Geodesic Restrictions of Maaß Forms and Moments of Hecke L-Functions

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The Eigenvalue Problem for the Laplacian

Let (M, g) be a compact (or, more generally, finite volume) n-dimensional Riemannian manifold without boundary.

Example

- the n-sphere $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = n\}$
- the n-torus $\mathbb{T}^n = \{(x_1, \ldots, x_n) \in (\mathbb{R}/\mathbb{Z})^n\}$

We shall study Laplacian eigenfunctions: L^2-normalised solutions $f \in L^2(M)$ to the eigenvalue problem

$$\Delta f = \lambda f,$$

where the Laplace–Beltrami operator Δ is

$$\Delta := -\frac{1}{\sqrt{|\det g|}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} g^{jk} \sqrt{|\det g|} \frac{\partial}{\partial x_k}$$

and $\lambda \in [0, \infty)$ is the Laplacian eigenvalue of f.

Examples: Torus

These eigenfunctions and eigenvalues need not have closed forms.

For particularly nice manifolds M, such as spheres or tori, we can explicitly describe the Laplacian eigenfunctions and eigenvalues.

Example

The Laplace–Beltrami operator on $\mathbb{T}^n$ is

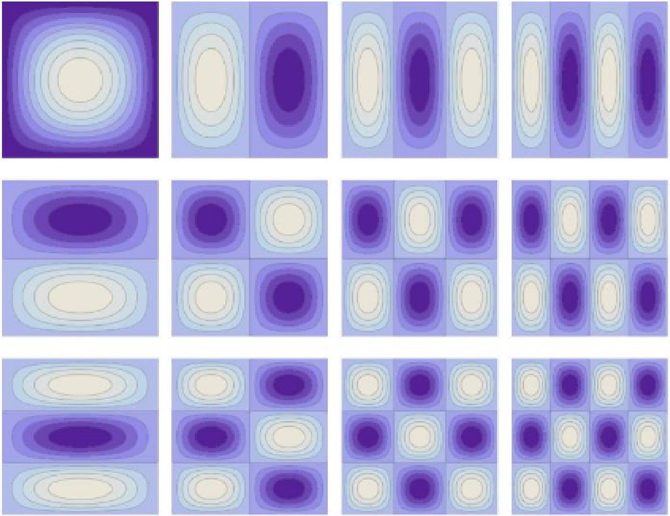
$$\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

A basis of $L^2(\mathbb{T}^n)$ consisting of Laplacian eigenfunctions $f(x_1, \ldots, x_n)$ is given by

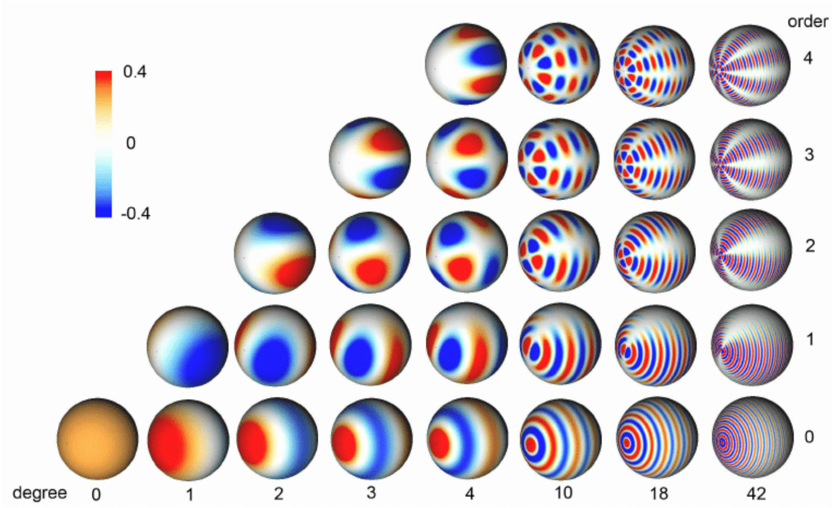
$$\{\sin(2\pi(x_1 y_1 + \cdots + x_n y_n)),\ \cos(2\pi(x_1 y_1 + \cdots + x_n y_n)) : (y_1, \ldots, y_n) \in \mathbb{Z}^n\}$$

with eigenvalues $4\pi^2(y_1^2 + \cdots + y_n^2)$.

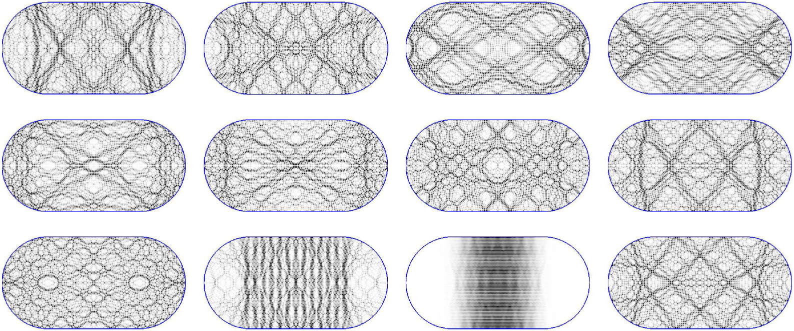
Examples: Torus



Examples: Sphere



Examples: Bunimovich Stadium



Examples: Modular Surface

Interesting setting for number theorists:
Riemannian locally symmetric spaces $M = \Gamma \backslash G/K$;

- G is a Lie group,
- K is a maximal compact subgroup of G,
- Γ is a lattice in G.

Simplest interesting case: $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}(2)$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$;

- $G/K \cong \mathbb{H}$ is the upper half-plane
$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\},$$
- $\Gamma \backslash G/K \cong \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ is the modular surface
$$\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} = \left\{z = x + iy \in \mathbb{H} : -\frac{1}{2} < x < \frac{1}{2},\ x^2 + y^2 > 1\right\},$$
- Laplacian eigenfunctions are automorphic forms.

Examples: Modular Surface

$\mathbb{H}$ is a negatively curved hyperbolic surface.

$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ inherits a hyperbolic metric from $\mathbb{H}$.

The Laplacian is $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.

The volume measure on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is $d\mathrm{vol}(z) = \frac{3}{\pi}\frac{dx\,dy}{y^2}$.

Nonconstant eigenfunctions of Δ on $\mathbb{H}$ that are $\mathrm{SL}_2(\mathbb{Z})$-invariant (equivalently, nonconstant Laplacian eigenfunctions on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$) are *Maaß forms*:

- type of automorphic form closely related to classical holomorphic modular forms.

Examples: Modular Surface

For each $k \in 2\mathbb{N}$, one can instead define the weight k Laplacian

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \frac{\partial}{\partial x}.$$

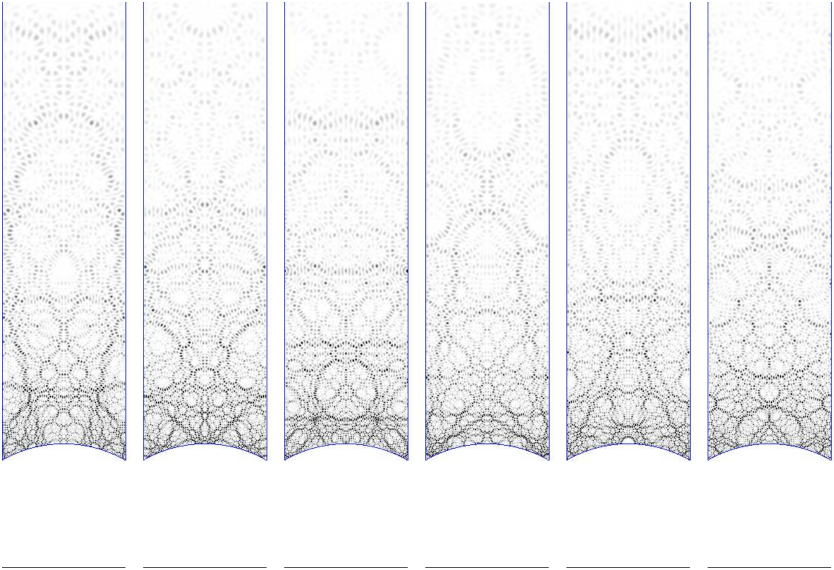
If $g \in L^2 \left(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}\right)$ satisfies the automorphic eigenvalue problem

$$\Delta_k g = \frac{k}{2} \left(1 - \frac{k}{2}\right) g,$$

$$g\left(\frac{az+b}{cz+d}\right) = \left(\frac{cz+d}{|cz+d|}\right)^k g(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

the function $G(z) := y^{-k/2} g(x+iy)$ is a holomorphic modular form of weight k.

Examples: Modular Surface



L^2-Restriction Bounds for Laplacian Eigenfunctions

Question

In the large eigenvalue limit, how big is the restriction of a Laplacian eigenfunction to a chosen submanifold?

Natural way to study the size of restrictions to a submanifold N of eigenfunctions f on a manifold M is to estimate their restricted L^2-norms in terms of λ:

$$\|f|_N\|_2 := \left(\int_N |f(x)|^2 \, d\mathrm{vol}_g(x)\right)^{\frac{1}{2}}.$$

Heuristic

If M and N are *arithmetic*, then $\|f|_N\|_2^2$ is equal to a weighted moment of L-functions.

Bilinear L^2-Norm Bounds for Maaß Forms

Example

Take $M = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \times \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ and $N = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ (embedded diagonally).

- Laplacian eigenfunctions on M are of the form $F(z_1, z_2) = f_1(z_1) f_2(z_2)$, where f_1, f_2 are Maaß forms.
- The restricted L^2-norm of F is

$$\|F|_N\|_2^2 = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} |f_1(z) f_2(z)|^2 \, \frac{3}{\pi} \frac{dx\,dy}{y^2}.$$

  In particular, if $f_1 = f_2$, this is the L^4-norm of f_1.
- By Parseval's identity for $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ and the Watson–Ichino triple product formula,

$$\|F|_N\|_2^2 \approx \sum_f \frac{\Lambda\left(\frac{1}{2}, f \otimes f_1 \otimes f_2\right)}{\Lambda(1, \mathrm{sym}^2 f)\Lambda(1, \mathrm{sym}^2 f_1)\Lambda(1, \mathrm{sym}^2 f_2)}.$$

Geodesic L^2-Restriction Bounds

Question

Let M be a 2-dimensional compact (or, more generally, finite volume) Riemannian manifold, and let N be a finite length geodesic segment on M. In the large eigenvalue limit, how large is $\|f|_N\|_2$?

Theorem (Burq–Gérard–Tzvetkov (2007))

We have that

$$\|f|_N\|_2 \ll \lambda^{\frac{1}{8}}.$$

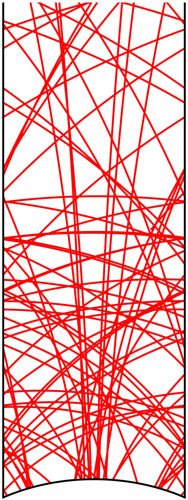
Moreover, this is sharp if $M = S^2$.

This is the *convexity bound*.

Question

Can one do better for *arithmetic* surfaces M?

Geodesics on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$



Geodesic L^2-Restriction Bounds on the Modular Surface

Geodesics on $\mathbb{H}$ are either vertical lines or semicircles centred on the real axis.

Geodesics on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ are the projections of geodesics on $\mathbb{H}$.

Theorem (Marshall (2016))

Let M be an arithmetic 2-dimensional compact Riemannian manifold and N be a finite length geodesic segment on M. Then

$$\|f|_N\|_2 \ll_\varepsilon \lambda^{\frac{1}{8}-\frac{1}{56}+\varepsilon}.$$

Proof uses the amplified pre-trace formula; no connection to moments of L-functions.

Geodesic L^2-Restriction Bounds on the Modular Surface

Question

Can one do better if not only the surface M is arithmetic but the geodesic segment N is *also* arithmetic?

Theorem (Ghosh–Reznikov–Sarnak (2013))

Let $M = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ *and* N *be a finite length geodesic segment of the vertical geodesic from* 0 *to* $i\infty$*. Then* $\|f|_N\|_2 \ll_\varepsilon \lambda^\varepsilon$*.*

Proof uses moments!

Sketch of proof.

We show more generally that

$$\int_0^\infty |f(iy)|^2 \, \frac{dy}{y} \ll_\varepsilon \lambda^\varepsilon.$$

By Parseval's identity for $L^2((0,\infty))$ (i.e. for the Mellin transform), LHS is

$$\frac{1}{2\pi} \int_{-\infty}^\infty \left| \int_0^\infty f(iy) y^{it} \, \frac{dy}{y} \right|^2 \, dt.$$

The inner squared integral is equal to

$$\frac{\left|\Lambda\left(\frac{1}{2}+it, f\right)\right|^2}{\Lambda(1, \operatorname{sym}^2 f)} \approx \frac{\left|L\left(\frac{1}{2}+it, f\right)\right|^2}{L(1, \operatorname{sym}^2 f)} \frac{1}{(1+|t_f - t|)^{1/2}(1+|t_f+t|)^{1/2}} \times \begin{cases} 1 & \text{if } |t| \leq t_f, \\ e^{-\pi(|t|-t_f)} & \text{if } |t| \geq t_f, \end{cases}$$

where $\lambda = 1/4 + t_f^2$.

Vertical Geodesic L^2-Restriction Bounds

Sketch of proof (cont'd).

Since $1/L(1, \mathrm{sym}^2 f) \ll_\varepsilon \lambda^\varepsilon$, just need to show

$$\int_{-t_f}^{t_f} \frac{\left|L\left(\frac{1}{2}+it, f\right)\right|^2}{(1+|t_f-t|)^{1/2}(1+|t_f+t|)^{1/2}}\, dt \ll_\varepsilon \lambda^\varepsilon.$$

Trivially holds under the Lindelöf hypothesis; need to show this unconditionally.

Aside

Were f the Eisenstein series $E(z, 1/2+it_f)$, then $L(s,f) = \zeta(s+it_f)\zeta(s-it_f)$. Would need to show

$$\int_{-t_f}^{t_f} \frac{\prod_{\pm_1,\pm_2} \zeta\left(\frac{1}{2} \pm_1 it_f \pm_2 it\right)}{(1+|t_f-t|)^{1/2}(1+|t_f+t|)^{1/2}}\, dt \ll_\varepsilon t_f^\varepsilon.$$

Shifted fourth moment of the Riemann zeta function!

Vertical Geodesic L^2-Restriction Bounds

Sketch of proof (cont'd).

Need to show

$$\int_{-t_f}^{t_f} \frac{\left|L\left(\frac{1}{2}+it,f\right)\right|^2}{(1+|t_f-t|)^{1/2}(1+|t_f+t|)^{1/2}}\,dt \ll_\varepsilon \lambda^\varepsilon.$$

Approximate functional equations: insert the Dirichlet polynomial

$$L\left(\frac{1}{2}+it,f\right) \approx \sum_{n\leq(1+|t_f-t|)^{1/2}(1+|t_f+t|)^{1/2}} \frac{\lambda_f(n)}{n^{\frac{1}{2}+it}}.$$

Interchange integration and double summation, yielding

$$\sum_{m,n} \frac{\lambda_f(m)\lambda_f(n)}{\sqrt{mn}} \int \frac{1}{(1+|t_f-t|)^{1/2}(1+|t_f+t|)^{1/2}} \left(\frac{m}{n}\right)^{it} dt.$$

Estimate the integral via integration by parts: small unless m is close to n. Use Cauchy–Schwarz to separate sum over m and n. Finally, bound each sum via the fact (due to Iwaniec) that $\sum_{n\leq x} |\lambda_f(n)|^2 \ll_\varepsilon t_f^\varepsilon x$. □

Closed Geodesic L^2-Restriction Bounds

Question

Can one do better for *infinitely* many geodesics on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$?

Theorem (H.–Thorner (2022+))

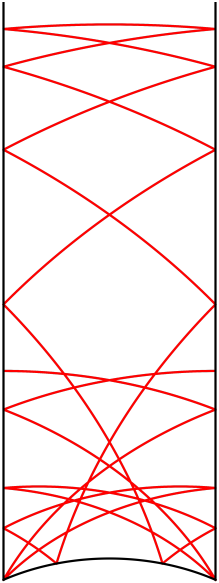
Let N be a closed *geodesic on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Then*

$$\|f|_N\|_2 \ll_\varepsilon \lambda^{\vartheta+\varepsilon},$$

where $\vartheta = \frac{7}{64}$ is the best known exponent towards the generalised Ramanujan conjecture for Maaß forms.

Assuming the generalised Ramanujan conjecture, so that $\vartheta = 0$, this is essentially sharp.

Closed Geodesics on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$



Closed Geodesics on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$

Key properties of closed geodesics:

- Bijective correspondence with narrow ideal classes of real quadratic number fields $\mathbb{Q}(\sqrt{D})$ (*arithmetic* submanifold)
- Length is $2\log\epsilon$, where ϵ is the fundamental unit of $\mathbb{Q}(\sqrt{D})$
- Infinitely many closed geodesics
- Union of all closed geodesics is dense in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$
- Topologically equivalent to a circle

Fourier Analysis on Closed Geodesics

First step of proof: Parseval's identity for $L^2(N)$.

Since closed geodesics are topologically circles, we get

$$\int_N |f(z)|^2 \, ds = \sum_{m=-\infty}^{\infty} \left| \int_N f(z) e^{-2\pi i m \theta(z)} \, ds \right|^2 .$$

Analogue of Parseval for Fourier series.

First key arithmetic fact: each character $e^{-2\pi i m \theta(z)}$ of N corresponds to a Hecke Größencharakter on $\mathbb{Q}(\sqrt{D})$:

$$\psi_m((a + b\sqrt{D})\mathcal{O}) = \left(\frac{a + b\sqrt{D}}{a - b\sqrt{D}} \right)^{\frac{\pi i m}{\log \epsilon}} .$$

Base Change

Second key arithmetic fact: associated to a Maaß form f on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is a Hilbert Maaß form F on $\mathrm{SL}_2(\mathcal{O})\backslash\mathbb{H}\times\mathbb{H}$.

F is the base change of f:

$$L(s,F) = L(s,f)L(s,f\otimes\chi_D).$$

LHS is a degree 2 L-function over $\mathbb{Q}(\sqrt{D})$.
RHS is a degree 4 L-function over $\mathbb{Q}$ (product of two degree 2 L-functions over $\mathbb{Q}$).

Hecke eigenvalues of F and f are related:

$$\sum_{\substack{\mathfrak{n}\subseteq\mathcal{O}\\ \mathrm{N}(\mathfrak{n})=n}} \lambda_F(\mathfrak{n}) = \sum_{ab=n} \lambda_f(a)\lambda_f(b)\chi_D(b).$$

Waldspurger's Formula

Theorem (Waldspurger (1985))

We have that

$$\left|\int_N f(z) e^{-2\pi i m\theta(z)}\,ds\right|^2 \approx \frac{\Lambda\left(\frac{1}{2}, F \otimes \psi_m\right)}{\Lambda(1, \operatorname{sym}^2 f)}$$

$$\approx \frac{L\left(\frac{1}{2}, F \otimes \psi_m\right)}{L(1, \operatorname{sym}^2 f)} \frac{1}{(1 + |t_f - \frac{\pi m}{\log \epsilon}|)^{1/2}(1 + |t_f + \frac{\pi m}{\log \epsilon}|)^{1/2}}$$

$$\times \begin{cases} 1 & \text{if } \frac{\pi |m|}{\log \epsilon} \leq t_f, \\ e^{-\pi\left(\frac{\pi |m|}{\log \epsilon} - t_f\right)} & \text{if } \frac{\pi |m|}{\log \epsilon} \geq t_f. \end{cases}$$

$L(s, F \otimes \psi_m)$ is a degree 2 L-function over $\mathbb{Q}(\sqrt{D})$.

Remark

Only true because N is a *closed* geodesic! No connection to L-functions otherwise.

Reduction to Weighted Moment of L-Functions

Goal

To bound $\int_N |f(z)|^2 \, ds$, we need to bound

$$\sum_{|m| \leq \frac{\log \epsilon}{\pi} t_f} \frac{L\left(\frac{1}{2}, F \otimes \psi_m\right)}{(1 + |t_f - \frac{\pi m}{\log \epsilon}|)^{1/2}(1 + |t_f + \frac{\pi m}{\log \epsilon}|)^{1/2}}.$$

Problem is reduced to a weighted first moment of L-functions.

Lindelöf hypothesis immediately implies the essentially optimal bound $O_\varepsilon(\lambda^\varepsilon)$.

Aside

May be possible to show that this is $O(1)$ under the Riemann hypothesis via Harper's method.

Approximate Functional Equation

Approximate functional equation: insert the Dirichlet polynomial

$$L\left(\frac{1}{2}, F \otimes \psi_m\right) \approx \sum_{\substack{\mathfrak{n} \subseteq \mathcal{O} \\ \mathrm{N}(\mathfrak{n}) \leq (1+|t_f - \frac{\pi n}{\log \epsilon}|)^{1/2}(1+|t_f + \frac{\pi n}{\log \epsilon}|)^{1/2}}} \frac{\lambda_F(\mathfrak{n}) \psi_m(\mathfrak{n})}{\sqrt{\mathrm{N}(\mathfrak{n})}}$$

$$\approx \sum_{\substack{(a,b) \in \mathbb{Z}^2 \\ \epsilon^{-1} < \frac{a+b\sqrt{D}}{a-b\sqrt{D}} \leq \epsilon \\ |a^2 - b^2\sqrt{D}| \leq (1+|t_f - \frac{\pi n}{\log \epsilon}|)^{1/2}(1+|t_f + \frac{\pi n}{\log \epsilon}|)^{1/2}}} \frac{\lambda_F((a+b\sqrt{D})\mathcal{O})}{\sqrt{|a^2 - b^2 D|}} \left(\frac{a+b\sqrt{D}}{a-b\sqrt{D}}\right)^{\frac{\pi i m}{\log \epsilon}}.$$

Interchanging Summation

Need to bound

$$\sum_{|m| \leq \frac{\log \epsilon}{\pi} t_f} \frac{L\left(\frac{1}{2}, F \otimes \psi_m\right)}{(1 + |t_f - \frac{\pi m}{\log \epsilon}|)^{1/2}(1 + |t_f + \frac{\pi m}{\log \epsilon}|)^{1/2}}.$$

Insert Dirichlet polynomial and interchange order of summation:

$$\sum_{(a,b)} \frac{\lambda_F((a + b\sqrt{D})\mathcal{O})}{\sqrt{|a^2 - b^2 D|}}$$

$$\times \sum_m \frac{1}{(1 + |t_f - \frac{\pi m}{\log \epsilon}|)^{1/2}(1 + |t_f + \frac{\pi m}{\log \epsilon}|)^{1/2}} \left(\frac{a + b\sqrt{D}}{a - b\sqrt{D}}\right)^{\frac{\pi i m}{\log \epsilon}}.$$

Estimate inner sum via summation by parts: small unless $a + b\sqrt{D}$ is close to $a - b\sqrt{D}$ (i.e. b small).

Estimate $\lambda_F((a + b\sqrt{D})\mathcal{O})$ pointwise via best bound towards Ramanujan conjecture: $|\lambda_F(\mathfrak{n})| \ll_\varepsilon \mathrm{N}(\mathfrak{n})^{\vartheta+\varepsilon}$.

Key Steps of Proof

Sketch of Proof.

1. Use Parseval's identity to expand L^2-norm as a sum of squares of Fourier coefficients.
2. Use Waldspurger's formula to relate to L-functions involving Hecke Größencharaktere and base change of f.
3. Insert Dirichlet polynomial for L-function.
4. Replace sum over ideals with sum over lattice points.
5. Interchange order of summation.
6. Estimate inner sum via summation by parts to restrict to the case that b is small.
7. Bound Hecke eigenvalues by estimates towards the Ramanujan conjecture.
8. Bound weighted sum of lattice points trivially. □

Generalisations

Question

Can one do better than the given upper bounds?

Conjecture (Restricted quantum unique ergodicity)

For any nice test function ψ,

$$\lim_{\lambda \to \infty} \int_N |f(z)|^2 \psi(z)\, ds = \frac{1}{\ell(N)} \int_N \psi(z)\, dz.$$

Special case of interest: $\psi \equiv 1$.

Theorem (Young (2018))

RQUE holds when f is an Eisenstein series and N is a geodesic segment of a vertical geodesic on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

When this vertical geodesic from 0 to $i\infty$, this boils down to *asymptotics* for shifted fourth moments of $\zeta(s)$.

Thank you!