Converse theorem - Selberg class

joint work with Andrew R. Booker and Michael Farmer

Modular forms and their L-functions

k, $\gamma \in GL_2^+(\mathbb{R})$ $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$(f|\gamma)(z) = (\det \gamma)^{\frac{k}{2}} (cz+d)^{-k} f(\gamma z)$

$= \chi(d) f(z)$

$f \in M_k(N, \chi)$. χ: nebentypus. , $\forall \gamma \in \Gamma_0(N)$.

$f(z) = \sum_{n \geq 0} f_n n^{\frac{k-1}{2}} e(nz)$ $\quad e(z) = e^{2\pi i z}$.

$\Rightarrow \Lambda(s, f) = \Gamma_{\mathbb{C}}(s + \frac{k-1}{2}) \sum_{n=1}^{\infty} \frac{f_n}{n^s}$ $\quad (\mathrm{Re}(s) > 1)$.

$= \Lambda(1-s, g) \cdot i^k N^{\frac{1}{2}-s}$.

$\Gamma_{\mathbb{C}}(s) = 2 \cdot (2\pi)^{-s} \Gamma(s)$.

$g(z) := \left(f \middle| \begin{pmatrix} 0 & -\sqrt{N}^{-1} \\ \sqrt{N} & 0 \end{pmatrix}\right)(z) = (\sqrt{N} z)^{-k} f\left(-\frac{1}{Nz}\right)$.

$\Lambda(s, f)$, $\Lambda(s, g)$ → entire fns of finite order

$\Lambda(s,f) = i^k N^{\frac{1}{2}-s} \Lambda(1-s, g)$.

$$\Lambda(s, f) = \int_0^\infty f(iy)\, y^{s+\frac{k-1}{2}} \frac{dy}{y}.$$

Converse theorems for modular forms

Hecke. (1936). $\boxed{N \le 4}$: $\boxed{T,\ W_N,\ -W_N,\ -I_2}$.

the modular forms of level $N \le 4$ can be characterized by ⊛.

$\boxed{\{f_n\}}$ + ⊛. $f(z) = \sum_{n \ge 0} f_n\, n^{\frac{k-1}{2}} e(nz)$.

- $f|T = f$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
- $f|W_N = f$ $W_N: \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} T \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \in \Gamma_0(N)$

$N > 4$: $N = 5$. $\Gamma_0(5) = \langle T,\ W_5,\ \boxed{\begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}} \rangle$.

R. Steiner. (2019).

More functional equations

ψ Dirichlet char. mod q.

$$\Lambda(s,f,\psi) := \Gamma_{\mathbb{C}}\left(s+\frac{k-1}{2}\right)\cdot\sum_{n=1}^{\infty}\frac{f_n\cdot\psi(n)}{n^s}.$$

$\boxed{\text{Weil's conver thm. (1967)}}$ $\{f_n\}_{n\geq1}$.

analytic properties. $\Lambda(s,f,\psi)$, $\Lambda(s,f)$.

ψ primitive char. mod q. q prime, $\boxed{q\nmid N.}$

$\in$ "set".

$\boxed{\text{Booker – Krishnamurthy (2013).}}$

$\exists$ a poly. $P(s)\neq 0$. s.t. $\underline{P(s)}\,\Lambda(s,f,\psi)$.

$\boxed{\text{Razar's converse thm. (1977)}}$ $\{f_n\}_{n\geq1}$.

analytic properties $\Lambda(s,f,\psi)$; ψ char. mod $q \in \mathbb{N}\mathbb{Z}_{\geq1}$.

$$N\leq q\leq N^2.$$

Generalizations

- J.-L.

- W. Li

$\vdots$

Trace formula

Petersson formula.

+

Voronoi summation. ($\Leftrightarrow$ f. e. additive twist L-fun).

Petersson Formula

$g \in H_k(N, \chi)$ ← o.n.b. $\subset$ $\boxed{S_k(N, \chi)}$

$k \geq 4$ $\quad m, n \in \mathbb{Z}_{\geq 1}$ $\quad K_n(s, f)$

$$g(z) = \sum_{n=1}^{\infty} \rho_g(n)\, n^{\frac{k-1}{2}}\, e(nz)$$

$e\left(\frac{ma + na}{q}\right)$

$$\zeta^{(N)}(2s) \sum_{m=1}^{\infty} \frac{f_m \overline{\rho_g}(m)}{m^s} = L(s, f \times \bar{g})$$

$$:= \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{g \in H_k(N,\chi)} \rho_g(n)$$

$$= \boxed{\delta_{m=n}} + 2\pi i^{-k} \sum_{\substack{q \geq 1 \\ N \mid q}} \zeta^{(N)}(2s) \sum_{m=1}^{\infty} \frac{f_m}{m^s} \frac{S_\chi(m, n; q)}{q} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{q}\right)$$

$\zeta^{(N)}(2s)\, f_n n^{-s}$

$\{f_n\}_{n \geq 1}$ $\quad \sum_{n=1}^{\infty} \frac{f_n}{n^s}$ abs. conv. $\mathrm{Re}(s) > 1$

Additive twists

$\underline{\alpha \in \mathbb{Q}}$, $\boxed{\Lambda(s,f,\alpha)} := \underbrace{\Gamma_{\mathbb{C}}(s+\tfrac{k-1}{2})}\ \underbrace{\sum_{n=1}^{\infty} \frac{f_n\, e(n\alpha)}{n^s}}$.

$\updownarrow$

mult. twists

Thm. (Booker - Farmer - L.) $\{f_n\}_{n\ge 1}$. N, χ, $\omega \in \mathbb{C}$, $\gamma(s)$.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{f_n}{n^s}$ converges abs. $\mathrm{Re}(s)>1$.

(2). $\gamma(s) = Q^s \displaystyle\prod_{j=1}^{r} \Gamma(\alpha_j s + \mu_j)$. $Q, \alpha_j \in \mathbb{R}_{>0}$, $\displaystyle\sum_{j=1}^{r} \alpha_j = 1$.

$\mu_j \in \mathbb{C}$, , $\mathrm{Re}(\mu_j) > -\frac{1}{2}\alpha_j$.

$$\Lambda(s, f, \alpha) := \gamma(s) \sum_{n=1}^{\infty} \frac{f_n e(n\alpha)}{n^s}.$$

Suppose. $\forall q \in N\mathbb{Z}_{>0}$, $\forall a, \bar{a} \in \mathbb{Z}$, $a\bar{a} \equiv 1 \ (q)$,

$\Lambda(s, f, \frac{a}{q})$, $\Lambda(s, f, -\frac{\bar{a}}{q})$ entire fcn of finite order.

$$\Lambda(s, f, \tfrac{a}{q}) = \omega \chi(\bar{a}) q^{1-2s} \Lambda(1-s, f, -\tfrac{\bar{a}}{q}).$$

$\Rightarrow \exists k \in \mathbb{Z}_{>0}$. s.t. $f(z) = \displaystyle\sum_{n=1}^{\infty} f_n n^{\frac{k-1}{2}} e(nz) \in M_k(N, \chi)$.

Remarks

- Venkatesh (2002). $\Lambda(s, f, \frac{\omega}{\infty})$.

$\underline{N=1}$.

$\gamma(s) = \Gamma_{\mathbb{C}}(s + \frac{k-1}{2})$.

$+$

a finite set of poles on $Re(s) = \frac{1}{2}$.

- Maass fcns ?

-

Selberg class S. the set of fcn's F. satisfying.

(1). Dirichlet series: $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$, $F(s) = \sum_{n=1} \frac{a_n}{n^s}$ conv. abs. $\mathrm{Re}(s) > 1$.

(2). Analytic continuation:

$\exists\, m \in \mathbb{Z}_{\geq 0}$. $(s-1)^m F(s)$ cont. to an entire fcn of finite order.

(3). Functional eq.: $\exists\, k \in \mathbb{Z}_{\geq 0}$. Q, $\alpha_j \in \mathbb{R}_{>0}$, $\mu_j \in \mathbb{C}$. $\mathrm{Re}(\mu_j) \geq 0$.

$\omega \in \mathbb{C}$, $|\omega| = 1$.

$$\Phi(s) = \varepsilon Q^s \prod_{j=1}^{r} \Gamma(\alpha_j s + \mu_j)\, F(s) = \overline{\Phi(1-\bar{s})}.$$

(4) Ramanujan Hypothesis: $a_n \ll_{\varepsilon} n^{\varepsilon}$, $\forall \varepsilon > 0$.

(5). Euler product: $a_1 = 1$, $\log F(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ $b_n = 0$ if n not a prime power, $b_n \ll n^{\theta}$. $\theta < \frac{1}{2}$.

J. Kaczorowski, A. Perelli (2022).

classified the elements of the Selberg class $\underbrace{\text{degree } 2}_{2\sum_{j=1}^{r} \lambda_j = d}$, of $\underbrace{\text{conductor } 1}_{L = (2\pi)^{d} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}}$ without the need of any twists.

Idea

$$K_n(s,f) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{g \in H_k(N,\chi)} \rho_g(n)\, L(s, f \times \bar{g}).$$

$$= \zeta^{(N)}(2s)\, f_n n^{-s}$$

$$+ 2\pi i^{-k} \zeta^{(N)}(2s) \sum_{q \in N\mathbb{Z}_{\geq 1}} \frac{1}{q} \sum_{m=1}^{\infty} \frac{f_m S_\chi(n,m;q)}{m^s} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{q}\right)$$

open.

$$\zeta^{(N)}(2s) f_n n^{-s} + \zeta(2s-1) \cdot f_n n^{s-1} \cdot * \quad F_k(s,1)$$

$F_k(1,1) \neq 0.$

+ off —.

$$F_k(s,x) = \int \cdots \frac{\gamma(1-s-u)}{\gamma(s+u)} x^u \, du.$$

$$K_n(s,f) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{g \in H_k(N,\chi)} \rho_g(n)\, L(s, f \times \bar{g}).$$

$$= \zeta^{(N)}(2s)\, f_n\, n^{-s} + \zeta(2s-1)\, N^{1-2s} \prod_{p|N} (1-p^{-1})\, i^{-k} \omega \cdot f_n\, n^{s-1}.$$

$$+ i^{-k} \omega \sum_{\substack{m \geq 1, \\ n \neq m}} \frac{f_m\, \sigma_{1-2s}(n-m;N)}{m^{1-s}} F_k\left(s, \frac{m}{n}\right)$$

$$\boxed{f_n = \sum_{g \in H_k(N, \chi)} \alpha_g \, P_g(n).}$$

$f \in M_k(N, \chi)$

$\underline{k \geq 4}$

$k < 4.$

$$\Lambda(s, f, \alpha) = \gamma(s) \cdot P_p(s) \, \Lambda(1-s, \ldots)$$