Selberg's Central Limit Theorem for Quadratic Dirichlet L-functions over Function Fields

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Distribution of Values for ζ

Theorem (Selberg, 1946)

Let V be a fixed positive real number. Then as $T \to \infty$, we have

$$\frac{1}{T}\#\left\{t \in [T, 2T] : \frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}} \geq v\right\} \sim \frac{1}{\sqrt{2\pi}}\int_v^{\infty} e^{-\frac{x^2}{2}}\,dx,$$

for $v \in [-V, V]$.

Radziwiłł & Soundararajan (2017) give a new proof of this fact.

Distribution of Values for Dirichlet L-functions

Theorem (Selberg, 1946)

Let χ be a primitive Dirichlet character and V a fixed positive real number. Then as $T \to \infty$, we have

$$\frac{1}{T}\#\left\{t \in [T, 2T] : \frac{\log|L(\frac{1}{2}+it,\chi)|}{\sqrt{\frac{1}{2}\log\log T}} \geq v\right\} \sim \frac{1}{\sqrt{2\pi}}\int_v^\infty e^{-\frac{x^2}{2}}\,dx,$$

for $v \in [-V, V]$.

Hsu & Wong (2020) provide a new proof of this fact.
Bombieri & Hejhal (1995) conditionally prove for a very general class of L-functions that their logarithms also satisfy a central limit theorem.
Luo (1995) verified the conditional assumptions in the above for Hecke L-functions associated with cusp forms.

Katz Sarnak Philosophy

• Central values $L(\frac{1}{2}+it, f)$ of an L-function belong in a family with a symmetry type governed by a classical compact matrix group.
• All the listed results so far have all been show to have approximately normal distributions with mean 0 and variance $\frac{1}{2}\log\log T$.
• They are all examples of families that correspond to Unitary matrices, U(N).
• So suppose now, instead of fixing an L-function and varying the points it is evaluated at, we look at $L(1/2, f)$ and vary the f in some family. Katz and Sarnak demonstrated that we can find families which have their symmetry type governed by $U_{\mathrm{Sp}}(N)$ and $O(N)$ instead.

Keating Snaith Conjectures

Let $S(D)$ represent a family of L-functions indexed by d that is expected to have symmetry type governed by the symplectic matrix group.
Define:

$$B(d) = \frac{1}{\sqrt{\log\log D}}\left(\log|L(\tfrac{1}{2},f_d)| - \frac{1}{2}\log\log D\right),$$

Random Matrix Theory predicts

$$\frac{1}{|S(D)|}\sum_{d\in S(D)}\Delta_{B(d)} \to N(0,1) \text{ as } D\to\infty,$$

Δ_x is the point mass function at x and $N(0,1)$ means normal distribution of mean 0 and standard deviation 1.

Partial Results

Theorem (Hough, 2014)

As $D \to \infty$,

$$\mathbb{P}\left(d \in s(D) : \frac{1}{\sqrt{\log\log D}}\left(\log\left|L\left(\frac{1}{2}, \chi_{8d}\right)\right| - \frac{1}{2}\log\log D\right) > A\right) \leq \frac{1}{\sqrt{2\pi}}\int_A^\infty e^{-\frac{x^2}{2}}\,dx + o_A(1),$$

$s(D)$ is the set of square-free odd d such that $\frac{D}{2} \leq d \leq D$.

Partial Results

Theorem (Hough,2014)

Suppose the ''Low-lying zero hypothesis'' holds for $\{L(s,\chi_{8d})\}_{d\in s(D)}$ *and define*

$$B(d) = \frac{1}{\sqrt{\log\log D}}\left(\log|L(\tfrac{1}{2},\chi_{8d})| - \frac{1}{2}\log\log D\right),$$

$$\frac{1}{|s(D)|}\sum_{d\in S(D)}\Delta_{B(d)} \to N(0,1) \text{ as } D\to\infty,$$

Δ_x *is the point mass function at* x *and* $N(0,1)$ *means normal distribution of mean* 0 *and standard deviation* 1.

Conditions

Hypothesis (Low-lying zero hypothesis)

Assume $y = y(D) \to \infty$ *with* D*. Then for* d *such that* $8d$ *is a fundamental discriminant, we have*

$$\mathbb{P}\left(\frac{D}{2} \leq d \leq D : \gamma_{\min}(d) < \frac{2\pi}{y \log D}\right) = o(1) \text{ as } D \to \infty,$$

where

$$\gamma_{\min}(d) = \min_{\substack{L(\rho,\chi_{8d})=0 \\ \rho=\frac{1}{2}+\beta+i\gamma}} |\gamma|.$$

Transition to Function Fields

We are interested in this problem over $\mathbb{F}_q(t)$, $q = p^e$ an odd prime power.
Let $\mathcal{H}_n = \{D \in \mathbb{F}_q[t] : D \text{ is monic and square-free and } \deg(D) = n\}$.
Define

$$L(s, \chi_D) = \sum_{f \text{ monic}} \frac{\chi_D(f)}{|f|^s}, \quad \chi_D(\cdot) = \left(\frac{D}{\cdot}\right), \mathfrak{Re}(s) > 1$$

Using the change of variable $u = q^{-s}$, we have

$$\mathcal{L}(u, \chi_D) = \sum_{f \in \mathcal{M}} \chi_D(f)\, u^{d(f)}, \quad |u| < \frac{1}{q}.$$

Orthogonality gives us that $\mathcal{L}(u, \chi_D)$ is actually a polynomial of degree at most $\deg(D) - 1$.

Transition to function fields

Thus, we may write it as

$$\mathcal{L}(u, \chi_D) = (1-u)^{\lambda} \prod_{j=1}^{n-1-\lambda} (1 - u\sqrt{q}\alpha_j),$$

where $\sqrt{q}\alpha_j = \sqrt{q}e(-\theta_{j,D})$ are the reciprocals of the roots $q^{-\frac{1}{2}}e(\theta_{j,D})$ and $\lambda = 1$ if $\deg(D)$ is even and 0 otherwise.

Statement of Results

Theorem

Let Z be a real number. As $n \to \infty$, we have

$$\mathbb{P}\left(D \in \mathcal{H}_n : \frac{1}{\sqrt{\log n}}\left(\log\left|L\left(\frac{1}{2}, \chi_D\right)\right| - \frac{1}{2}\log n\right) > Z\right) \leq \frac{1}{\sqrt{2\pi}}\int_Z^\infty e^{-\frac{t^2}{2}}\,dt + o_Z(1).$$

Theorem

Let Z be a real number. As $n \to \infty$, we have

$$\mathbb{P}\left(D \in \mathcal{H}_n : \frac{1}{\sqrt{\log n}}\left(\log\left|L\left(\frac{1}{2}, \chi_D\right)\right| - \frac{1}{2}\log n\right) > Z\right) \geq \frac{19 - \cot(1/4)}{16}\frac{1}{\sqrt{2\pi}}\int_Z^\infty e^{-\frac{t^2}{2}}\,dt + o_Z(1).$$

Statement of results

Theorem

Suppose that the "Low-lying Zero Hypothesis" holds for $\{L(s,\chi_D)\}_{D\in\mathcal{H}_n}$. For $D \in \mathcal{H}_n$, we consider

$$\widetilde{A}(D) = \frac{1}{\sqrt{\log n}}\left(\log\left|L\left(\frac{1}{2},\chi_D\right)\right| - \frac{1}{2}\log n\right).$$

Then we have

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n}\Delta_{\widetilde{A}(D)} \to N(0,1), \quad n\to\infty.$$

Δ_x is the point mass function at x and $N(0,1)$ means normal distribution of mean 0 and standard deviation 1.

Conditions

Hypothesis (Low-lying Zero Hypothesis)

Let $\theta_{j,D}$ be the angles of zeros of $\mathcal{L}^*(u,\chi_D)$. If $y = y(g) \to \infty$ then as $g \to \infty$ we obtain

$$\frac{1}{|\mathcal{H}_n|}\left|\left\{D \in \mathcal{H}_n : \min_j |\theta_{j,D}| < \frac{1}{yg}\right\}\right| = o(1),$$

where g satisfies $2g = \deg(D) - 1 - \lambda$ with $\lambda = 1$ if $\deg(D)$ is even and 0 otherwise.

Ideas in the proof

Theorem

Let g be defined by $2g = \deg(D) - 1 - \lambda$ and $\sigma_o = \sigma_o(g)$ be a function of g, tending to zero as $g \to \infty$ in such a way that $\sigma_o g \to \infty$ but $\frac{\sigma_o g}{\sqrt{\log g}} \to 0$. For $D \in \mathcal{H}_n$, we consider

$$A(D) = \frac{1}{\sqrt{\log n}} \left(\log \left| L\left(\frac{1}{2} + \sigma_o, \chi_D\right) \right| - \frac{1}{2} \log n \right).$$

Then, as $n \to \infty$

$$\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_n} \Delta_{A(D)} \to N(0,1),$$

where Δ_x is the point mass at x and $N(0,1)$ is the standard normal distribution.

Ideas in Proof

Proposition

Let $X \geq 1$, $\sigma_0 = \frac{c}{X}$, *with* $0 < c < \frac{1}{2\log q}$. *Then*

$$\log L(1/2 + \sigma_0, \chi_D) = \sum_{f \in \mathcal{M}_{\leq X}} \frac{\Lambda(f)\chi_D(f)}{d(f)|f|^{\sigma}} + O\left(\frac{1}{X^3} \sum_{f \in \mathcal{M}_{\leq 3X}} \frac{\Lambda_X(f)\chi_D(f)}{|f|^{\frac{1}{2}+\sigma_0}}\right) + O\left(\frac{g}{X} + \frac{\lambda}{X(X+2)}\right).$$

Ideas in Proof

We define

$$\mathcal{H}_{n,0} = \left\{ D \in \mathcal{H}_n : \mathcal{L}(q^{-\frac{1}{2}} e(\theta_{j,D}), \chi_D) = 0 \Rightarrow \min_j |\theta_{j,D}| > \frac{1}{yg} \right\}.$$

Proposition

Let $X \leq \frac{n}{4k}$ but goes to infinity with n. Suppose that $g\,\sigma_o = o(\sqrt{\log g})$ as $g \to \infty$. Then, we have

$$\frac{1}{|\mathcal{H}_n|} \sum_{D \in \mathcal{H}_{n,0}} \left| \log \left| L\left(\frac{1}{2}, \chi_D\right) \right| - \log \left| L\left(\frac{1}{2} + \sigma_o, \chi_D\right) \right| \right| = o\left(\sqrt{\log g}\right).$$

Ideas for the unconditional lower bound

Proposition

Let $X \geq 1$. If we know that $L(\frac{1}{2}, \chi_D) \neq 0$ then

$$\log L\left(\frac{1}{2}, \chi_D\right) = \sum_{f \in \mathcal{M}_{\leq X}} \frac{\Lambda(f)\chi_D(f)}{d(f)|f|^{\frac{1}{2}}} + O\left(\frac{1}{X^2} \sum_{X < d(f) \leq 3X} \frac{\Lambda_X(f)\chi_D(f)}{d(f)|f|^{\frac{1}{2}}}\right) + O\left(\frac{g}{X^3} + \frac{\lambda}{X(X+2)}\right).$$

Ideas for the unconditional lower bound

Theorem (Bui and Florea, 2016)

We have

$$\frac{1}{|\mathcal{H}_{2g+1}|}|\{D \in \mathcal{H}_{2g+1} : L(\frac{1}{2}, \chi_D) \neq 0\}| \geq \frac{19 - \cot(\frac{1}{4})}{16} + o(1) = 0.9427... + o(1),$$

as $g \to \infty$.

Future/Ongoing work

✓ ✓ A new collaborator has been added Fatma Çiçek.
Consider the following new scenario:
Let a_1, a_2 be fixed real numbers and t_1, t_2 be fixed numbers s.t. $|t_j| \leq c/X$.
Define

$$P_{\mathcal{L}}\left(\frac{1}{2}+i\mathbf{t}, \chi_D\right) := a_1 \mathfrak{Re} P_X\left(\frac{1}{2}+it_1, \chi_D\right) + a_l \mathfrak{Re} P_X\left(\frac{1}{2}+it_l, \chi_D\right)$$

Q: Can we obtain a central limit theorem for $P_{\mathcal{L}}\left(\frac{1}{2}+i\mathbf{t}, \chi_D\right)$??

Initial Investigations and Surprises

We begin by considering $P_{\mathcal{L}}\left(\frac{1}{2}+\sigma_0+i\mathbf{t},\chi_D\right)$ where $\sigma_0 := \sigma_0(g) \to 0$ as $g \to \infty$ in such a way that $\sigma_o g \to \infty$ but $\frac{\sigma_o g}{\sqrt{\log g}} \to 0$.

Lemma

Assume that $1 \leq X \leq \frac{n}{2k}$*. Uniformly for all even natural numbers* $k \ll (\log n)^{\frac{1}{3}}$*, we have*

$$\frac{1}{|\mathcal{H}_n|}\sum_{D\in\mathcal{H}_n} P_{\mathcal{L}}\left(\frac{1}{2}+\sigma_o+i\mathbf{t},\chi_D\right)^k = \frac{k!}{(k/2)!2^{\frac{k}{2}}}\left((a_1+a_2)^2\log X\right)^{\frac{k}{2}}\left(1+O\left(\frac{k^3}{\log X}\right)\right).$$