Distributions of sums of the divisor function over function fields

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Moments of L-functions Workshop
University of Northerm British Columbia

July 25, 2022

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Polynomials over finite fields

Let q be a power of a prime, $\mathbb{F}_q$ the field with q elements.

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$$f(T) = a_nT^n + \cdots + a_0 = a_nP_1^{e_1}P_2^{e_2}\cdots P_s^{e_s}$$

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The Riemann zeta function over finite fields

$$\zeta_q(s) = \sum_{\substack{f \in \mathbb{F}_q[T] \\ f \text{ monic}}} \frac{1}{|f|^s} = \prod_{\substack{\ell \text{ irreducible} \\ \text{monic}}} \left(1 - \frac{1}{|\ell|^s}\right)^{-1}$$

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$\zeta_q(s)$ has no zeros, the Riemann hypothesis is true!

What is a function field?

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Examples

$$\mathbb{F}_q(T), \qquad \mathbb{F}_q[x,y]/(y^2 - x^3 + x) = \mathbb{F}_q(x)(\sqrt{x^3 - x}).$$

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Primes in function fields

A prime of the function field K is a discrete valuation ring R with maximal ideal P such that $\mathbb{F}_q \subseteq R \subset K$ and the quotient field of R is K.

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For $\mathbb{F}_q(T)$, we get

$$\{\text{monic irreducible polynomials}\} \cup P_\infty.$$

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Some background on $\zeta_K(s)$

$$\zeta_K(s) := \sum_{A \geq 0} \frac{1}{|A|^s} = \prod_P \left(1 - \frac{1}{|P|^s}\right)^{-1}, \qquad \mathrm{Re}(s) > 1,$$

where $A \geq 0$ are effective divisors (formal products of primes with positive exponents).

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Example

$$\zeta_{\mathbb{F}_q(T)}(s) := \prod_P \left(1 - \frac{1}{|P|^s}\right)^{-1} = \frac{1}{(1-q^{-s})(1-q^{1-s})}$$

The extra factor (compared to $\zeta_q(s)$) comes from P_∞, which has degree 1.

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The Weil conjectures (Dwork, Grothendieck and Deligne)

- Rationality

$$\zeta_K(s) = \frac{L_K(q^{-s})}{(1-q^{-s})(1-q^{1-s})} \qquad L_K(u) \in \mathbb{Z}[u],$$

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- Riemann hypothesis

$$L_K(q^{-s}) = \prod_{j=1}^{2g}(1-\pi_j q^{-s}), \qquad |\pi_j| = \sqrt{q}.$$

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Dirichlet L-functions over function fields

A Dirichlet character is a map

$$\chi : (\mathbb{F}_q[T]/(D(T)))^* \to \mathbb{C}^*$$

extended to $\mathbb{F}_q[T]$ by periodicity and with the condition that

$$\chi(A) = 0 \text{ when } (A, D) \neq 1.$$

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Since $\sum_{\deg f=n} \chi(f) = 0$ for $n \geq \deg D$, we get a finite sum with finitely many zeros:

$$L(s,\chi) = \sum_{\deg f < \deg D} \frac{\chi(f)}{|f|^s}.$$

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Quadratic Dirichlet characters

$q \equiv 1 \pmod 4$. Let $f \in \mathbb{F}_q[T]$, P monic irreducible,

$$\left(\frac{f}{P}\right) \equiv f^{\frac{|P|-1}{2}} \pmod P \text{ if } P \nmid f$$

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Extend multiplicatively (Jacobi symbol).

$$\left(\frac{f}{D}\right) = \prod_{j=1}^{n} \left(\frac{f}{P_j}\right)^{e_j}, \qquad D = \alpha \prod_{j=1}^{n} P_j^{e_j}.$$

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Odd iff $\deg(D)$ is odd, since

$$\left(\frac{\alpha}{P}\right) = \alpha^{\frac{q-1}{2}\deg(P)} \qquad \alpha \in \mathbb{F}_q.$$

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Quadratic Dirichlet L-functions

D $\square$-free. Consider the hyperelliptic curve

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If $\deg D = 2g+1$,

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The Weil conjectures imply

- $\mathcal{L}(u, \chi_D)$ is a polynomial of degree $2g$.
- Functional equation $\mathcal{L}(u, \chi_D) = (qu^2)^g \mathcal{L}\left(\frac{1}{qu}, \chi_D\right)$.
- Riemann Hypothesis, the zeros of $\mathcal{L}(u, \chi_D)$ at $|u| = \frac{1}{\sqrt{q}}$.

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L-functions and Random Matrix Models

$$\mathcal{L}(u,\chi)=\prod_{j=1}^{2g}(1-\pi_j(\chi)u)=\det(1-uq^{\frac{1}{2}}\Theta_\chi),$$

where $\Theta_\chi \in \mathrm{U}(2g)$ is a unitary matrix ($UU^*=U^*U=I$).

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Katz and Sarnak (1999): the statistics for the zeroes in families of L-functions, in the limit when the conductor of the L-functions gets large, follow the distribution laws of classical random matrices.

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The k-divisor function

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$$\sum_{n \leq x} d_k(n) = \mathrm{Res}_{s=1} \frac{x^s \zeta(s)^k}{s} + \Delta_k(x) = x P_{k-1}(\log x) + \Delta_k(x).$$

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$$\frac{1}{X} \int_X^{2X} \Delta_k(x)^2 dx \sim c_k X^{1-\frac{1}{k}}.$$

Cramér (1922) $k = 2$, Tong (1956) $k \geq 3$ under RH.

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The divisor function - Two types of arithmetic questions

- Short intervals

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$$\mathcal{S}_{d_k;X;Q}(A) := \sum_{\substack{n\leq x \\ n\equiv A \pmod{Q}}} d_k(n)$$

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$$\mathrm{Var}(\mathcal{S}_{d_k;X;Q}) := \frac{1}{\Phi(Q)} \sum_{\substack{A \pmod{Q} \\ (A,Q)=1}} \left|\mathcal{S}_{d_k;X;Q}(A) - \langle \mathcal{S}_{d_k;X;Q}\rangle\right|^2$$

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Distributions over arithmetic progressions

Selberg $Q < X^{2/3-\varepsilon}$,

$$\mathcal{S}_{d_2;X;Q}(A) = \frac{Xp_Q(\log(X))}{\Phi(Q)} + O(X^{1/3+o(1)})$$

$\mathrm{Var}(\mathcal{S}_{d_2;X;Q})$ studied by Motohashi (1973), Blomer (2008), and Lau and Zhao (2012), for Q prime:

- If $1 \leq Q < X^{1/2+\varepsilon}$,

$$\mathrm{Var}(\mathcal{S}_{d_2;X;Q}) \ll X^{1/2} + \left(\frac{X}{Q}\right)^{2/3+\varepsilon}$$

- If $X^{1/2} < Q < X$,

$$\mathrm{Var}(\mathcal{S}_{d_2;X;Q}) = \frac{X}{Q}p_3\left(\log\left(\frac{Q^2}{X}\right)\right) + O\left(\left(\frac{X}{Q}\right)^{5/6}(\log X)^3\right)$$

and Kowalski and Ricotta (2014) for $k \geq 3$ ($Q^{k-\frac{1}{2}+\varepsilon} < X < Q^{k-\varepsilon}$).

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Distributions over arithmetic progressions

Conjecture (Keating, Rodgers, Roditty-Gershon, and Rudnick (2018))

For Q prime, $Q^{1+\varepsilon} < X < Q^{k-\varepsilon}$, as $X \to \infty$,

$$\mathrm{Var}(\mathcal{S}_{d_k;X;Q}) \sim \frac{X}{Q} a_k \gamma_k \left(\frac{\log X}{\log Q}\right) (\log Q)^{k^2-1}.$$

where

$$\gamma_k(c) = \frac{1}{k! G(1+k)^2} \int_{[0,1]^k} \delta_c(w_1 + \cdots + w_k) \prod_{i<j} (w_i - w_j)^2 d^k w,$$

where $\delta_c(w) = \delta(w-c)$ is the delta distribution translated by c, and G is the Barnes G-function $G(1+k) = 1! \cdots (k-1)!$ and

$$a_k = \prod_p \left(\left(1 - \frac{1}{p}\right)^{k^2} \sum_{j=0}^{\infty} \left(\frac{\Gamma(k+j)}{\Gamma(k) j!}\right)^2 \frac{1}{p^j} \right).$$

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The divisor function over function fields

$f \in \mathbb{F}_q[T]$ monic, q odd prime power.

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$$\zeta_q(s)^k = \sum_{\substack{f \in \mathbb{F}_q[T] \\ f \text{ monic}}} \frac{d_k(f)}{|f|^s}, \qquad \mathrm{Re}(s) > 1.$$

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The secular coefficients

Let U be an $N \times N$ matrix. The secular coefficients $\mathrm{Sc}_j(U)$ are the coefficients of the characteristic polynomial of U:

$$\det(1 + xU) = \sum_{j=0}^{N} \mathrm{Sc}_j(U) x^j.$$

Distribution on arithmetic progressions

$$\mathcal{S}_{d_k;n;Q}(A) := \sum_{\substack{f \text{ monic}, \deg(f)=n \\ f \equiv A \,(\mathrm{mod}\, Q)}} d_k(f).$$

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Distribution on arithmetic progressions

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$$\langle \mathcal{S}_{d_k;n;Q} \rangle \sim \frac{q^n \binom{n+k-1}{k-1}}{\Phi(Q)}$$

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Theorem (Keating, Rodgers, Roditty-Gershon, and Rudnick (2018))

If Q is $\square$-free and $n \leq k(\deg(Q)-1)$, as $q \to \infty$

$$\mathrm{Var}^*(\mathcal{S}_{d_k;n;Q}) \sim \frac{q^n}{|Q|} \int_{\mathrm{U}(\deg(Q)-1)} \left| \sum_{\substack{j_1+\cdots+j_k=n \\ 0 \leq j_1,\ldots,j_k \leq \deg(Q)-1}} \mathrm{Sc}_{j_1}(U) \cdots \mathrm{Sc}_{j_k}(U) \right|^2 dU,$$

where $\mathrm{U}(N)$ is the set of $N \times N$ unitary matrices ($UU^* = U^*U = I$) with respect to the Haar probability measure.

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Computing the integral

Keating, Rodgers, Roditty-Gershon, and Rudnick (2018) work with the integral

$$I_k(n;N) = \int_{\mathrm{U}(N)} \left| \sum_{\substack{j_1+\cdots+j_k=n \\ 0\le j_1,\ldots,j_k\le N}} \mathrm{Sc}_{j_1}(U)\cdots\mathrm{Sc}_{j_k}(U) \right|^2 dU$$

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and find that

$$I_k(n;N) = \gamma_k(c)N^{k^2-1} + O_k(N^{k^2-2}),$$

where $c = n/N$ and $\gamma_k(c)$ is a piecewise continuous polynomial function.

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$$\gamma_2(c) = \begin{cases} \frac{c^3}{3!} & 0 \le c \le 1, \\ \frac{(2-c)^3}{3!} & 1 \le c \le 2. \end{cases}$$

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Unitary and symplectic matrices

The results described in the previous slides correspond to unitary matrices ($UU^* = U^*U = I$).

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What about symplectic distributions?

Unitary and symplectic matrices

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What about symplectic distributions?

A $2N \times 2N$ matrix S with real coefficients is symplectic if

$$M^T \Omega M = \Omega,$$

where

$$\Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}.$$

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THEY DID THE UNITARY CASE, SO WE'RE DOING THE SYMPLECTIC.
WHY?
BECAUSE IT'S SYMPLER-TIC.
imgflip.com

Finding a problem leading to symplectic distributions...

Keating, Rodgers, Roditty-Gershon, and Rudnick (2018) work with odd primitive characters with conductor Q $\square$-free to pick up the congruence condition.

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First attempt: Quadratic characters with conductor $D \in \mathbb{F}_q[T]$ monic and $\square$-free.

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$$\sum_{\substack{f \text{ monic}, \deg(f)=n \\ f \equiv \square \pmod{D} \\ D \nmid f}} d_k(f)$$

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☹ It is not easy to detect squares with $\chi_D = \left(\frac{D}{\cdot}\right)$.

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☹ It is not easy to detect squares with $\chi_D = \left(\frac{D}{\cdot}\right)$.

☺ Second attempt: $\chi_P = \left(\frac{P}{\cdot}\right)$, P monic and irreducible.

Distribution of quadratic residues modulo P

$$\mathcal{S}^{S}_{d_k,n}(P) := \sum_{\substack{f \text{ monic}, \deg(f)=n \\ f \equiv \square \pmod{P} \\ P \nmid f}} d_k(f),$$

where P is a monic irreducible polynomial of degree $2g+1$.

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Theorem (Kuperberg and L. (2022))

As $q \to \infty$,

$$\mathcal{S}^S_{d_k,n}(P) \sim \frac{1}{2} \sum_{\substack{f \text{ monic}, \deg(f)=n \\ P \nmid f}} d_k(f) \sim \frac{q^n}{2} \binom{k+n-1}{k-1}.$$

Let $n \leq 2gk$. As $q \to \infty$,

$$\mathrm{Var}^*(\mathcal{S}^S_{d_k,n}) \sim \frac{q^n}{4} \int_{\mathrm{Sp}(2g)} \left| \sum_{\substack{j_1+\cdots+j_k=n \\ 0 \leq j_1,\ldots,j_k \leq 2g}} \mathrm{Sc}_{j_1}(U) \cdots \mathrm{Sc}_{j_k}(U) \right|^2 dU.$$

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Ingredients in the proof - the mean

P monic irreducible of degree $2g+1$.

$$\mathcal{S}^S_{d_k,n}(P) = \sum_{\substack{f \text{ monic} \\ \deg(f)=n, P\nmid f}} d_k(f) \left(\frac{1+\chi_P(f)}{2} \right)$$

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Look for the coefficient of u^n in the generating function.

$$\sum_{\substack{f \text{ monic} \\ P\nmid f}} d_k(f)u^{\deg(f)} = \left(\frac{1-u^{2g+1}}{1-qu}\right)^k$$

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Ingredients in the proof - the L-function

$$\mathcal{L}(u,\chi_D)=\sum_{f\,\text{monic}}\chi_D(f)u^{\deg(f)}=\det(1-uq^{1/2}\Theta_{\chi_D}).$$

Ingredients in the proof - the L-function

$$\mathcal{L}(u,\chi_D) = \sum_{f\,\text{monic}} \chi_D(f) u^{\deg(f)} = \det(1 - uq^{1/2}\Theta_{\chi_D}).$$

Use

$$\mathcal{L}(u,\chi_D)^k = \sum_{f\,\text{monic}} d_k(f)\chi_D(f) u^{\deg(f)}$$

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Use

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to express

$$M(n;d_k\chi_D):=\sum_{\substack{f\,\text{monic}\\ \deg(f)=n}}d_k(f)\chi_D(f)$$

$$=(-1)^kq^{n/2}\sum_{\substack{j_1+\cdots+j_k=n\\ 0\le j_1,\dots j_k\le 2g}}\mathrm{Sc}_{j_1}(\Theta_{\chi_D})\cdots\mathrm{Sc}_{j_k}(\Theta_{\chi_D})$$

for $n\le 2gk$.

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for $n\leq 2gk$.

It is $O\left(q^{\frac{n}{2}}\right)$ by the Riemann Hypothesis.

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Ingredients in the proof - Extraction of the variance

$$\mathrm{Var}^*(\mathcal{S}^S_{d_k,n}) = \frac{1}{\#\mathcal{P}_{2g+1}} \sum_{P\in\mathcal{P}_{2g+1}} \left| \mathcal{S}^S_{d_k,n}(P) - \frac{1}{2} \sum_{\substack{f \text{ monic} \\ \deg(f)=n, P\nmid f}} d_k(f) \right|^2$$

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$$= \frac{q^n}{4\#\mathcal{P}_{2g+1}} \sum_{P\in\mathcal{P}_{2g+1}} \left| \sum_{\substack{j_1+\cdots+j_k=n \\ 0\le j_1,\dots,j_k\le 2g}} \mathrm{Sc}_{j_1}(\Theta_{\chi_P}) \cdots \mathrm{Sc}_{j_k}(\Theta_{\chi_P}) \right|^2$$

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Ingredients in the proof - Equidistribution

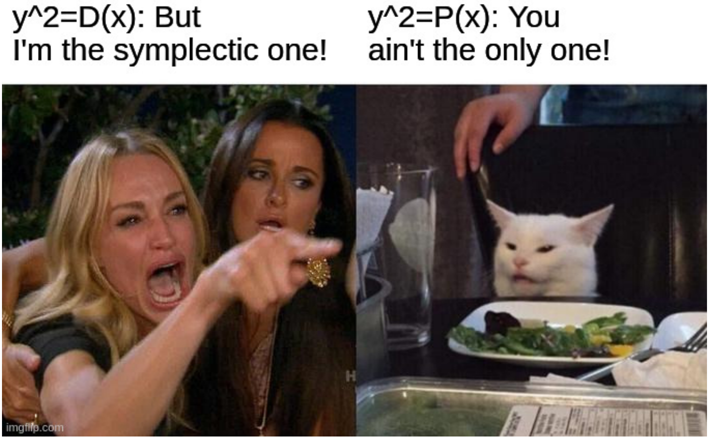
$$\mathcal{H}_{2g+1} = \square\text{-free monic polynomials of degree } 2g+1$$

Theorem (Katz & Sarnak (1999))

Let F be $\mathbb{C}$-valued central continuous function on $\mathrm{Sp}(2g)$. Then

$$\lim_{q\to\infty} \frac{1}{\#\mathcal{H}_{2g+1}} \sum_{Q\in\mathcal{H}_{2g+1}} F(Q) = \int_{\mathrm{Sp}(2g)} F(U)dU.$$

Ingredients in the proof - Equidistribution



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Ingredients in the proof - Monodromy and equidistribution

Theorem (Katz, personal communication)

Let F be $\mathbb{C}$-valued central continuous function on $\mathrm{Sp}(2g)$ and let σ be any fixed partition of $2g+1$. Then

$$\lim_{q\to\infty} \frac{1}{\#\mathcal{H}_\sigma} \sum_{Q\in\mathcal{H}_\sigma} F(Q) = \int_{\mathrm{Sp}(2g)} F(U)dU.$$

$$\mathcal{H}_{d_1,\dots,d_n} = \left\{ f \in \mathcal{H}_{2g+1}, : f = \prod_i^n f_{d_i}, f_{d_i} \text{ monic}, \deg(f_{d_i}) = d_i \right\}$$

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The monodromy group of $y^2 = f(x)$ with $f(x) \in \mathcal{H}_{d_1,\dots,d_n}$ is a subgroup of $\mathrm{Sp}(2g)$.

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It contains the monodromy of $y^2 = (x-t)f_{2g}(x)$ with $f_{2g}(x) \in \mathcal{H}_{d_1-1,\dots,d_n}$ and parameter t, known to be $\mathrm{Sp}(2g)$.

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Ingredients in the proof - Monodromy and equidistribution

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Use inclusion-exclusion to prove the same for

$$\mathcal{P}_{d_1,\dots,d_n} = \left\{ f \in \mathcal{H}_{2g+1}, : f = \prod_i^n f_{d_i}, f_{d_i} \text{ monic and irreducible } \deg(f_{d_i}) = d_i \right\}.$$

Understanding the integral

$$I^S_{d_k,2}(n;N) := \int_{\mathrm{Sp}(2N)} \left| \sum_{\substack{j_1+\cdots+j_k=n \\ 0\le j_1,\ldots,j_k\le 2N}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_k}(U) \right|^{2} dU.$$

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Understanding the integral - Symmetric function theory

Bump and Gamburd (2006), Medjedovic and Rubinstein (2021)

$$\int_{\mathrm{Sp}(2N)} \prod_{i=1}^{r} \det(1 + x_i U) dU = \sum_{\substack{\lambda_1 \leq 2N \\ \lambda \text{ even}}} s_\lambda(x_1, \dots, x_r),$$

where s_λ is a Schur function, a generating polynomial counting semi-standard Young tableaux. (A similar identity exists for Unitary matrices, also due to Bump and Gamburd.)

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$$\int_{\mathrm{Sp}(2N)} \det(1 + xU)^k \det(1 + yU)^k dU = \sum_{\substack{\lambda_1 \leq 2N \\ \lambda \text{ even}}} s_\lambda(\underbrace{x, \dots, x}_{k}, \underbrace{y, \dots, y}_{k})$$

$$= \sum_{m,n=0}^{2Nk} J^S_{d_k,2}(m, n; N) x^m y^n.$$

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$$I^S_{d_k,2}(n; N) := J^S_{d_k,2}(n, n; N).$$

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Understanding the integral - Symmetric function theory

- Let $c = \frac{n}{2N}$. Then

$$I^S_{d_k,2}(n;N) = \#\left(\Lambda_{2k^2+k-2} \cap (2N \cdot V^S_c)\right),$$

where V^S_c is a convex polytope.

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By Ehrhart theory, $I^S_{d_k,2}(cN;N)$ is a polynomial in N of degree $2k^2 + k - 2$.

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By Ehrhart theory, $I^S_{d_k,2}(cN;N)$ is a polynomial in N of degree $2k^2+k-2$.

- Medjedovic and Rubinstein (2021) use the Desarmenien–Stembridge–Proctor formula,

$$\sum_{\substack{\lambda_1 \leq 2N \\ \lambda \text{ even}}} s_\lambda(x_1,\dots,x_r) = \prod_{i=1}^{r} \frac{1}{1-x_i^2} \prod_{i<j} \frac{1}{1-x_ix_j} \frac{\det_{1\leq i,j\leq r}[x_i^{j-1} - x_i^{2N+2r+1-j}]}{\Delta(x_1,\dots,x_r)}.$$

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Computing the integral - 0-swaps terms

For $n \leq N$, we have

$$I^S_{d_k,2}(n;N) = \frac{1}{G(1+k)} \sum_{\substack{\ell=0 \\ \ell \equiv n \pmod 2}}^{n} \binom{\frac{n-\ell}{2} + \binom{k+1}{2} - 1}{\binom{k+1}{2} - 1}^2 \binom{\ell + k^2 - 1}{k^2 - 1}.$$

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$k = 1$

$$\int_{\mathrm{Sp}(2N)} \mathrm{Sc}_n(U)^2 dU = \frac{2n+3}{4} + \frac{(-1)^n}{4} \qquad \gamma^S_{d_1}(c) = \begin{cases} c & 0 \leq c \leq \frac{1}{2}, \\ 1 - c & \frac{1}{2} \leq c \leq 1. \end{cases}$$

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$k = 2$

$$\int_{\mathrm{Sp}(2N)} \left| \sum_{\substack{j_1+j_2=n \\ 0 \le j_1, j_2 \le 2N}} \mathrm{Sc}_{j_1}(U)\mathrm{Sc}_{j_2}(U) \right|^2 dU \to \gamma^S_{d_2}(c) = \begin{cases} \frac{c^8}{840} & 0 \le c \le \frac{1}{2}, \\ \frac{(2-c)^8}{840} & \frac{3}{2} \le c \le 2. \end{cases}$$

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Back to the rationals

Conjecture (Kuperberg and L. (2022++))

Let p be a prime and define

$$\mathcal{S}^{S}_{d_k;x}(p) = \sum_{\substack{n\leq x\\ n\equiv\square \,(mod\, p)\\ p\nmid n}} d_k(n).$$

Let $x^{1/k} \leq y$. *For* $y \leq p \leq 2y$,

$$\mathrm{Var}_{p\in[y,2y]}\left(\mathcal{S}^{S}_{d_k;x}(p)\right) \sim a_k^S \frac{x}{4}\gamma^S_{d_k}\left(\frac{\log x}{\log y}\right)(\log y)^{2k^2+k-2},$$

where

$$a_k^S = \prod_p \left\{\left(1-\frac{1}{p}\right)^{k(2k-1)}\left(1+\frac{1}{p}\right)^{-1}\left[\frac{1}{2}\left(\left(1-\frac{1}{\sqrt{p}}\right)^{-2k}+\left(1+\frac{1}{\sqrt{p}}\right)^{-2k}\right)+\frac{1}{p}\right]\right\}.$$

$$\mathrm{Var}_{p\in[y,2y]}\left(\mathcal{S}^S_{d_k;x}(p)\right) := \frac{1}{y}\sum_{y<p\leq 2y}\left(\sum_{\substack{n\leq x\\ n\equiv\square \pmod{p}\\ p\nmid n}} d_k(n) - \left\langle \mathcal{S}^S_{d_k;x}(p)\right\rangle\right)^2,$$

where

$$\left\langle \mathcal{S}^S_{d_k;x}(p)\right\rangle := \frac{1}{y}\sum_{y<p\leq 2y}\sum_{\substack{n\leq x\\ n\equiv\square \pmod{p}\\ p\nmid n}} d_k(n).$$

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Thank you very much for your attention!