L^p-norm bounds for automorphic forms

Rizwan Khan

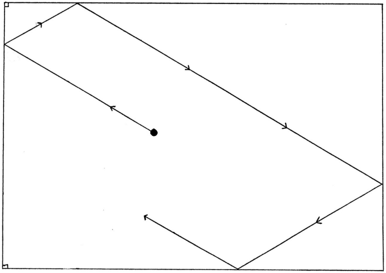
University of Mississippi

(joint work with Peter Humphries)

Dynamical systems

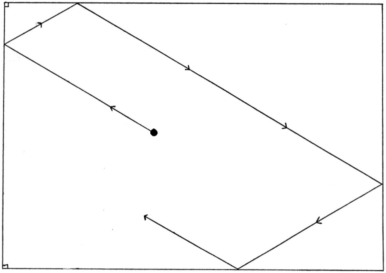
Dynamical systems

Billiards on a rectangular table



Dynamical systems

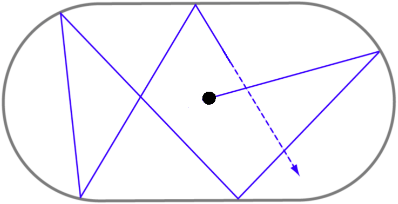
Billiards on a rectangular table



• Predictable: the trajectories of two billiard balls with almost identical initial conditions are similar.

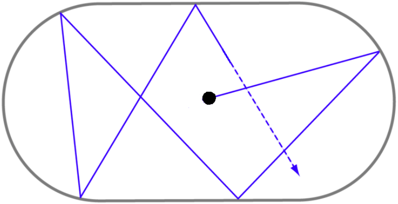
A different billiards table

A different billiards table



Stadium

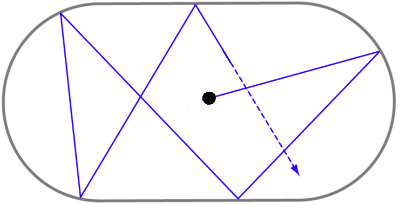
A different billiards table



Stadium

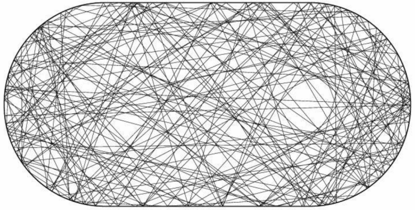
- Trajectories generally very sensitive to initial conditions

A different billiards table

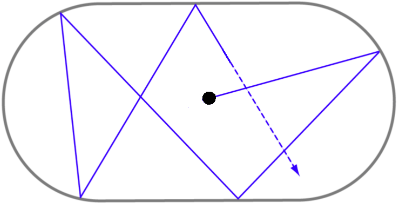


Stadium

- Trajectories generally very sensitive to initial conditions
- Trajectories generally fill out stadium in an even manner.

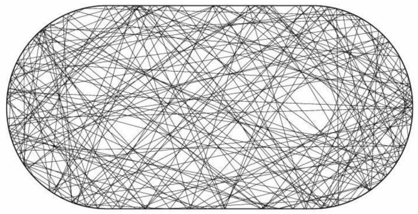


A different billiards table



Stadium

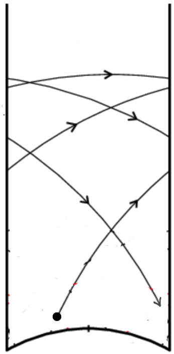
- Trajectories generally very sensitive to initial conditions
- Trajectories generally fill out stadium in an even manner.



Chaotic dynamical system

Negatively curved surface

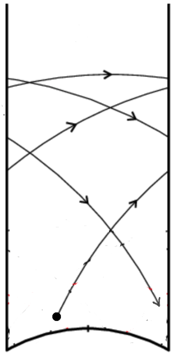
Negatively curved surface



Billards on $X = SL_2(\mathbb{Z})\backslash\mathbb{H}$

Geodesics are vertical lines and circular arcs $\perp$ real axis.

Negatively curved surface



Billards on $X = SL_2(\mathbb{Z})\backslash\mathbb{H}$

Geodesics are vertical lines and circular arcs $\perp$ real axis.

Chaotic dynamical system

Quantum chaos

Quantum chaos

These were examples in classical dynamics.

Quantum chaos

These were examples in classical dynamics.

Quantum mechanics: a wave ψ takes the place of the billiard ball,
governed by a Schrödinger type equation.

Quantum chaos

These were examples in classical dynamics.

Quantum mechanics: a wave ψ takes the place of the billiard ball,
governed by a Schrödinger type equation.

Probability of finding wave in region *A* given by integrating $|\psi|^2$ over *A*.

Quantum chaos

These were examples in classical dynamics.

Quantum mechanics: a wave ψ takes the place of the billiard ball, governed by a Schrödinger type equation.

Probability of finding wave in region *A* given by integrating $|\psi|^2$ over *A*.

Q. Do the waves exhibit characteristics of chaos, when the underlying classical billiards is chaotic?

Arithmetic Quantum chaos

Arithmetic Quantum chaos

$X = SL_2(\mathbb{Z})\backslash\mathbb{H}$ (classical dynamics is chaotic)

Arithmetic Quantum chaos

$X = SL_2(\mathbb{Z})\backslash\mathbb{H}$ (classical dynamics is chaotic)

Waves are (square-integrable) eigenfunctions of Laplacian

Arithmetic Quantum chaos

$X = SL_2(\mathbb{Z})\backslash\mathbb{H}$ (classical dynamics is chaotic)

Waves are (square-integrable) eigenfunctions of Laplacian

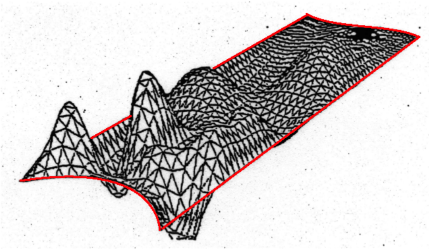
Assume also that waves are real valued and eigenfunctions of the Hecke operators.

Arithmetic Quantum chaos

$X = SL_2(\mathbb{Z})\backslash\mathbb{H}$ (classical dynamics is chaotic)

Waves are (square-integrable) eigenfunctions of Laplacian

Assume also that waves are real valued and eigenfunctions of the Hecke operators.



Hecke Maass cusp forms

(closely related to classical holomorphic modular forms)

Quantum Unique Ergodicity

Quantum Unique Ergodicity

Theorem (Lindenstrauss 2006, Soundararajan 2010)

Let f be a Hecke Maass cusp form with Laplacian eigenvalue λ_f, normalized so that $\int_X f(z)^2 d\mu_z = \int_X 1 d\mu_z$.

Quantum Unique Ergodicity

Theorem (Lindenstrauss 2006, Soundararajan 2010)

Let f be a Hecke Maass cusp form with Laplacian eigenvalue λ_f, normalized so that $\int_X f(z)^2 d\mu_z = \int_X 1 d\mu_z$.

Quantum Unique Ergodicity

Theorem (Lindenstrauss 2006, Soundararajan 2010)

Let f be a Hecke Maass cusp form with Laplacian eigenvalue λ_f, normalized so that $\int\limits_X f(z)^2 d\mu_z = \int\limits_X 1 d\mu_z$.

Then for any nice compact set $A \subset X$,

$$\frac{\int\limits_A f(z)^2 d\mu_z}{\int\limits_A 1 d\mu_z} \to 1 \qquad \textit{as} \quad \lambda_f \to \infty.$$

Quantum Unique Ergodicity

Theorem (Lindenstrauss 2006, Soundararajan 2010)

Let f be a Hecke Maass cusp form with Laplacian eigenvalue λ_f, normalized so that $\int_X f(z)^2 d\mu_z = \int_X 1 d\mu_z$.

Then for any nice compact set $A \subset X$,

$$\frac{\int_A f(z)^2 d\mu_z}{\int_A 1 d\mu_z} \to 1 \qquad \text{as} \quad \lambda_f \to \infty.$$

QUE says that the probability of finding a wave in A depends only on A.

Quantum Unique Ergodicity

Theorem (Lindenstrauss 2006, Soundararajan 2010)

Let f be a Hecke Maass cusp form with Laplacian eigenvalue λ_f, normalized so that $\int\limits_X f(z)^2 d\mu_z = \int\limits_X 1 d\mu_z$.

Then for any nice compact set $A \subset X$,

$$\frac{\int\limits_A f(z)^2 d\mu_z}{\int\limits_A 1 d\mu_z} \to 1 \qquad as \quad \lambda_f \to \infty.$$

QUE says that the probability of finding a wave in A depends only on A.

In other words, waves of high energy become evenly spread out.

Random Wave Conjecture

QUE is part of a bigger conjecture.

Random Wave Conjecture

QUE is part of a bigger conjecture.

Conjecture (Berry 1977)

Let f be a Hecke Maass cusp form with Laplacian eigenvalue λ_f, normalized so that $\int\limits_X f(z)^2 d\mu_z = \int\limits_X 1 d\mu_z$.

Random Wave Conjecture

QUE is part of a bigger conjecture.

Conjecture (Berry 1977)

Let f be a Hecke Maass cusp form with Laplacian eigenvalue λ_f, normalized so that $\int_X f(z)^2 d\mu_z = \int_X 1 d\mu_z$.

Random Wave Conjecture

QUE is part of a bigger conjecture.

Conjecture (Berry 1977)

Let f be a Hecke Maass cusp form with Laplacian eigenvalue λ_f, normalized so that $\int\limits_X f(z)^2 d\mu_z = \int\limits_X 1 d\mu_z$.

Then for any nice compact set $A \subset X$,

$$\frac{\int\limits_A f(z)^k d\mu_z}{\int\limits_A 1 d\mu_z} \to c_k \qquad as \quad \lambda_f \to \infty,$$

Random Wave Conjecture

QUE is part of a bigger conjecture.

Conjecture (Berry 1977)

Let f be a Hecke Maass cusp form with Laplacian eigenvalue λ_f, normalized so that $\int_X f(z)^2 d\mu_z = \int_X 1 d\mu_z$.

Then for any nice compact set $A \subset X$,

$$\frac{\int_A f(z)^k d\mu_z}{\int_A 1 d\mu_z} \to c_k \qquad \text{as} \quad \lambda_f \to \infty,$$

where $c_k = k^{th}$ moment of a standard Normal random variable.

Random Wave Conjecture

QUE is part of a bigger conjecture.

Conjecture (Berry 1977)

Let f be a Hecke Maass cusp form with Laplacian eigenvalue λ_f, normalized so that $\int_X f(z)^2 d\mu_z = \int_X 1 d\mu_z$.

Then for any nice compact set $A \subset X$,

$$\frac{\int_A f(z)^k d\mu_z}{\int_A 1 d\mu_z} \to c_k \qquad as \quad \lambda_f \to \infty,$$

where $c_k = k^{th}$ moment of a standard Normal random variable.

Hecke Maass cusp forms are expected to behave like random waves for large Laplacian eigenvalue!

Random Wave Conjecture

QUE is part of a bigger conjecture.

Conjecture (Berry 1977)

Let f be a Hecke Maass cusp form with Laplacian eigenvalue λ_f, normalized so that $\int\limits_X f(z)^2 d\mu_z = \int\limits_X 1 d\mu_z$.

Then for any nice compact set $A \subset X$,

$$\frac{\int\limits_A f(z)^k d\mu_z}{\int\limits_A 1 d\mu_z} \to c_k \qquad as \quad \lambda_f \to \infty,$$

where $c_k = k^{th}$ moment of a standard Normal random variable.

Hecke Maass cusp forms are expected to behave like random waves for large Laplacian eigenvalue!
Can take $A = X$ for small values of k.

What's known

What's known

$k = 2 \ldots \checkmark\checkmark$ This is QUE.

What's known

$k = 2 \ldots \checkmark\checkmark$ This is QUE.

$k = 3 \ldots \checkmark\checkmark$ $A = X$ by Watson (2002), compact A by Huang (2022).

What's known

$k = 2 \ldots \checkmark\checkmark$ This is QUE.

$k = 3 \ldots \checkmark\checkmark$ $A = X$ by Watson (2002), compact A by Huang (2022).

$k = 4 \ldots \checkmark$ Partial results, for $A = X$.

What's known

$k = 2 \ldots \checkmark\checkmark$ This is QUE.

$k = 3 \ldots \checkmark\checkmark$ $A = X$ by Watson (2002), compact A by Huang (2022).

$k = 4 \ldots \checkmark$ Partial results, for $A = X$.

Proven for Eisenstein series (Djanković-K 2020)

What's known

$k = 2 \ldots \checkmark\checkmark$ This is QUE.

$k = 3 \ldots \checkmark\checkmark$ $A = X$ by Watson (2002), compact A by Huang (2022).

$k = 4 \ldots \checkmark$ Partial results, for $A = X$.

Proven for Eisenstein series (Djanković-K 2020)

Proven for Dihedral Maass cusp forms (Humphries-K 2020)

What's known

$k = 2 \ldots \checkmark\checkmark$ This is QUE.

$k = 3 \ldots \checkmark\checkmark$ $A = X$ by Watson (2002), compact A by Huang (2022).

$k = 4 \ldots \checkmark$ Partial results, for $A = X$.

Proven for Eisenstein series (Djanković-K 2020)

Proven for Dihedral Maass cusp forms (Humphries-K 2020)

Conditionally proven (on GLH) for general Hecke Maass cusp forms (Buttcane-K 2017).

L^p-norm Upper bounds

L^p-norm Upper bounds

For p relatively small, one would expect

Conjecture (Iwaniec-Sarnak 1995)

$$\|f\|_p = \Big(\int_X |f(z)|^p d\mu_z \Big)^{\frac{1}{p}} \ll \lambda_f^{\epsilon}.$$

L^p-norm Upper bounds

For p relatively small, one would expect

Conjecture (Iwaniec-Sarnak 1995)

$$\|f\|_p = \Big(\int_X |f(z)|^p d\mu_z \Big)^{\frac{1}{p}} \ll \lambda_f^{\epsilon}.$$

L^p-norm Upper bounds

For p relatively small, one would expect

Conjecture (Iwaniec-Sarnak 1995)

$$\|f\|_p = \Big(\int_X |f(z)|^p d\mu_z \Big)^{\frac{1}{p}} \ll \lambda_f^{\epsilon}.$$

Warning: Such small bounds are not expected in general.

L^p-norm Upper bounds

For p relatively small, one would expect

Conjecture (Iwaniec-Sarnak 1995)

$$\|f\|_p = \Big(\int_X |f(z)|^p d\mu_z \Big)^{\frac{1}{p}} \ll \lambda_f^{\epsilon}.$$

Warning: Such small bounds are not expected in general.

Sogge (1988) has proven L^p-norm bounds for Laplacian eigenfunctions ψ of compact n-dimensional Riemannian manifolds.

L^p-norm Upper bounds

For p relatively small, one would expect

Conjecture (Iwaniec-Sarnak 1995)

$$\|f\|_p = \Big(\int_X |f(z)|^p d\mu_z \Big)^{\frac{1}{p}} \ll \lambda_f^{\epsilon}.$$

Warning: Such small bounds are not expected in general.

Sogge (1988) has proven L^p-norm bounds for Laplacian eigenfunctions ψ of compact n-dimensional Riemannian manifolds.

His bounds are not $\|\psi\|_p \ll \lambda_\psi^{\epsilon}$, yet they are sharp for the n-sphere S^n.

L^4-norm bounds

The L^4-norm is special due to its connection to L-functions.

L^4-norm bounds

The L^4-norm is special due to its connection to L-functions.

Sarnak and Watson conjectured: $\|f\|_4 \ll \lambda_f^\epsilon$.

L^4-norm bounds

The L^4-norm is special due to its connection to L-functions.

Sarnak and Watson conjectured: $\|f\|_4 \ll \lambda_f^{\epsilon}$.

Until now, the best known was Sogge's bound $\|f\|_4 \ll \lambda_f^{\frac{1}{16}+\epsilon}$.

L^4-norm bounds

The L^4-norm is special due to its connection to L-functions.

Sarnak and Watson conjectured: $\|f\|_4 \ll \lambda_f^{\epsilon}$.

Until now, the best known was Sogge's bound $\|f\|_4 \ll \lambda_f^{\frac{1}{16}+\epsilon}$.

Theorem (Humphries-K)

$$\|f\|_4 \ll \lambda_f^{\frac{3}{304}+\epsilon}.$$

L^4-norm bounds

The L^4-norm is special due to its connection to L-functions.

Sarnak and Watson conjectured: $\|f\|_4 \ll \lambda_f^{\epsilon}$.

Until now, the best known was Sogge's bound $\|f\|_4 \ll \lambda_f^{\frac{1}{16}+\epsilon}$.

Theorem (Humphries-K)

$$\|f\|_4 \ll \lambda_f^{\frac{3}{304}+\epsilon}.$$

L^4-norm bounds

The L^4-norm is special due to its connection to L-functions.

Sarnak and Watson conjectured: $\|f\|_4 \ll \lambda_f^{\epsilon}$.

Until now, the best known was Sogge's bound $\|f\|_4 \ll \lambda_f^{\frac{1}{16}+\epsilon}$.

Theorem (Humphries-K)

$$\|f\|_4 \ll \lambda_f^{\frac{3}{304}+\epsilon}.$$

If we assume the Lindelöf Hypothesis for $L(\frac{1}{2}, u_j)$ *and* $\zeta(\frac{1}{2}+it)$, *then* $\|f\|_4 \ll \lambda_f^{\epsilon}$.

L^4-norm bounds

The L^4-norm is special due to its connection to L-functions.

Sarnak and Watson conjectured: $\|f\|_4 \ll \lambda_f^{\epsilon}$.

Until now, the best known was Sogge's bound $\|f\|_4 \ll \lambda_f^{\frac{1}{16}+\epsilon}$.

Theorem (Humphries-K)

$$\|f\|_4 \ll \lambda_f^{\frac{3}{304}+\epsilon}.$$

If we assume the Lindelöf Hypothesis for $L(\frac{1}{2}, u_j)$ *and* $\zeta(\frac{1}{2}+it)$, *then* $\|f\|_4 \ll \lambda_f^{\epsilon}$.

Interpolating with Iwaniec and Sarnak's bound $\|f\|_\infty \ll \lambda_f^{\frac{5}{24}+\epsilon}$, gives new L^p-norm bounds for all $p \geq 4$.

L-functions

L-functions

Let f be a Hecke Maass cusp form with Laplace eigenvalue λ_f

L-functions

Let f be a Hecke Maass cusp form with Laplace eigenvalue $\lambda_f = \frac{1}{4} + t_f^2$.

L-functions

Let f be a Hecke Maass cusp form with Laplace eigenvalue $\lambda_f = \frac{1}{4} + t_f^2$.

Let $\{u_j : j \geq 1\}$ be an orthonormal basis of Hecke Maass cusp forms with Laplace eigenvalue $\lambda_j = \frac{1}{4} + t_j^2$.

L-functions

Let f be a Hecke Maass cusp form with Laplace eigenvalue $\lambda_f = \frac{1}{4} + t_f^2$.

Let $\{u_j : j \geq 1\}$ be an orthonormal basis of Hecke Maass cusp forms with Laplace eigenvalue $\lambda_j = \frac{1}{4} + t_j^2$.

$$\int_X f(z)^4 d\mu_z$$

L-functions

Let f be a Hecke Maass cusp form with Laplace eigenvalue $\lambda_f = \frac{1}{4} + t_f^2$.

Let $\{u_j : j \geq 1\}$ be an orthonormal basis of Hecke Maass cusp forms with Laplace eigenvalue $\lambda_j = \frac{1}{4} + t_j^2$.

$$\int_X f(z)^4 d\mu_z$$

$$= \langle f^2, f^2 \rangle$$

Let f be a Hecke Maass cusp form with Laplace eigenvalue $\lambda_f = \frac{1}{4} + t_f^2$.

Let $\{u_j : j \geq 1\}$ be an orthonormal basis of Hecke Maass cusp forms with Laplace eigenvalue $\lambda_j = \frac{1}{4} + t_j^2$.

$$\int_X f(z)^4 d\mu_z$$

$$= \langle f^2, f^2 \rangle = \sum_{j \geq 1} |\langle f^2, u_j \rangle|^2 + \ldots$$ by Parseval

L-functions

Let f be a Hecke Maass cusp form with Laplace eigenvalue $\lambda_f = \frac{1}{4} + t_f^2$.

Let $\{u_j : j \geq 1\}$ be an orthonormal basis of Hecke Maass cusp forms with Laplace eigenvalue $\lambda_j = \frac{1}{4} + t_j^2$.

$$\int_X f(z)^4 d\mu_z$$

$$= \langle f^2, f^2 \rangle = \sum_{j \geq 1} |\langle f^2, u_j \rangle|^2 + \dots \text{ by Parseval}$$

$$= \sum_{t_j < 2t_f} \frac{L(\frac{1}{2}, u_j) L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)}{t_j t_f} + \dots \text{ by Watson's formula}$$

L-functions

Let f be a Hecke Maass cusp form with Laplace eigenvalue $\lambda_f = \frac{1}{4} + t_f^2$.

Let $\{u_j : j \geq 1\}$ be an orthonormal basis of Hecke Maass cusp forms with Laplace eigenvalue $\lambda_j = \frac{1}{4} + t_j^2$.

$$\int_X f(z)^4 d\mu_z$$

$$= \langle f^2, f^2 \rangle = \sum_{j \geq 1} |\langle f^2, u_j \rangle|^2 + \dots \text{ by Parseval}$$

$$= \sum_{t_j < 2t_f} \frac{L(\frac{1}{2}, u_j) L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)}{t_j t_f} + \dots \text{ by Watson's formula}$$

- Mean value of product of GL_2 and $GL_2 \times GL_3$ L-functions.

L-functions

Let f be a Hecke Maass cusp form with Laplace eigenvalue $\lambda_f = \frac{1}{4} + t_f^2$.

Let $\{u_j : j \geq 1\}$ be an orthonormal basis of Hecke Maass cusp forms with Laplace eigenvalue $\lambda_j = \frac{1}{4} + t_j^2$.

$$\int_X f(z)^4 d\mu_z$$

$$= \langle f^2, f^2 \rangle = \sum_{j \geq 1} |\langle f^2, u_j \rangle|^2 + \dots \text{ by Parseval}$$

$$= \sum_{t_j < 2t_f} \frac{L(\frac{1}{2}, u_j) L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)}{t_j t_f} + \dots \text{ by Watson's formula}$$

- Mean value of product of GL_2 and $GL_2 \times GL_3$ L-functions.
- Central values are real and ≥ 0.

L-functions

Let f be a Hecke Maass cusp form with Laplace eigenvalue $\lambda_f = \frac{1}{4} + t_f^2$.

Let $\{u_j : j \geq 1\}$ be an orthonormal basis of Hecke Maass cusp forms with Laplace eigenvalue $\lambda_j = \frac{1}{4} + t_j^2$.

$$\int_X f(z)^4 d\mu_z$$
$$= \langle f^2, f^2 \rangle = \sum_{j \geq 1} |\langle f^2, u_j \rangle|^2 + \dots \text{ by Parseval}$$
$$= \sum_{t_j < 2t_f} \frac{L(\frac{1}{2}, u_j) L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)}{t_j t_f} + \dots \text{ by Watson's formula}$$

- Mean value of product of GL_2 and $GL_2 \times GL_3$ L-functions.
- Central values are real and ≥ 0.
- Reminiscent of the fourth moment problem $\sum_{t_j < T} L(\frac{1}{2}, u_j)^4$.

L-functions

Let f be a Hecke Maass cusp form with Laplace eigenvalue $\lambda_f = \frac{1}{4} + t_f^2$.

Let $\{u_j : j \geq 1\}$ be an orthonormal basis of Hecke Maass cusp forms with Laplace eigenvalue $\lambda_j = \frac{1}{4} + t_j^2$.

$$\int_X f(z)^4 d\mu_z$$

$$= \langle f^2, f^2 \rangle = \sum_{j \geq 1} |\langle f^2, u_j \rangle|^2 + \ldots \text{ by Parseval}$$

$$= \sum_{t_j < 2t_f} \frac{L(\frac{1}{2}, u_j) L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)}{t_j t_f} + \ldots \text{ by Watson's formula}$$

- Mean value of product of GL_2 and $GL_2 \times GL_3$ L-functions.
- Central values are real and ≥ 0.
- Reminiscent of the fourth moment problem $\sum_{t_j < T} L(\frac{1}{2}, u_j)^4$.
- Dihedral & Eisenstein: the $GL_2 \times GL_3$ L-function factors further.

Getting a feel

Getting a feel

$$\|f\|_4^4 \approx \sum_{t_j < 2t_f} \frac{L(\frac{1}{2}, u_j) L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)}{t_j t_f}$$

Getting a feel

$$\|f\|_4^4 \approx \sum_{t_j < 2t_f} \frac{L(\frac{1}{2}, u_j) L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)}{t_j t_f} \overset{?}{\ll} t_f^{\epsilon}.$$

Getting a feel

$$\|f\|_4^4 \approx \sum_{t_j < 2t_f} \frac{L(\frac{1}{2}, u_j) L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)}{t_j t_f} \overset{?}{\ll} t_f^{\epsilon}.$$

Working in dyadic intervals $T \leq t_j \leq 2T$,

$$\sum_{t_j \sim T} \frac{L(\frac{1}{2}, u_j) L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)}{T t_f} \overset{?}{\ll} t_f^{\epsilon}.$$

Getting a feel

$$\|f\|_4^4 \approx \sum_{t_j<2t_f} \frac{L(\frac{1}{2},u_j)L(\frac{1}{2},u_j\times \mathrm{sym}^2 f)}{t_j t_f} \overset{?}{\ll} t_f^{\epsilon}.$$

Working in dyadic intervals $T \leq t_j \leq 2T$,

$$\sum_{t_j\sim T} \frac{L(\frac{1}{2},u_j)L(\frac{1}{2},u_j\times \mathrm{sym}^2 f)}{T t_f} \overset{?}{\ll} t_f^{\epsilon}.$$

The sum is over T^2 forms and the convexity bound is
$L(\frac{1}{2},u_j)L(\frac{1}{2},u_j\times \mathrm{sym}^2 f) \ll T t_f$

Getting a feel

$$\|f\|_4^4 \approx \sum_{t_j<2t_f} \frac{L(\frac{1}{2},u_j)L(\frac{1}{2},u_j\times \mathrm{sym}^2 f)}{t_j t_f} \overset{?}{\ll} t_f^{\epsilon}.$$

Working in dyadic intervals $T \le t_j \le 2T$,

$$\sum_{t_j\sim T} \frac{L(\frac{1}{2},u_j)L(\frac{1}{2},u_j\times \mathrm{sym}^2 f)}{T t_f} \overset{?}{\ll} t_f^{\epsilon}.$$

The sum is over T^2 forms and the convexity bound is
$L(\frac{1}{2},u_j)L(\frac{1}{2},u_j\times \mathrm{sym}^2 f) \ll T t_f$

- For $T \asymp 1$, convexity is enough.

Getting a feel

$$\|f\|_4^4 \approx \sum_{t_j<2t_f} \frac{L(\frac{1}{2},u_j)L(\frac{1}{2},u_j\times \mathrm{sym}^2 f)}{t_j t_f} \overset{?}{\ll} t_f^{\epsilon}.$$

Working in dyadic intervals $T \le t_j \le 2T$,

$$\sum_{t_j\sim T} \frac{L(\frac{1}{2},u_j)L(\frac{1}{2},u_j\times \mathrm{sym}^2 f)}{T t_f} \overset{?}{\ll} t_f^{\epsilon}.$$

The sum is over T^2 forms and the convexity bound is
$L(\frac{1}{2},u_j)L(\frac{1}{2},u_j\times \mathrm{sym}^2 f) \ll T t_f$

- For $T \asymp 1$, convexity is enough.
- For $T \ll t_f$, need to prove certain subconvexity bound on average.

Getting a feel

$$\|f\|_4^4 \approx \sum_{t_j<2t_f} \frac{L(\frac{1}{2},u_j)L(\frac{1}{2},u_j\times \mathrm{sym}^2 f)}{t_j t_f} \overset{?}{\ll} t_f^{\epsilon}.$$

Working in dyadic intervals $T \le t_j \le 2T$,

$$\sum_{t_j\sim T} \frac{L(\frac{1}{2},u_j)L(\frac{1}{2},u_j\times \mathrm{sym}^2 f)}{T t_f} \overset{?}{\ll} t_f^{\epsilon}.$$

The sum is over T^2 forms and the convexity bound is
$L(\frac{1}{2},u_j)L(\frac{1}{2},u_j\times \mathrm{sym}^2 f) \ll T t_f$

- For $T \asymp 1$, convexity is enough.
- For $T \ll t_f$, need to prove certain subconvexity bound on average.
- For $T \asymp t_f$, need to prove Lindelöf on average.

Reciprocity formulae for L-functions

These are certain identities between two different moments of central values of L-functions.

Reciprocity formulae for L-functions

These are certain identities between two different moments of central values of L-functions.

Goes back to work of Kuznetsov and Motohashi.

Reciprocity formulae for L-functions

These are certain identities between two different moments of central values of L-functions.

Goes back to work of Kuznetsov and Motohashi.

We need to prove new reciprocity formulae and use them in a hybrid form (f is not fixed).

$GL_2 \times GL_3 \rightsquigarrow GL_3, GL_1$ product

$GL_2 \times GL_3 \rightsquigarrow GL_3, GL_1$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) h(t_j) = Diag + \int_{-\infty}^{\infty} L(\tfrac{1}{2} + it, \mathrm{sym}^2 f)\zeta(\tfrac{1}{2} + it)\widetilde{h}(t)dt$$

$GL_2 \times GL_3 \rightsquigarrow GL_3, GL_1$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) h(t_j) = Diag + \int_{-\infty}^{\infty} L(\tfrac{1}{2} + it, \mathrm{sym}^2 f) \zeta(\tfrac{1}{2} + it) \widetilde{h}(t) dt$$

$GL_2 \times GL_3 \rightsquigarrow GL_3, GL_1$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) h(t_j) = Diag + \int_{-\infty}^{\infty} L(\tfrac{1}{2} + it, \mathrm{sym}^2 f)\zeta(\tfrac{1}{2} + it)\widetilde{h}(t) dt$$

- If $t_j \sim T$ on LHS, then $t \sim t_f^2 / T^2$ on RHS.

$GL_2 \times GL_3 \rightsquigarrow GL_3, GL_1$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) h(t_j) = Diag + \int_{-\infty}^{\infty} L(\tfrac{1}{2} + it, \mathrm{sym}^2 f)\zeta(\tfrac{1}{2} + it)\widetilde{h}(t)dt$$

- If $t_j \sim T$ on LHS, then $t \sim t_f^2/T^2$ on RHS.
- Similar formula also recently proven by Kwan.

$GL_2 \times GL_3 \rightsquigarrow GL_3, GL_1$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) h(t_j) = Diag + \int_{-\infty}^{\infty} L(\tfrac{1}{2} + it, \mathrm{sym}^2 f)\zeta(\tfrac{1}{2} + it)\widetilde{h}(t)dt$$

- If $t_j \sim T$ on LHS, then $t \sim t_f^2/T^2$ on RHS.
- Similar formula also recently proven by Kwan.
- Compare with Motohashi's formula

$$\sum L(\tfrac{1}{2}, u_j)^3 \rightsquigarrow \int |\zeta(\tfrac{1}{2} + it)|^4 dt.$$

$GL_2 \times GL_3 \rightsquigarrow GL_3, GL_1$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) h(t_j) = Diag + \int_{-\infty}^{\infty} L(\tfrac{1}{2} + it, \mathrm{sym}^2 f)\zeta(\tfrac{1}{2} + it)\widetilde{h}(t)dt$$

- If $t_j \sim T$ on LHS, then $t \sim t_f^2/T^2$ on RHS.
- Similar formula also recently proven by Kwan.
- Compare with Motohashi's formula

$$\sum L(\tfrac{1}{2}, u_j)^3 \rightsquigarrow \int |\zeta(\tfrac{1}{2} + it)|^4 dt.$$

- RHS can be bounded by inserting absolute values to kill the transform function,

$GL_2 \times GL_3 \rightsquigarrow GL_3, GL_1$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) h(t_j) = Diag + \int_{-\infty}^{\infty} L(\tfrac{1}{2} + it, \mathrm{sym}^2 f)\zeta(\tfrac{1}{2} + it)\widetilde{h}(t)dt$$

- If $t_j \sim T$ on LHS, then $t \sim t_f^2/T^2$ on RHS.
- Similar formula also recently proven by Kwan.
- Compare with Motohashi's formula

$$\sum L(\tfrac{1}{2}, u_j)^3 \rightsquigarrow \int |\zeta(\tfrac{1}{2} + it)|^4 dt.$$

- RHS can be bounded by inserting absolute values to kill the transform function, then apply Cauchy-Schwarz, & the Large Sieve.

$GL_2, GL_2 \times GL_3$ product $\rightsquigarrow$ $GL_2, GL_2 \times GL_3$ product

$GL_2, GL_2 \times GL_3$ product $\rightsquigarrow$ $GL_2, GL_2 \times GL_3$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) h(t_j) = Diag + \sum_{t_j} L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) \widetilde{h}(t_j).$$

$GL_2, GL_2 \times GL_3$ product $\rightsquigarrow$ $GL_2, GL_2 \times GL_3$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) h(t_j) = Diag + \sum_{t_j} L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) \widetilde{h}(t_j).$$

$GL_2, GL_2 \times GL_3$ product $\rightsquigarrow$ $GL_2, GL_2 \times GL_3$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j)L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f)h(t_j) = Diag + \sum_{t_j} L(\tfrac{1}{2}, u_j)L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f)\widetilde{h}(t_j).$$

- If $t_j \sim T$ on LHS, then $t_j \sim t_f/T$ on RHS.

$GL_2, GL_2 \times GL_3$ product $\rightsquigarrow$ $GL_2, GL_2 \times GL_3$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) h(t_j) = Diag + \sum_{t_j} L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) \widetilde{h}(t_j).$$

- If $t_j \sim T$ on LHS, then $t_j \sim t_f / T$ on RHS.
- Compare with Kuznetsov: $\sum L(\frac{1}{2}, u_j)^4 \rightsquigarrow \sum L(\frac{1}{2}, u_j)^4$.

$GL_2, GL_2 \times GL_3$ product $\rightsquigarrow$ $GL_2, GL_2 \times GL_3$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) h(t_j) = Diag + \sum_{t_j} L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) \widetilde{h}(t_j).$$

- If $t_j \sim T$ on LHS, then $t_j \sim t_f / T$ on RHS.
- Compare with Kuznetsov: $\sum L(\frac{1}{2}, u_j)^4 \rightsquigarrow \sum L(\frac{1}{2}, u_j)^4$.
- See also work of Blomer-Li-Miller ($GL_2 \times GL_4$), Humphries-K. ($GL_2 \times GL_2, GL_2 \times GL_2$ product).

$GL_2, GL_2 \times GL_3$ product $\rightsquigarrow$ $GL_2, GL_2 \times GL_3$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) h(t_j) = Diag + \sum_{t_j} L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) \widetilde{h}(t_j).$$

- If $t_j \sim T$ on LHS, then $t_j \sim t_f / T$ on RHS.
- Compare with Kuznetsov: $\sum L(\frac{1}{2}, u_j)^4 \rightsquigarrow \sum L(\frac{1}{2}, u_j)^4$.
- See also work of Blomer-Li-Miller ($GL_2 \times GL_4$), Humphries-K. ($GL_2 \times GL_2, GL_2 \times GL_2$ product).
- On RHS, we insert absolute values to kill the transform function,

$GL_2, GL_2 \times GL_3$ product $\rightsquigarrow$ $GL_2, GL_2 \times GL_3$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) h(t_j) = Diag + \sum_{t_j} L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) \widetilde{h}(t_j).$$

- If $t_j \sim T$ on LHS, then $t_j \sim t_f / T$ on RHS.
- Compare with Kuznetsov: $\sum L(\frac{1}{2}, u_j)^4 \rightsquigarrow \sum L(\frac{1}{2}, u_j)^4$.
- See also work of Blomer-Li-Miller ($GL_2 \times GL_4$), Humphries-K. ($GL_2 \times GL_2, GL_2 \times GL_2$ product).
- On RHS, we insert absolute values to kill the transform function, then leave it as it is since central values are ≥ 0. Important!

$GL_2, GL_2 \times GL_3$ product $\rightsquigarrow$ $GL_2, GL_2 \times GL_3$ product

Theorem (Humphries-K)

$$\sum_{t_j} L(\tfrac{1}{2}, u_j)L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f)h(t_j) = Diag + \sum_{t_j} L(\tfrac{1}{2}, u_j)L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f)\widetilde{h}(t_j).$$

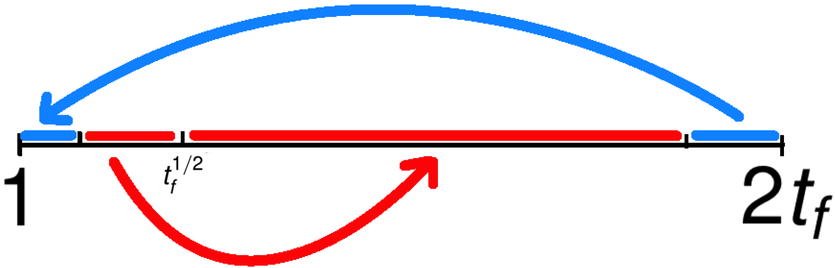
- If $t_j \sim T$ on LHS, then $t_j \sim t_f/T$ on RHS.
- Compare with Kuznetsov: $\sum L(\frac{1}{2}, u_j)^4 \rightsquigarrow \sum L(\frac{1}{2}, u_j)^4$.
- See also work of Blomer-Li-Miller ($GL_2 \times GL_4$), Humphries-K. ($GL_2 \times GL_2, GL_2 \times GL_2$ product).
- On RHS, we insert absolute values to kill the transform function, then leave it as it is since central values are ≥ 0. Important!
- LHS and RHS sums have same length when $T = t_f^{\frac{1}{2}}$.

Application to the L^4-norm problem

Application to the L^4-norm problem

Need to understand $\sum_{t_j < 2t_f} L(\frac{1}{2}, u_j) L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)$.

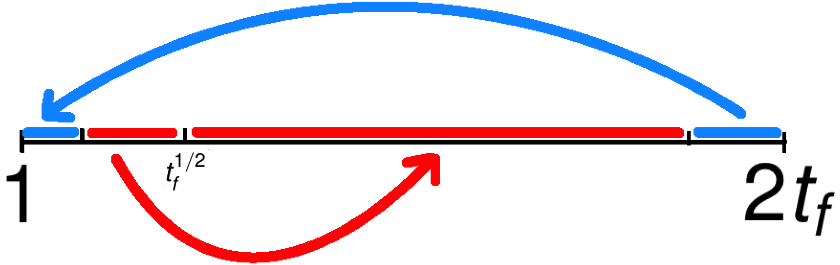
Need to understand $\sum_{t_j<2t_f} L(\frac{1}{2}, u_j)L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)$.



Application to the L^4-norm problem

Need to understand $\sum_{t_j < 2t_f} L(\frac{1}{2}, u_j) L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)$.

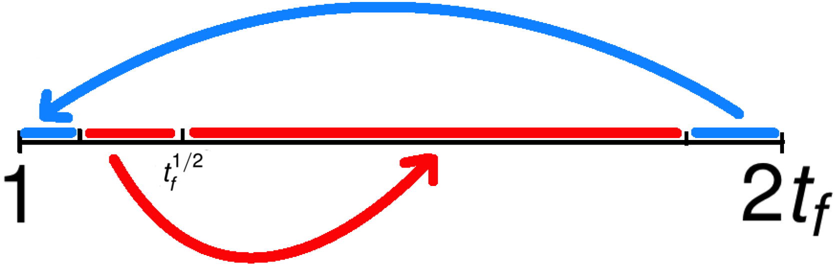
Use reciprocity to pass to a short dual moment, then trivially bound using the Large Sieve.



Application to the L^4-norm problem

Need to understand $\sum_{t_j<2t_f} L(\frac{1}{2}, u_j)L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)$.

Use reciprocity to pass to a short dual moment, then trivially bound using the Large Sieve.

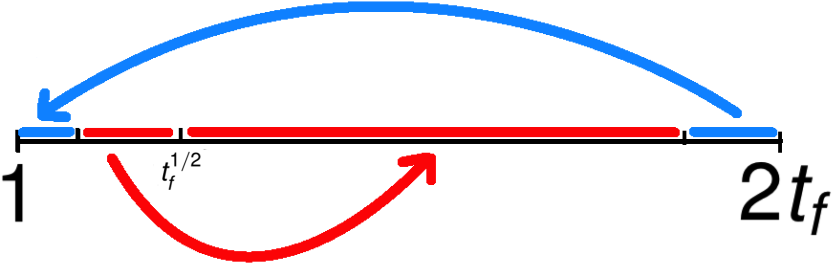


Average is too short to get a hold of. Use reciprocity to pass to a long dual moment.

Application to the L^4-norm problem

Need to understand $\sum_{t_j<2t_f} L(\frac{1}{2}, u_j)L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)$.

Use reciprocity to pass to a short dual moment, then trivially bound using the Large Sieve.



Average is too short to get a hold of. Use reciprocity to pass to a long dual moment. Apply Hölder to get

$$\Big(\sum L(\tfrac{1}{2}, u_j)^{12}\Big)^{\frac{1}{12}} \Big(\sum L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f)\Big)^{\frac{10}{12}} \Big(\sum L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f)^2\Big)^{\frac{1}{12}},$$

and apply Jutila's bound, reciprocity, and the large sieve.

Rough idea for second reciprocity formula

Rough idea for second reciprocity formula

$L(\frac{1}{2}, u_j) = 0$ if root number $\lambda_j(-1) = -1$.

Rough idea for second reciprocity formula

$L(\frac{1}{2}, u_j) = 0$ if root number $\lambda_j(-1) = -1$.

$$\sum_j L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f)$$

Rough idea for second reciprocity formula

$L(\frac{1}{2}, u_j) = 0$ if root number $\lambda_j(-1) = -1$.

$$\sum_j L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) = \sum_j \lambda_j(-1) L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f)$$

Rough idea for second reciprocity formula

$L(\frac{1}{2}, u_j) = 0$ if root number $\lambda_j(-1) = -1$.

$$\sum_j L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) = \sum_j \lambda_j(-1) L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) = \sum_j \lambda_j(-1) \sum_{n,m} \frac{\lambda_j(n)\lambda_j(m)\lambda_f(m^2)}{\sqrt{nm}}$$

Rough idea for second reciprocity formula

$L(\frac{1}{2}, u_j) = 0$ if root number $\lambda_j(-1) = -1$.

$$\begin{aligned}
\sum_j L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) \\
&= \sum_j \lambda_j(-1) L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) \\
&= \sum_j \lambda_j(-1) \sum_{n,m} \frac{\lambda_j(n)\lambda_j(m)\lambda_f(m^2)}{\sqrt{nm}} \\
&= \sum_{n,m} \frac{1}{\sqrt{nm}} \sum_c \frac{S(-n,m,c)}{c} \lambda_f(m^2)
\end{aligned}$$

Rough idea for second reciprocity formula

$L(\frac{1}{2}, u_j) = 0$ if root number $\lambda_j(-1) = -1$.

$$\begin{aligned}
\sum_j L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) \\
&= \sum_j \lambda_j(-1) L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) \\
&= \sum_j \lambda_j(-1) \sum_{n,m} \frac{\lambda_j(n)\lambda_j(m)\lambda_f(m^2)}{\sqrt{nm}} \\
&= \sum_{n,m} \frac{1}{\sqrt{nm}} \sum_c \frac{S(-n,m,c)}{c} \lambda_f(m^2) \\
&= \sum_{n,m} \frac{1}{\sqrt{nm}} \sum_c \frac{S(n,m,c)}{c} \lambda_f(m^2)
\end{aligned}$$

Rough idea for second reciprocity formula

$L(\frac{1}{2}, u_j) = 0$ if root number $\lambda_j(-1) = -1$.

$$\sum_j L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f)$$
$$= \sum_j \lambda_j(-1) L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f)$$
$$= \sum_j \lambda_j(-1) \sum_{n,m} \frac{\lambda_j(n)\lambda_j(m)\lambda_f(m^2)}{\sqrt{nm}}$$
$$= \sum_{n,m} \frac{1}{\sqrt{nm}} \sum_c \frac{S(-n,m,c)}{c} \lambda_f(m^2)$$
$$= \sum_{\mathcal{n},\mathcal{m}} \frac{1}{\sqrt{\mathcal{n}\mathcal{m}}} \sum_c \frac{S(\mathcal{n},\mathcal{m},c)}{c} \lambda_f(\mathcal{m}^2)$$
$$= \sum_{\mathcal{j}} L(\tfrac{1}{2}, u_{\mathcal{j}}) L(\tfrac{1}{2}, u_{\mathcal{j}} \times \mathrm{sym}^2 f)$$

Rough idea for second reciprocity formula

$L(\frac{1}{2}, u_j) = 0$ if root number $\lambda_j(-1) = -1$.

$$
\begin{aligned}
\sum_j L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) \\
&= \sum_j \lambda_j(-1) L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) \\
&= \sum_j \lambda_j(-1) \sum_{n,m} \frac{\lambda_j(n)\lambda_j(m)\lambda_f(m^2)}{\sqrt{nm}} \\
&= \sum_{n,m} \frac{1}{\sqrt{nm}} \sum_c \frac{S(-n,m,c)}{c} \lambda_f(m^2) \\
&= \sum_{\mathcal{n},\mathcal{m}} \frac{1}{\sqrt{\mathcal{n}\mathcal{m}}} \sum_c \frac{S(\mathcal{n},\mathcal{m},c)}{c} \lambda_f(\mathcal{m}^2) \\
&= \sum_{\mathcal{j}} L(\tfrac{1}{2}, u_{\mathcal{j}}) L(\tfrac{1}{2}, u_{\mathcal{j}} \times \mathrm{sym}^2 f)
\end{aligned}
$$

- Work with Dirichlet series and careful analytic continuation

Rough idea for second reciprocity formula

$L(\frac{1}{2}, u_j) = 0$ if root number $\lambda_j(-1) = -1$.

$$\begin{aligned}
\sum_j L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) \\
&= \sum_j \lambda_j(-1) L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f) \\
&= \sum_j \lambda_j(-1) \sum_{n,m} \frac{\lambda_j(n)\lambda_j(m)\lambda_f(m^2)}{\sqrt{nm}} \\
&= \sum_{n,m} \frac{1}{\sqrt{nm}} \sum_c \frac{S(-n,m,c)}{c} \lambda_f(m^2) \\
&= \sum_{\mathcal{n},\mathcal{m}} \frac{1}{\sqrt{\mathcal{n}\mathcal{m}}} \sum_c \frac{S(\mathcal{n},\mathcal{m},c)}{c} \lambda_f(\mathcal{m}^2) \\
&= \sum_{\mathcal{j}} L(\tfrac{1}{2}, u_{\mathcal{j}}) L(\tfrac{1}{2}, u_{\mathcal{j}} \times \mathrm{sym}^2 f)
\end{aligned}$$

- Work with Dirichlet series and careful analytic continuation
- The various steps get encoded in the final transform function, which takes work to understand.

Weight function

Let $h(t_j)$ be weight function on LHS, and $\widetilde{h}(t_j)$ the transform function on the RHS.

Weight function

Let $h(t_j)$ be weight function on LHS, and $\widetilde{h}(t_j)$ the transform function on the RHS.

$\widetilde{h}$ looks like the inverse Mellin transform of

Weight function

Let $h(t_j)$ be weight function on LHS, and $\widetilde{h}(t_j)$ the transform function on the RHS.

$\widetilde{h}$ looks like the inverse Mellin transform of

The Mellin transform of the Bessel transform of h arising from Kuznetsov's formula

$\times$

Weight function

Let $h(t_j)$ be weight function on LHS, and $\widetilde{h}(t_j)$ the transform function on the RHS.

$\widetilde{h}$ looks like the inverse Mellin transform of

The Mellin transform of the Bessel transform of h arising from Kuznetsov's formula

$\times$

Gamma functions which arise from Voronoi and Poisson summation

$\times$

Weight function

Let $h(t_j)$ be weight function on LHS, and $\widetilde{h}(t_j)$ the transform function on the RHS.

$\widetilde{h}$ looks like the inverse Mellin transform of

The Mellin transform of the Bessel transform of h arising from Kuznetsov's formula

$\times$

Gamma functions which arise from Voronoi and Poisson summation

$\times$

The Mellin transform of the Bessel kernels from Kuznetsov's formula for Kloosterman sums (these are also gamma functions).

Transition region

Transition region

More care is needed than I have let on.

Transition region

More care is needed than I have let on.

$L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)$ experiences 'conductor-dropping' for t_j close to $2t_f$.

Transition region

More care is needed than I have let on.

$L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)$ experiences 'conductor-dropping' for t_j close to $2t_f$.

So we need to understand short interval sums

$$\sum_{|t_j - 2t_f| \sim t_f^{\alpha}} L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f),$$

for which the transform function is tricky to understand.

Transition region

More care is needed than I have let on.

$L(\frac{1}{2}, u_j \times \mathrm{sym}^2 f)$ experiences 'conductor-dropping' for t_j close to $2t_f$.

So we need to understand short interval sums

$$\sum_{|t_j - 2t_f| \sim t_f^\alpha} L(\tfrac{1}{2}, u_j) L(\tfrac{1}{2}, u_j \times \mathrm{sym}^2 f),$$

for which the transform function is tricky to understand.

Reminiscent of work of Jutila:

$$\sum_{|t_j - T| \sim T^\alpha} L(\tfrac{1}{2}, u_j)^4 \quad \text{(sharp bounds only for } \alpha > \tfrac{1}{3}.\text{)}$$

End

Thank you!