

Among Scholars this is known as the You Tube commentators' fallacy

(Saturday Morning Breakfast Cereal Comics)

Several topics

What is going on?

1. sentential logic
   a. When is an ‘if-then sentence’ true?
   b. What related sentence is equivalent to an ‘if-then sentence’?
      - The contrapositive of an implication is equivalent to the implication.
      - The converse of an implication doesn’t usually imply the implication.
2. modern logic:
   - Predicates of more than one object
   - quantification

Unfortunately true story

In a class with rather unprepared students the instructor tried to explain.

'all squares are rectangles'

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All rectangles are parallelograms, all parallelograms are quadrilaterals

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It was clear that the analogy the students needed was, 'All dogs are animals'.

Accessible examples

Example 1

If Alex is a cat then Alex is an animal.
contrapositive: If Alex is not an animal then Alex is not a cat.

Example 2

If today is Wednesday, then yesterday was Tuesday.
If yesterday wasn't Tuesday then today isn't Wednesday

Definition

A conditional is TRUE if whenever the premise is true so is the conclusion

Truth of conditionals

SCHOOL **Determine the truth value of the following statement for each set of conditions.**

If you get 100% on your test, then your teacher will give you an A.

a. You get 100%; your teacher gives you an A.

The hypothesis is true since you got 100%, and the conclusion is true because the teacher gave you an A. Since what the teacher promised is true, the conditional statement is true.

b. You get 100%; your teacher gives you a B.

The hypothesis is true, but the conclusion is false. Because the result is not what was promised, the conditional statement is false.

c. You get 98%; your teacher gives you an A.

The hypothesis is false, and the conclusion is true. The statement does not say what happens if you do not get 100% on the test. You could still get an A. It is also possible that you get a B. In this case, we cannot say that the statement is false. Thus, the statement is true.

d. You get 85%; your teacher gives you a B.

As in part c, we cannot say that the statement is false. Therefore, the conditional statement is true.

[BE02]

Defining the truth of a conditional

The resulting truth values in Example 3 can be used to create a truth table for conditional statements. Notice that a conditional statement is true in all cases except where the hypothesis is true and the conclusion is false.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

How not to explain equivalence

Key Concept — *Related Conditionals*

Statement	Formed by	Symbols	Examples
Conditional	given hypothesis and conclusion	$p \to q$	If two angles have the same measure, then they are congruent.
Converse	exchanging the hypothesis and conclusion of the conditional	$q \to p$	If two angles are congruent, then they have the same measure.
Inverse	negating both the hypothesis and conclusion of the conditional	$\sim p \to \sim q$	If two angles do not have the same measure, then they are not congruent.
Contrapositive	negating both the hypothesis and conclusion of the converse statement	$\sim q \to \sim p$	If two angles are not congruent, then they do not have the same measure.

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TOO FAST

Statements with the same truth values are said to be **logically equivalent**. So, a conditional and its contrapositive are logically equivalent as are the converse and inverse of a conditional. These relationships are summarized below.

p	q	Conditional $p \to q$	Converse $q \to p$	Inverse $\sim p \to \sim q$	Contrapositive $\sim q \to \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

source: Glencoe Geometry around 2000

Basic Inference

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Modern Logic

The 'You-Tube commenters fallacy' has long been known as the confusion between 'converse' and 'contrapositive'

Converse

$T(x, y)$ 'x tells the truth about y'
$O(x, y)$ 'x offends y'

$$1)\ (\forall y)T(\mathcal{I}, y) \rightarrow (\exists z)O(\mathcal{I}, z)$$

If I always tell the truth $\mathcal{I}$ will offend someone

$$2)\ (\exists z)O(\mathcal{I}, z) \rightarrow (\forall y)T(\mathcal{I}, y)$$

If $\mathcal{I}$ offend someone then $\mathcal{I}$ always tell the truth.

Modern Logic II

Contrapositive

$$3)\ \neg(\exists z) O(\mathcal{I}, z) \rightarrow \neg(\forall y) T(\mathcal{I}, y)$$

Recall: $\neg(\exists)A$ iff $\forall\neg A$ and $\neg\forall A$ iff $(\exists)\neg A$

$$4)(\forall z)\neg O(\mathcal{I}, z) \rightarrow (\forall y)\neg T(\mathcal{I}, y)$$

If $\mathcal{I}$ offend no one then $\mathcal{I}$ always lie.

Deduction in Algebra

(Similar to final exam of course for 8th grade teachers preparing to teach algebra I.)

A student solved the quadratic equation $\frac{5x}{x-2} = 7 + \frac{10}{x-2}$ in the following way:

Given	$\frac{5x}{x-2}$	$=$	$7 + \frac{10}{x-2}$
multiply by $x-2$	$\frac{5x}{x-2}(x-2)$	$=$	$(7 + \frac{10}{x-2})(x-2)$
distributive law	$5x$	$=$	$7x - 14 + 10$
combining like terms	$5x$	$=$	$7x - 4$
subtract $7x$ from both sides	$-2x$	$=$	-4
divide	x	$=$	2

1. Is the solution correct?
2. If not, where is the mistake?
3. Why do you think the student made the error?

DISCUSS

Deduction in Algebra

The mistake is failing to check.

The solution steps to a quadratic equations are proofs that the only solutions are among the numbers isolated.

A check is not only a search for errors in arithmetic but an essential step to avoid ‘false roots’.

Variables and Quantifiers

What's happening here?

What is the difference between these two equations?

$$x^2 + 5x + 6 = 0$$

$$xy = yx$$

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$$x^2 + 5x + 6 = 0$$

$$xy = yx$$

$$(\forall x)(\forall y)xy = yx$$

Free Variables

A variable that is not in the scope of a quantifier is free.

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An expression with free variables is a question.

What elements can be substituted for the free variables and give a true statement?

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$$x^2 = -1$$

The Angle Problem

The following statement is taken from a high school trigonometry text.

What does it mean?

$\sin A = \sin B$ if and only if
$A = B + 360K$ or $A = (180 - B) + 360K$.

The Angle Problem

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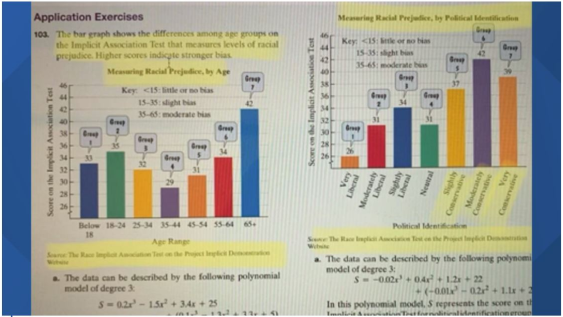
What does it mean?

$\sin A = \sin B$ if and only if
$A = B + 360K$ or $A = (180 - B) + 360K$.

$(\exists K)(A = B + 360K)$ or $(\exists K)A = (180 - B) + 360K$.

Engaging with real situations

‘statistical models’



Credit: Florida Department of Education

What? Me? Racist? More than 2 million people have tested their racial prejudice using an online version of the Implicit Association Test. Most groups' average scores fall between "slight" and "moderate" bias, but the differences among groups, by age and by political identification, are intriguing.

In this section's Exercise Set (Exercises 103 and 104), you will be working with models that measure bias:

$$S = 0.3x^3 - 2.8x^2 + 6.7x + 30$$
$$S = -0.03x^3 + 0.2x^2 + 2.3x + 24.$$

In each model, S represents the score on the Implicit Association Test. (Higher scores indicate stronger bias.) In the first model (see Exercise 103), x represents age group. In the second model (see Exercise 104), x represents political identification.

Published Problem

Here is the actual problem.
The bar graph shows the differences among age groups on the Implict Association Test that measures levels of racial prejudice. Higher scores indicate stronger bias.The data can be described by the following polynomial model of degree 3:

$$S = 0.2x^3 - 1.5x^2 + 3.4x + 25 + (0.1x^3 - 1.3x^2 + 3.3x + 5)$$

.
In this polynomial model, S represents the score on the Implicit Association Test for age group x. Simplify the model.
In the first model x represents age; in the second model it represents political group.

This isn't just political

What's wrong here?

In each category find examples of malpractice:

1. education
2. mathematics
3. statistics

https://www.washingtonpost.com/education/2022/04/21/4-math-textbook-problems-florida-prohibited/

DISCUSS

Some answers

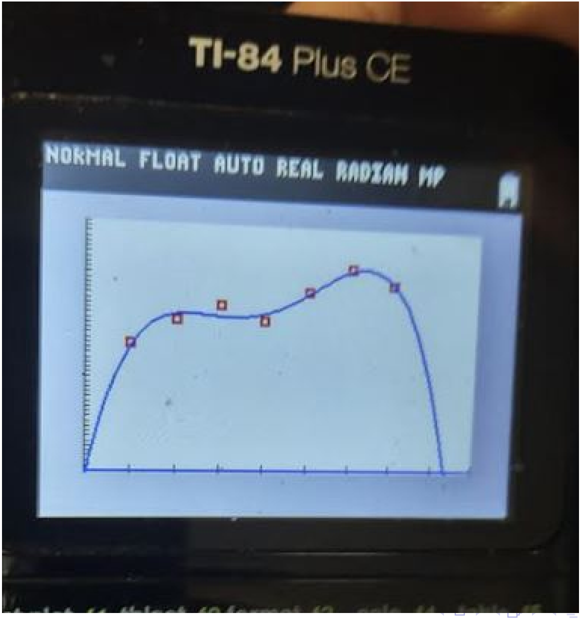
1. education: An absurdly complicated setting for a problem that would not arise.
2. mathematics:
   (a) There is no x value for the second example; the argument is a category not a number.
   (b) It makes no sense to assign a polynomial graph based only curve fitting. There must be an argument for why the curve is degree three.
   (c) The calculator can fit closely curves of any degree strictly greater than the number of maxima and minima. Unfortunately, this curve exhibits 3, so the (quadratic) derivative would have 3 distinct roots.

answers continued

3. statistics:
   (a) The data is apparently collected from people who take the exam on line. There is no randomization. Even without randomization, statistical inference should be made on a *well defined population*.
   (b) In the first graph, the curve is fit through the mean score of groups with different number of ages (and no notion of whether the number of participants is the same for each age in the group). But the curve is predicting the value for each age.
   (c) The second group has no obvious numerical domain; it would be an equally challenging problem for statistical sociology to assign a numerical rank to each person's political views.

[BBT18]

4th degree approximation



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