Quantum variance for automorphic forms

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July 27, 2022

Moments of L-functions Workshop

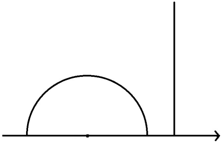
Plan

1. Automorphic forms
2. Quantum variances for Hecke–Maass cusp forms and Eisenstein series
3. Dihedral Maass forms and their quantum variance
4. Moments of L-functions

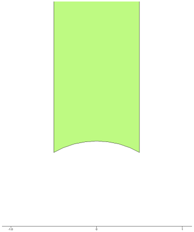
Hyperbolic surfaces

- $\mathbb{H} = \{z = x + iy : y > 0\}$ the upper half-plane with measure $\mathrm{d}\mu z = \mathrm{d}x\mathrm{d}y/y^2$.
- $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ the Laplace operator.
- Gauss curvature on $\mathbb{H}$ is negative ($= -1$).
- Geodesics are semicircles subtended on $y = 0$ and vertical lines.

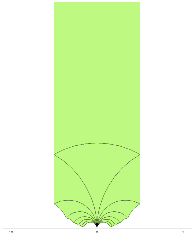
- $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_0(D) = \{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{D}\}$.
  $\Gamma \curvearrowright \mathbb{H}$ as fractional linear transforms.
- $\mathbb{X} = \Gamma_0(D)\backslash\mathbb{H}$ a hyperbolic surface.
- Gauss curvature on $\mathbb{X}$ is negative $\Rightarrow$ The geodesic flow on $T^1\mathbb{X}$ is chaotic.



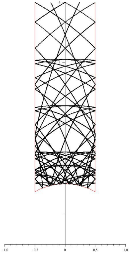
Geodesics in $\mathbb{H}$



$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$



$\Gamma_0(13)\backslash\mathbb{H}$



A projected geodesic

Hecke–Maass forms

A *cuspidal Hecke–Maass newform* ϕ of level D with a nebentypus character χ of modulus D satisfies the automorphy condition

$$\phi(\gamma z) = \chi(d)\phi(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D), \quad z \in \mathbb{H},$$

and is an eigenfunction of the Laplace operator Δ with eigenvalue λ_ϕ, and of the Hecke operators, and has no zero-th term in the Fourier expansion at any cusp. Define the spectral parameter $t_\phi \geq 0$ by $\lambda_\phi = 1/4 + t_\phi^2$.

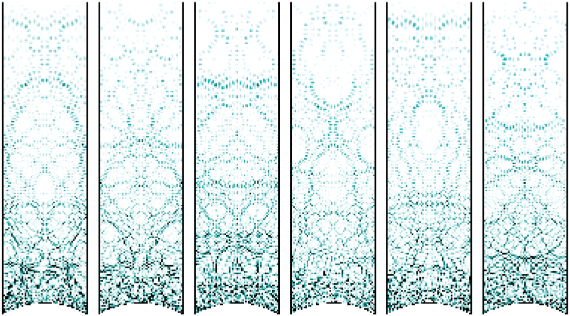
Weyl's law (Selberg):

$$\#\{\phi : t_\phi \leq T\} \sim \frac{\mathrm{vol}(\mathbb{X})}{4\pi} T^2.$$

Here we have $\mathrm{vol}(\mathbb{X}) = \frac{\pi}{3} D \prod_{p|D} (1 + p^{-1})$.

Let $\{u_j\}_{j=1}^\infty$ be an orthonormal basis of the cuspidal Hecke–Maass forms, which corresponding to the discrete spectrum. The continuous spectrum for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is parametrized by Eisenstein series $E(z, s)$.

Value distribution



This depicts the densities of a sequence of Maass forms on the hyperbolic surface.

Quantum Unique Ergodicity

For a test function $\psi : \mathbb{X} \to \mathbb{C}$, define

$$\mu_j(\psi) := \langle \psi, |u_j|^2 \rangle = \int_{\mathbb{X}} \psi(z) |u_j(z)|^2 \frac{\mathrm{d}x\mathrm{d}y}{y^2}, \quad \mu_t(\psi) := \langle \psi, |E(*, 1/2 + it)|^2 \rangle.$$

Quantum Unique Ergodicity (Rudnick–Sarnak conjecture 1994):

$$\mu_j(\psi) \sim \frac{3}{\pi} \int_{\mathbb{X}} \psi(z) \frac{\mathrm{d}x\mathrm{d}y}{y^2}, \quad \text{as } j \to \infty.$$

- Luo–Sarnak 1995:
$$\mu_t(\psi) \sim \frac{6}{\pi} \log t \int_{\mathbb{X}} \psi(z) \frac{\mathrm{d}x\mathrm{d}y}{y^2}, \quad \text{as } t \to \infty.$$
- Sarnak 2001 & Liu–Ye 2002: QUE holds for dihedral Maass forms.
- Lindenstrauss 2006 & Soundararajan 2010: QUE holds for Hecke–Maass cusp forms.
- Holowinsky and Soundararajan 2010: QUE holds for holomorphic Hecke eigenforms.

Quantum variance for cusp forms

Luo–Sarnak 1995, Zhao 2010, Sarnak–Zhao 2019, and Nelson 2016-2019 computed the quantum variance for the discrete spectrum. E.g.

Theorem (Luo–Sarnak 2004; Zhao 2010)

Define the quantum variance for cusp forms by

$$Q_C(\phi,\psi) := \lim_{T\to\infty} \frac{1}{T} \sum_{t_j \sim T} \mu_j(\phi)\overline{\mu_j(\psi)},$$

for fixed $\phi, \psi \in \{u_j\}$. Then we have

$$Q_C(\phi,\psi) = \begin{cases} c(\phi)L(1/2,\phi)\mathrm{V}_{\mathrm{cl}}(\phi), & \text{if } \phi=\psi \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}$$

with the classical variance

$$\mathrm{V}_{\mathrm{cl}}(\phi) = \frac{\left|\Gamma(\frac{1}{4}+\frac{it_\phi}{2})\right|^4}{2\pi|\Gamma(\frac{1}{2}+it_\phi)|^2}.$$

Tools: Watson's formula, trace formulas, Poincaré series, Hecke operators, ...

Quantum variance for Eisenstein series

Recall that $\mu_t(\psi) = \langle \psi, |E(*, 1/2 + it)|^2 \rangle$.

Theorem (H. 2021)

Define the quantum variance for Eisenstein series by

$$Q_E(\phi, \psi) := \lim_{T \to \infty} \frac{1}{\log T} \int_T^{2T} \mu_t(\phi) \overline{\mu_t(\psi)} \mathrm{d}t,$$

for $\phi, \psi \in \{u_j\}$. Then we have

$$Q_E(\phi, \psi) = \begin{cases} C(\phi) L(1/2, \phi)^2 \mathrm{V}_{\mathrm{cl}}(\phi), & \text{if } \phi = \psi \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}$$

where $C(\phi)$ is an explicit constant depending on ϕ.

Remark:

- In fact, we proved asymptotic formula with quantitative error terms.
- Rudnick–Soundararajan (2005) showed higher moments blow up (no CLT).

$D > 0$ squarefree and $D \equiv 1 \pmod 4$.

$F = \mathbb{Q}(\sqrt{D})$ be a fixed real quadratic fields with discriminant D.

For simplicity, we assume that F has the narrow class number 1, and D is a product of two distinct primes congruent to $3 \pmod 4$.
For example $D = 21$.

$\omega_D = \frac{1+\sqrt{D}}{2}$.
$\epsilon_D > 1$ the fundamental unit of F.
$\mathcal{O}_F = \mathbb{Z}[\omega_D]$ the ring of integers of F.
$U_F = \{\pm 1\} \times \epsilon_D^{\mathbb{Z}}$ the group of units.

For integer $k \neq 0$, we have the **Hecke Grössencharacter** Ξ_k of F defined by

$$\Xi_k((\alpha)) := \left|\frac{\alpha}{\tilde{\alpha}}\right|^{\frac{\pi i k}{\log \epsilon_D}} \quad \text{for ideal } (\alpha) \subset \mathcal{O}_F \text{ with generator } \alpha,$$

where $\tilde{\alpha}$ is the conjugate of α under the nontrivial automorphism of F.

Dihedral Maass forms

Let $\mathcal{B}_0^*(D, \chi_D)$ denote the set of L^2-normalized newforms of weight 0 for $\Gamma_0(D)$, with nebentypus character χ_D (the Kronecker symbol). Maass showed that the theta-like series associated to Ξ_k by

$$\phi_k(z) := \rho_k(1)\, y^{1/2} \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_F \\ \mathfrak{a}\neq\{0\}}} \Xi_k(\mathfrak{a}) K_{it_k}(2\pi\, \mathrm{N}(\mathfrak{a})y)\big(e(\mathrm{N}(\mathfrak{a})x) + e(-\mathrm{N}(\mathfrak{a})x)\big) \in \mathcal{B}_0^*(D, \chi_D),$$

where $z = x + iy \in \mathbb{H}$, $t_k := t_{\phi_k} = \pi k/\log \epsilon_D$ and ϕ_k has Laplace eigenvalue $1/4 + t_k^2$. Here $\mathrm{N}(\mathfrak{a}) = \#\mathcal{O}_F/\mathfrak{a}$ is the norm of a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_F$, $K_s(z)$ is the modified Bessel function, and $\rho_k(1)$ is the positive real number such that ϕ_k is L^2-normalized, i.e.,

$$\|\phi_k\|_2^2 = \int_{\Gamma_0(D)\backslash\mathbb{H}} |\phi_k(z)|^2 \frac{\mathrm{d}x\mathrm{d}y}{y^2} = 1.$$

Weyl's law: Let $t_k := \frac{\pi k}{\log \epsilon_D}$, then

$$\#\{\phi_k : 0 < t_k \leq T\} \sim \frac{\log \epsilon_D}{\pi} T.$$

Define

$$\mu_k(\psi) := \langle \psi, |\phi_k|^2 \rangle.$$

Define the quantum covariance for dihedral Maass forms by

$$Q(\psi_1, \psi_2; K; \Phi) := \sum_{k \in \mathbb{Z}} \mu_k(\psi_1)\overline{\mu_k(\psi_2)}\Phi\left(\frac{k}{K}\right)$$

for $\psi_1, \psi_2 \in L^2_{\mathrm{cusp}}(\mathbb{X})$.

Define the harmonic weighted quantum covariance by

$$Q^{\mathrm{h}}(\psi_1, \psi_2; K; \Phi) := \sum_{k \in \mathbb{Z}} L(1, \phi_{2k})^2 \mu_k(\psi_1)\overline{\mu_k(\psi_2)}\Phi\left(\frac{k}{K}\right),$$

where $L(s, \phi_{2k})$ is the L-function of ϕ_{2k}.

Quantum variance for dihedral Maass forms

Theorem (H.–Lester 2020)

Let ψ be an even Hecke–Maass cuspidal newform on $\Gamma_0(D)$. Then as $K \to \infty$ we have that

$$Q^{\mathrm{h}}(\psi, \psi; K; \Phi) = \widetilde{\Phi}(0) A^{\mathrm{h}}(\psi)\, L(\tfrac{1}{2}, \psi) L(\tfrac{1}{2}, \psi \times \chi_D) \mathrm{V}_{\mathrm{cl}}(\psi) + o(1),$$

where

$$A^{\mathrm{h}}(\psi) = \frac{\pi \log \epsilon_D}{2D^2 \zeta_D(2) L(1, \chi_D)} \left(1 + \frac{\lambda_\psi(p_1)}{\sqrt{p_1}} + \frac{\lambda_\psi(p_2)}{\sqrt{p_2}} + \frac{\lambda_\psi(D)}{\sqrt{D}}\right).$$

Assume the Generalized Ramanujan Conjecture (GRC). Then as $K \to \infty$ we have that

$$Q(\psi, \psi; K; \Phi) = \widetilde{\Phi}(0) A(\psi)\, L(\tfrac{1}{2}, \psi) L(\tfrac{1}{2}, \psi \times \chi_D) V(\psi) + o(1),$$

where $A(\psi) = A^{\mathrm{h}}(\psi) C'_{D,\psi}$, with an explicit $C'_{D,\psi}$.

Application: If $A^{\mathrm{h}}(\psi)\, L(\frac{1}{2}, \psi) L(\frac{1}{2}, \psi \times \chi_D) \neq 0$, then $\mu_k(\psi) = \Omega(k^{-1/2-\varepsilon})$.
Conjecture: $\mu_k(\psi) = O(k^{-1/2+\varepsilon})$. GRH implies this conjecture.

Quantum covariance for dihedral Maass forms

Theorem (H.–Lester 2020)

Assume Generalized Riemann Hypothesis (GRH). Let ψ_1, ψ_2 be two orthogonal even Hecke–Maass cuspidal newforms. Then we have as $K \to \infty$ that

$$Q(\psi_1, \psi_2; K; \Phi) \longrightarrow 0.$$

In particular, the quadratic form $Q = \lim_{K\to\infty} Q(\cdot, \cdot; K; \Phi)$ is diagonalized by the orthonormal basis of Hecke–Maass cuspidal newforms on $\mathcal{B}_0^*(D)$.

Using the method of Rudnick–Soundararajan for lower bounds for moments of L-functions one can show the moments of $\mu_k(\psi)$ blow up.

Rankin–Selberg and Watson–Ichino

Let ϕ, u_j be Hecke–Maass forms of level 1, ϕ_k be a dihedral Maass form of level D, and ψ a Hecke–Maass newform of level D with trivial nebentypus.

- Rankin–Selberg method:

$$|\mu_t(\phi)|^2 = \frac{|\Lambda(1/2+2it,\phi)|^2\Lambda(1/2,\phi)^2}{2|\xi(1+2it)|^4\Lambda(1,\operatorname{sym}^2\phi)},$$

  where Λ means the corresponding completed L-functions and $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

- Watson–Ichino formula:

$$|\mu_j(\phi)|^2 = \frac{\Lambda(1/2,\operatorname{sym}^2 u_j\times\phi)\Lambda(1/2,\phi)}{8\Lambda(1,\operatorname{sym}^2 u_j)^2\Lambda(1,\operatorname{sym}^2\phi)},$$

$$|\mu_k(\psi)|^2 = \frac{1}{8\sqrt{D}}\frac{\Lambda(\frac{1}{2},\psi)\Lambda(\frac{1}{2},\psi\times\chi_D)\Lambda(\frac{1}{2},\psi\times\phi_{2k})}{\Lambda(1,\operatorname{sym}^2\psi)\Lambda(1,\chi_D)^2\Lambda(1,\phi_{2k})^2}.$$

Proof ingredients: Moments of L-functions

Proposition (H.–Lester 2020)

Let ψ be an even Hecke–Maass cuspidal newform on $\Gamma_0(D)$ with trivial nebentypus and $\eta_\psi(D)$ denote the W_D-eigenvalue of ψ, where W_D is the Atkin-Lehner operator. Suppose $\eta_\psi(D) = 1$. Let w be a Schwartz function with compact support in $[\frac{1}{2}, 2]$ such that $w^{(j)}(x) \ll P^j$, where $P \geq 1$ is a large parameter. Then there exists $A_0 > 0$ such that

$$\sum_{k \in \mathbb{Z}} L(\tfrac{1}{2}, \psi \times \phi_{2k})\, w\left(\frac{k}{K}\right) = \tilde{w}(1) \cdot C_{D,\psi} \cdot K + O(P^{A_0} \cdot K^{\frac{1}{2}+\vartheta+\varepsilon}),$$

where the implied constant depends at most on ψ, D. Here ϑ is the bound toward the Ramanujan–Selberg conjecture and

$$C_{D,\psi} = 2 \cdot \frac{L(1, \chi_D)}{\zeta_D(2)} L(1, \operatorname{sym}^2 \psi) \left(1 + \frac{\lambda_\psi(p_1)}{\sqrt{p_1}} + \frac{\lambda_\psi(p_2)}{\sqrt{p_2}} + \frac{\lambda_\psi(D)}{\sqrt{D}}\right)$$

where $\zeta_D(s) = \zeta(s) \prod_{p|D}(1 - p^{-s})$. Recall that $\tilde{w}(s) := \int_0^\infty w(x) x^{s-1}\, dx$ is the Mellin transform of w.

Proof ingredients: Twisted moments of L-functions

Proposition (H.–Lester 2020)

Assume GRC. Suppose $\eta_\psi(D) = 1$. Then there exists $A_0 > 0$ such that for $n \in \mathbb{N}$

$$\sum_{k \in \mathbb{Z}} L(\tfrac{1}{2}, \psi \times \phi_{2k}) \cdot \lambda_{2k}(n)\, w\left(\frac{k}{K}\right) = \tilde{w}(1) \cdot C_{D,\psi} \cdot h\left(\frac{n}{(n,D)}\right) \cdot K + O((Pn)^{A_0} \cdot K^{\frac{1}{2}+\vartheta+\varepsilon}),$$

for certain multiplicative function h.

Proposition (H.–Lester 2020)

Assume GRC. Suppose $\eta_\psi(D) = 1$. Also, suppose $P \le K^\delta$ for some $\delta > 0$ sufficiently small. Then for any $A \ge 1$ we have that

$$\sum_{k \in \mathbb{Z}} \frac{L(\frac{1}{2}, \psi \times \phi_{2k})}{L(1, \phi_{2k})^2} w\left(\frac{k}{K}\right) = \tilde{w}(1) \cdot C'_{D,\psi} \cdot C_{D,\psi} \cdot K + O\left(\frac{K}{(\log K)^A}\right)$$

where $C'_{D,\psi}$ is an explicit constant depending on D and ψ.

Proof ingredients: Conditional upper bounds

Theorem (Soundararajan 2009)

Let $M_k(T) := \int_0^T |\zeta(1/2+it)|^{2k} \mathrm{d}t$. Assume RH. Then we have

$$T(\log T)^{k^2} \ll_k M_k(T) \ll_{k,\varepsilon} T(\log T)^{k^2+\varepsilon}.$$

Proposition (H.–Lester 2020)

Assume GRH. Let $n \geq 1$. Also, let $\psi_1, \ldots, \psi_n$ be pairwise orthogonal Hecke–Maass cuspidal newforms on $\Gamma_0(D)$ with trivial nebentypus. Then for any real numbers $\ell_1, \cdots, \ell_n > 0$ we have that

$$\sum_{K<k\leq 2K} L(\tfrac{1}{2}, \psi_1 \times \phi_{2k})^{\ell_1} \cdots L(\tfrac{1}{2}, \psi_n \times \phi_{2k})^{\ell_n} \ll K \cdot (\log K)^{\frac{\ell_1(\ell_1-1)}{2}+\cdots+\frac{\ell_n(\ell_n-1)}{2}+\varepsilon}.$$

$n = 2,\ \ell_1 = \ell_2 = 1/2 \quad \Rightarrow \quad$ Quantum covariance for dMF vanishes.

Thank you for your attention!