# Double Square Moments and Bounds for Resonance Sums for Cusp Forms

Joint Work with Professors
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Workshop on Moments of L-functions 2022

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## Introduction

- $f$ and $g$ holomorphic cusp forms for $SL_2(\mathbb{Z})$ of weights $k_1$ and $k_2$ respectively.
- Fix $0 < \beta < 1$ and $\alpha \neq 0$.
- $\varphi$ smooth with compact support in (1,2).
- Define ''resonance sum''

$$S_X(f, g, \alpha, \beta) = \sum_n \lambda_f(n)\lambda_g(n)e(\alpha n^\beta)\varphi\Big(\frac{n}{X}\Big).$$

- These weights correspond to the coefficients from the Rankin-Selberg $L$-function

$$L(s, f \times g) = \zeta(2s) \sum_{n \geq 1} \frac{\lambda_f(n)\lambda_g(n)}{n^s},$$

absolutely convergent for $Re(s) > 1$.

- The goal is to examine the oscillatory behavior of the coefficients $\lambda_f(n)\lambda_g(n)$.

## Hypothesis S

- In 1976, Vinogradov proved

$$\sum_{\substack{p\le x\\ p\equiv a(\bmod c)}} e\Big(\frac{2\sqrt{p}}{c}\Big) \ll_\varepsilon x^{7/8+\varepsilon}.$$

- In 2001, Iwaneic, Luo, Sarnak conjectured as Hypothesis S that the exponent can be improved from 7/8 to 1/2 for any $x \ge 1$, $c \ge 1$, $a$ with $(a,c)=1$, and any $\varepsilon > 0$.
- One can also consider a smoothed sum

$$S_q(X) = \sum_n a_n e\big(-2\sqrt{nq}\big)\varphi\big(\frac{n}{X}\big),$$

with $X$ and $qX$ large, $\varphi \in C_c^\infty(1,2)$, with $\varphi(1)=\varphi(2)=0$, and $\varphi^{(j)}(t) \ll_j 1$ for $j \ge 0$ and $(a_n)$ is an arithmetically defined sequence of complex numbers satisfying

$$a_n \ll n^\varepsilon,\ \ \varepsilon > 0.$$

- In 2010, Ren and Ye showed the square root cancellation predicted by Hypothesis S is obstructed by a main term of size $X^{\frac{3}{4}}$ that we refer to as *resonance barrier*.
- It is believed that one might be able to break the resonance barrier if the cusp form $f$ is allowed to move.
- In this direction, Ye in 2022 proved the first known non-trivial bound for $S_X(f;\alpha,\beta)$

when the weight $k$ of $f$ tends to infinity with $X$.

- In this project we try to find a nontrivial bound considering 2 holomorphic cusp forms.

## Existing Results

- Sums of the form $\sum_n \lambda_f(n) e(\alpha n^\beta) \varphi\left(\frac{n}{X}\right)$ were first considered by Iwaniec-Luo-Sarnak (2000) for $f$ a normalized Hecke eigenform for $SL(2, \mathbb{Z})$ with $\alpha = 2\sqrt{q}$ for $q \in \mathbb{Z}_+$ and $\beta = 1/2$.
- Ren-Ye (2010) investigated resonance for $SL(2, \mathbb{Z})$ holomorphic cusp forms with no restrictions on $\alpha$ and $\beta$.
- Sun-Wu (2014) showed the same results but for Maass cusp forms for $SL(2, \mathbb{Z})$.
- Ren-Ye (2014) gave resonance results for $SL(3, \mathbb{Z})$ Maass cusp forms.
- Ren-Ye (2016) studied resonance for $SL(m, \mathbb{Z})$ Maass cusp forms.
- Czarnecki (2016) showed $S_X(f, g, \alpha, \beta)$ when $\beta = 1/4$ and $\alpha$ is close or equal to $\pm 4q^{\frac{1}{4}}$ for positive integer $q$, the average has a main term of size $|\lambda_f(q)\lambda_g(q)| X^{\frac{1}{8}+\frac{1}{2}}$. Otherwise, when $\alpha$ is fixed and $0 < \beta < \frac{1}{4}$ it decays rapidly.
- So in the case for the Rankin-Selberg $L$-function the ''resonance barrier'' that we would like to break is $X^{\frac{5}{8}}$.

# Alternate Formulation

- Let $H_k$ denote an orthogonal basis of Hecke eigenforms for the holomorphic forms of weight $k$ of $SL(2,\mathbb{Z})$ where each form is normalized so that the first Fourier coefficient is equal to $1$.
- Following Sarnak, for a non-negative test function $g_0 \in C^\infty(-1,1)$, $g_0(0)=1$ we want non-trivial bound for

$$\sum_{2|k_1}\sum_{2|k_2} \quad g_0\Big(\frac{k_1-K_1}{L_1}\Big)g_0\Big(\frac{k_2-K_2}{L_2}\Big) \times \sum_{f\in H_{k_1}}\sum_{g\in H_{k_2}} |S_X(f,g,\alpha,\beta)|^2.$$

- Alternatively, we normalize

$$\sum\nolimits_{K_1L_1}^{K_2L_2}$$

$$= K_1K_2 \sum_{2|k_1}\sum_{2|k_2} g_0\Big(\frac{k_1-K_1}{L_1}\Big)g_0\Big(\frac{k_2-K_2}{L_2}\Big)$$

$$\times \sum_{f\in H_{k_1}} \frac{2\pi^2}{(k_1-1)L(1,Sym^2 f)} \sum_{g\in H_{k_2}} \frac{2\pi^2}{(k_2-1)L(1,Sym^2 g)}$$

$$\times |S_X(f,g,\alpha,\beta)|^2$$

  allowing a negligible discrepancy bounded by $K_i^{-\varepsilon}$ and $K_i^{\varepsilon}$, $i=1,2$.
- Using Deligne's estimate in Ramanujan conjecture we obtain a trivial bound

$$S_X(f,g,\alpha,\beta) \ll X^{1+\varepsilon}$$

where the implied constant is independent of $f$ and $g$. This gives a trivial bound of $O(K_1L_1K_2L_2X^{2+\varepsilon})$ for $\sum_{K_1L_1}^{K_2L_2}$.

- Our goal is to break this trivial bound.
- Note:non-trivial bounds are known for $S_X(f,g,\alpha,\beta)$, $f$ and $g$ both fixed but not for their weights tending to infinity.

## Main Theorem

- (Gillespie-S-Ye)

  For $j = 1,2$ assume $K_j^{\varepsilon} \leq L_j \leq K_j^{1-\varepsilon}$. Then

$$\sum\nolimits_{K_1L_1}^{K_2L_2}$$

$$\ll K_1L_1K_2L_2X^{1+\varepsilon} \text{ if } K_1L_1 \geq X^{1+\varepsilon} \text{ and } K_2 \geq X^{\frac{1}{2}+\varepsilon}$$

$$\ll K_1^2L_1L_2X^{1+\varepsilon} + \frac{X^{3+\varepsilon}}{K_1}$$

$$\text{if } K_1L_1, K_2L_2 \leq X^{1+\varepsilon}, \ K_1 = K_2,$$

$$\text{and } K_1^2L_1L_2 \geq X^{1+\beta+\varepsilon}.$$

- First line gives non-trivial bounds for square moments of $S_X(f,g,\alpha,\beta)$ in both the $f$ and $g$ aspects. Since the number of terms in $\sum_{K_1L_1}^{K_2L_2}$ is $\asymp K_1L_1K_2L_2$ the average size of $S_X(f,g,\alpha,\beta)$ is $O(X^{\frac{1}{2}+\varepsilon})$ when $K_1L_1$ and $K_2$ are large.

- The second line gives a non-trivial bound for the single square moment

$$\sum_{2|k_1} g_0\Big(\frac{k_1-K_1}{L_1}\Big) \sum_{f\in H_{k_1}} |S_X(f,g,\alpha,\beta)|^2 \ll K_1L_1k_2X^{1+\varepsilon} + \frac{X^{3+\varepsilon}}{k_2},$$

when $L_2 = K_2^{\varepsilon}$, $K_1L_1, K_2L_2 \le X^{1+\varepsilon}$, $k_2 \ge X^{\frac{1+\delta}{2}+\varepsilon}$, $K_1 \le X^{1-\delta}$.

## Overview of the Proof

- Expanding $|S_X(f,g,\alpha,\beta)|^2$ and applying Petersson trace formula twice to $\sum_{K_1L_1}^{K_2L_2}$ we get

$$K_1K_2 \sum_{2|k_1}\sum_{2|k_2} g_0\Big(\frac{k_1-K_1}{L_1}\Big) g_0\Big(\frac{k_2-K_2}{L_2}\Big)$$
$$\times \sum_n \sum_m e(\alpha n^\beta - \alpha m^\beta)\varphi\Big(\frac{n}{X}\Big)\bar{\varphi}\Big(\frac{m}{X}\Big)$$
$$\times\Big(\delta(n,m) + 2\pi i^{k_1} \sum_{c_1\ge 1} \frac{S(m,n,c_1)}{c_1} J_{k_1-1}\Big(\frac{4\pi\sqrt{mn}}{c_1}\Big)\Big)$$
$$\times\Big(\delta(n,m) + 2\pi i^{k_2} \sum_{c_2\ge 1} \frac{S(m,n,c_2)}{c_2} J_{k_2-1}\Big(\frac{4\pi\sqrt{mn}}{c_2}\Big)\Big)$$
$$=: \ D_{00} + D_{01} + D_{10} + D_{11}.$$

- Here $S(m,n,c)$ is the Kloosterman sum

$$= \sum_{z \bmod c}^{*} e\Big(\frac{mz+n\bar{z}}{c}\Big).$$

- The diagonal sum

$$D_{00} = K_1K_2\sum_{2|k_1}\sum_{2|k_2} g_0\Big(\frac{k_1-K_1}{L_1}\Big)g_0\Big(\frac{k_2-K_2}{L_2}\Big) \times \sum_n \Big|\varphi\Big(\frac{n}{X}\Big)\Big|^2$$

  has a trivial bound of $K_1L_1K_2L_2X$.

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$$\begin{aligned} D_{11} &= K_1K_2\sum_{2|k_1}\sum_{2|k_2} g_0\Big(\frac{k_1-K_1}{L_1}\Big)g_0\Big(\frac{k_2-K_2}{L_2}\Big) \\ &\times \sum_n\sum_m e(\alpha n^\beta - \alpha m^\beta)\varphi\Big(\frac{n}{X}\Big)\bar{\varphi}\Big(\frac{m}{X}\Big) \\ &\times 4\pi^2 i^{k_1+k_2}\sum_{c_1\geq 1}\frac{S(m,n,c_1)}{c_1}J_{k_1-1}\Big(\frac{4\pi\sqrt{mn}}{c_1}\Big) \\ &\times \sum_{c_2\geq 1}\frac{S(m,n,c_2)}{c_2}J_{k_2-1}\Big(\frac{4\pi\sqrt{mn}}{c_2}\Big). \end{aligned}$$

- We write $\sum_{2|k_j} i^{k_j} g_0\Big(\frac{k_j-K_j}{L_j}\Big)J_{k_j-1}(x)$s as oscillatory integrals and follow Sarnak, and Salazar-Ye to write their asymptotic expansions.

- We focus on the main terms in the expansions and obtain

$$
\begin{aligned}
T_{11}^{\eta_1\eta_2} &= K_1L_1K_2L_2\sum_n\sum_m e(\alpha n^\beta - \alpha m^\beta)\\
&\times\varphi(\frac{n}{X})\bar{\varphi}(\frac{m}{X})\sum_{c_j\le\frac{X}{K_j^{1-\varepsilon}L_j}}\frac{1}{c_1c_2}\\
&\times\sideset{}{^*}\sum_{z_1 \bmod c_1} e\Big(\frac{mz_1+n\bar{z}_1}{c_1}\Big)\sideset{}{^*}\sum_{z_2 \bmod c_2} e\Big(\frac{mz_2+n\bar{z}_2}{c_2}\Big)\\
&\times h_1^{\eta_1}(m,n,dc_1)h_2^{\eta_2}(m,n,dc_2)\\
&\times e(\varphi_1^{\eta_1}(m,n,dc_1)+\varphi_2^{\eta_2}(m,n,dc_2)).
\end{aligned}
$$

where $h_j^{\eta}$s are weight functions and $\varphi_j^{\eta_j}$s are certain phase functions.

- We apply Poisson summation twice, for the $n$-sum and the $m$-sum and combine the two oscillatory integrals into a double integral that we bound using a two-dimensional second derivative test (Srinivasan).
- This gives us

$$
T_{11}^{\eta_1\eta_2} \ll \frac{X^{3+\varepsilon}}{K_1}
$$

- For other terms: We use a weighted first derivative test following McKee-Sun-Ye, after using Poisson summation on $n$ (or, $m$) and obtaining an oscillatory integral.

- These bounds on $\sum_{K_1L_1}^{K_2L_2}$ also imply new non-trivial bounds on the single square moments

$$\sum_{2|k_1} g_0\Big(\frac{k_1 - K_1}{L_1}\Big) \sum_{f \in H_{k_1}} |S_X(f, g, \alpha, \beta)|^2.$$

Thank you!