The Generalised Shanks' Conjecture

PIMS CRG Workshop on Moments of L-Functions

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Table of Contents

Section

Moments of $\zeta(s)$

The $2k^{th}$ (continuous) moment of the Riemann zeta function $\zeta(s)$ is given by

$$I_{2k}(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} \, dt$$

as $T \to \infty$. To leading order, we have that

$$I_2(T) \sim T \log T \text{ and } I_4(T) \sim \frac{1}{2\pi^2} T \log^4 T$$

as $T \to \infty$ with lower order terms known.
A reasonable conjecture for higher order moments then is

$$I_{2k}(T) \sim g_k a(k) T \log^{k^2} T$$

as $T \to \infty$.

RMT Conjecture for Moments of $\zeta(s)$

The main conjecture for higher moments comes from Keating and Snaith using RMT, which states that for $\Re(k) > -1/2$,

$$I_{2k}(T) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \sim \frac{G^2(k+1)}{G(2k+1)} a(k) T \log^{k^2} T$$

as $T \to \infty$, where $G(z)$ is the Barnes G-function and $a(k)$ is an arithmetic factor given by

$$a(k) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)}\right)^2 p^{-m}.$$

The lower order terms are also conjectured in the CFKRS paper.

Moments of $\zeta(s)$

The $2k^{th}$ (discrete) moment of the Riemann zeta function $\zeta(s)$ is given by

$$J_{2k}(T) = \sum_{0<\gamma\leq T} \left| \zeta' \left(\frac{1}{2} + i\gamma \right) \right|^{2k}$$

where $\rho = \beta + i\gamma$ is a zero of $\zeta(s)$. We know to leading order that

$$J_2(T) \sim \frac{1}{12} \frac{T}{2\pi} \log^4 T$$

with lower order terms known.
In fact, we know the second moment for all derivatives of $\zeta(s)$.
Gonek and Hejhal independently conjectured that

$$J_{2k}(T) \sim c_k T \log^{(k+1)^2} T.$$

RMT Conjecture for Moments of $\zeta(s)$

A similar conjecture to that of Keating and Snaith is known for all $2k^{th}$ discrete moments due to Hughes, Keating and O'Connell. It states that for $\Re(k) > -3/2$,

$$J_{2k}(T) = \sum_{0<\gamma\leq T} \left|\zeta'\left(\frac{1}{2}+i\gamma\right)\right|^{2k} \sim \frac{G^2(k+2)}{G(2k+3)} a(k) N(T) \log^{k(k+2)} T,$$

as $T \to \infty$, where $G(z)$ is the Barnes G-function, $a(k)$ is the same arithemtic factor as in the Keating-Snaith conjecture and $N(T)$ is the number of zeros of $\zeta(s)$ inside the critical strip up to a height T, given by

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Moments - Why do we care?

Moments have many useful applications to the study of $\zeta(s)$. As some examples of the applications of moments, we have the following:

1. **Levinson's Method**
2. **Multiplicity of Zeros**
3. **The Lindelöf Hypothesis**
   The Lindelöf Hypothesis is equivalent to the statement that

$$I_{2k} = \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt = O(T^{1+\varepsilon})$$

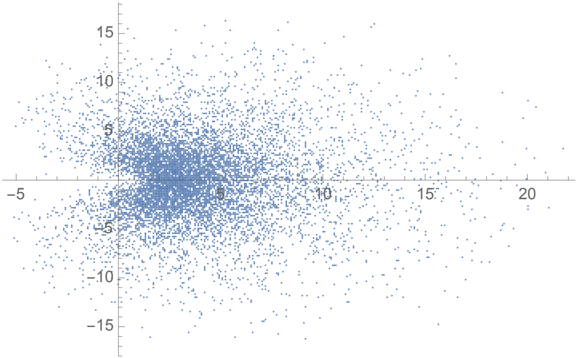
for all $k \in \mathbb{N}$ and all $\varepsilon > 0$.

Section

Shanks' Conjecture

A similar problem to the moments problem is known as Shanks' Conjecture, which states that for $\rho = \beta + i\gamma$ a non-trival zero of $\zeta(s)$,

$$\zeta'(\rho) \text{ is real and positive in the mean.}$$



Asymptotic Expansion

Fujii showed that

$$\sum_{0<\gamma\leq T}\zeta'(\rho) = \frac{T}{4\pi}\log^2\left(\frac{T}{2\pi}\right) + \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right)(-1+C_0) + \frac{T}{2\pi}(1-C_0-C_0^2+3C_1) + O\left(Te^{-C\sqrt{\log T}}\right),$$

with an improvement in the error term under RH.

Generalised Shanks' Conjecture

Write $\zeta^{(n)}(s)$ for the n^{th} derivative of $\zeta(s)$. We have shown that for ρ a non-trival zero of $\zeta(s)$,

$$\zeta^{(n)}(\rho) \text{ is real and } \begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix} \text{ in the mean if } n \text{ is } \begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix}.$$

Furthermore, we have been able to give a full asymptotic expansion which recovers Fujii's result as a special case, given on the next slide.

Asymptotic Expansion

The asymptotic formula from our result states

$$\sum_{0<\gamma\leq T} \zeta^{(n)}(\rho) = (-1)^{n+1}\frac{1}{n+1}\frac{T}{2\pi}\log^{n+1}\left(\frac{T}{2\pi}\right) + \frac{T}{2\pi}\mathcal{P}_n\left(\log\frac{T}{2\pi}\right) + E_n(T),$$

where $\mathcal{P}_n(x)$ is an explicit n^{th} degree polynomial in x that depends only on some known coefficients related to the Laurent series expansions of $\zeta(s)$ about $s = 1$.

A Brief History of the Results

First derivatives:

1. Shanks gives his conjecture in 1961.
2. Conrey, Ghosh and Gonek prove the conjecture to leading order in 1985. By doing so they give a simple proof that there are infinitely many simple zeros of $\zeta(s)$.
3. Fujii gives all lower order terms in 1994, correcting some lower order terms in 2012.
4. Trudgian gives an alternative proof in 2010.
5. Stopple gives an alternative proof in 2020.

All n^{th} order derivatives:

1. Kaptan, Karabulut and Yıldırım prove the result to leading order in 2011.
2. Hughes and I give all lower order terms in 2021, and state the Shanks' conjecture style statement.

A simple proof that there are infinitely many simple zeros of $\zeta(s)$

Using Cauchy's inequality,

$$\left| \sum_{0<\gamma\leq T} \zeta'(\rho) \right|^2 \leq \sum_{0<\gamma\leq T} 1 \sum_{0<\gamma\leq T} \left|\zeta'(\rho)\right|^2 .$$

The first sum is Shanks' conjecture and the third sum is Gonek's result on discrete moments.
The second sum here is $N^*(T)$, the number of simple zeros of $\zeta(s)$. It suffices to bound the first sum from below and the third sum from above. Combined, this gives

$$N^*(T) \gg T.$$

This result is due to Conrey, Ghosh and Gonek.

Section

Cauchy's Residue Theorem

Begin by considering the integral given by

$$S = \frac{1}{2\pi i} \int_R \frac{\zeta'}{\zeta}(s)\zeta^{(n)}(s)\ ds$$

where R is a suitable contour containing the non-trivial zeros of $\zeta(s)$ up to a height T.

By Cauchy's Residue Theorem, this equals

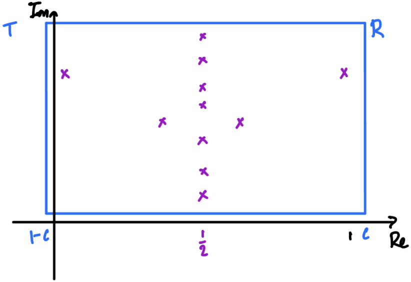
$$S = \sum_{0<\gamma\leq T} \zeta^{(n)}(\rho).$$

We then need to evaluate the integral in another way.

Basic Error Terms

We can show that the contribution from the top, right and bottom sides of the contour are small and the main contribution comes from the left side.

Bottom side: $O(1)$
Right-hand side: $O(\log^{n+3} T)$
Top side: $O(T^{\frac{1}{2}} \log^{n+2} T)$



The Left-Hand Side

Combining the previous steps and switching the integral along the LHS to the RHS, we have

$$S = \sum_{0<\gamma\leq T} \zeta^{(n)}(\rho) = -\bar{I} + O(T^{\frac{1}{2}} \log^{n+2} T)$$

where

$$I = \frac{1}{2\pi i} \int_{c+i}^{c+iT} \frac{\zeta'}{\zeta}(1-s)\zeta^{(n)}(1-s)\, ds.$$

We can use the functional equation

$$\zeta(1-s) = \chi(1-s)\zeta(s)$$

to derive the functional equation for $\zeta^{(n)}(s)$, so we can write I as

$$I = \frac{1}{2\pi i} \int_{c+i}^{c+iT} \chi(1-s)\frac{\zeta'}{\zeta}(1-s)f(n,s)\zeta^{(n)}(s)\, ds.$$

Gonek's Lemma

Lemma (Gonek)

Let $\{b_m\}_{m=1}^{\infty}$ *be a sequence of complex numbers such that for any* $\varepsilon > 0$, $b_m \ll m^{\varepsilon}$. *Let* $c > 1$ *be as before and let* k *be a non-negative integer. Then for* T *sufficiently large,*

$$\frac{1}{2\pi}\int_1^T \left(\sum_{m=1}^{\infty} b_m m^{-c-it}\right)\chi(1-c-it)\log^k\left(\frac{t}{2\pi}\right)\,dt = \sum_{1\le m\le \frac{T}{2\pi}} b_m \log^k m + O(T^{c-\frac{1}{2}}\log^k T).$$

Using this lemma and recombining, we have

$$S = -\bar{I} = (-1)^{n+1}\sum_{mr\le\frac{T}{2\pi}} \Lambda(r)\log^n r + O(T^{\frac{1}{2}}\log^{n+2} T).$$

Perron's formula

Using Perron's formula, we can switch the summation back into an integral, which we can then evaluate to give

$$\frac{1}{2\pi i}\int_{c-iV}^{c+iV}\left(\frac{\zeta'}{\zeta}(s)\right)^{(n)}\zeta(s)\frac{(T/2\pi)^s}{s}\,ds = \operatorname*{Res}_{s=1}\left(\frac{\zeta'}{\zeta}(s)\right)^{(n)}\zeta(s)\frac{(T/2\pi)^s}{s} + E_n(Y).$$

All that remains is to evaluate the residue.

Residue Calculations

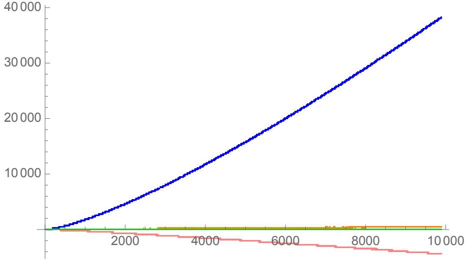
We can calculate the residue at $s = 1$ to obtain the full asymptotic expansion. Expanding each of the terms in

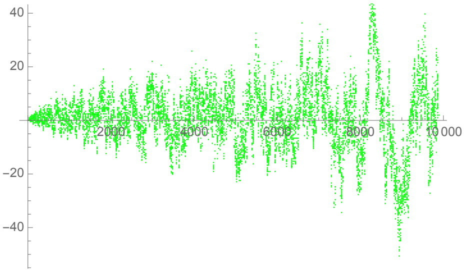
$$\operatorname*{Res}_{s=1} \left(\frac{\zeta'}{\zeta}(s) \right)^{(n)} \zeta(s) \frac{(T/2\pi)^s}{s}$$

using their Laurent expansions about $s = 1$ gives the full asymptotic formula stated in our theorem. This is done through a combinatorial argument to ensure we capture all terms in the calculation.

'Proof' for $n = 1$

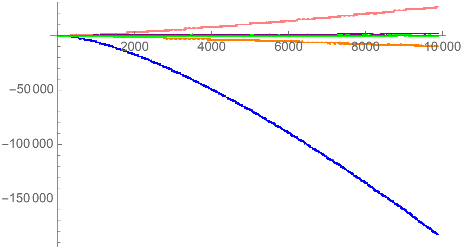
$$\sum_{0<\gamma\leq T} \zeta'(\rho) = \frac{T}{4\pi}\log^2\left(\frac{T}{2\pi}\right) + \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right)(-1 + C_0) + \frac{T}{2\pi}(1 - C_0 - C_0^2 + 3C_1) + E_1(T)$$

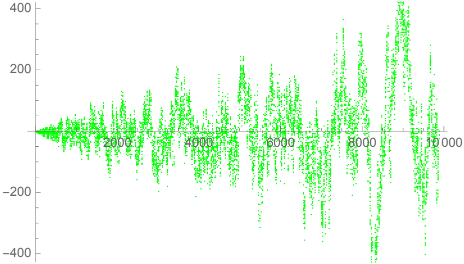




'Proof' for $n = 2$

$$\sum_{0<\gamma\leq T} \zeta''(\rho) = -\frac{T}{6\pi}\log^3\left(\frac{T}{2\pi}\right) - (-1+C_0)\frac{T}{2\pi}\log^2\left(\frac{T}{2\pi}\right) + (-2+2C_0-2C_1)\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) + (2-2C_0+2C_1+4C_2-6C_0C_1+2C_0^3)\frac{T}{2\pi} + E_2(T)$$





The Error Term

Unconditionally, we have shown that

$$E_n(T) = O\left(T\mathrm{e}^{-C\sqrt{\log T}}\right)$$

where C is a positive constant.
If we assume the Riemann hypothesis, we were able to show that

$$E_n(T) = O\left(T^{\frac{1}{2}}\log^{n+\frac{5}{2}} T\right).$$

We have since been able to do better than this, giving

$$E_n(T) = O\left(T^{\frac{1}{2}}\log^{n+\frac{9}{4}} T\right),$$

a saving of a fourth root of the logarithm in the error term.

Section

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