Setting

Let H_k be the set of primitive cusp forms of weight k and level 1, eigenfunctions of Hecke operators; namely

$$f(z) = \sum_{n\geq 1} \lambda(n)\, n^{\frac{k-1}{2}}\, e^{2\pi i n z}.$$

For $\mathrm{Re}(s) > 1$, the associated L-function is

$$L_f(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1}.$$

For $r \in \mathbb{N}$, we are interested in the r-th moment

$$M_r(k) = \sum_{f\in H_k}^{h} L_f\left(\tfrac{1}{2}\right)^r,$$

$k \to \infty$.

- $\sum^h$ indicates the usual harmonic weight arising from Petersson norm.
- Assume $4 \mid k$.

From the recipe (CFKRS), we expect $A = \{\alpha_1, \dots, \alpha_r\}$, $\alpha_i \ll (\log k)^{-1}$

$$\sum_{f\in H_k}^{h} \prod_{\alpha\in A} L_f\left(\tfrac{1}{2}+\alpha\right) \sim \sum_{V\subset A} \sum_{f\in H_k}^{h} \left(\frac{k}{4\pi^2}\right)^{-\sum_{\alpha\in V}\alpha} G(A\setminus V \cup V^-)$$

where

$$G(A) = \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sum_{m_1\cdots m_r = m} \frac{\prod_p \frac{2}{\pi}\int_0^{\pi} \prod_{i=1}^{r} U_{\nu_p(m_i)}(\cos\theta)\sin^2\theta\, d\theta}{m_1^{\alpha_1}\cdots m_r^{\alpha_r}}.$$

$V^- = \{-\alpha : \alpha\in V\}$ and $U_m(\cos\theta) = \sin((m+1)\theta)/\sin\theta$ Chebyshev polynomials.

Main result

Let $X > 0$ be a parameter, we look at

$$\sum_{f\in H_k}^{h} \sum_{n\leq X} \frac{\lambda_A(n)}{\sqrt{n}}$$

where $\lambda_A(n)$ are s.t.

$$\prod_{\alpha\in A} L_f(s+\alpha) = \sum_{n=1}^{\infty} \frac{\lambda_A(n)}{n^s}.$$

From the Conrey-Keating heuristic, we expect that

- if $k^{2\cdot 0} < X < k^2$ only the 0-swap term contributes i.e. $V = \phi$

- if $k^{2\cdot 0} < X < k^2$ only the 0-swap term contributes (i.e. $V=\phi$)
- if $k^2 < X < k^4$ also the 1-swaps start contributing (i.e. $V=\{\alpha_i\}$)
- if $k^4 < X < k^6$ also the 2-swaps ... (i.e. $V=\{\alpha_i,\alpha_j\}$) and so on!

THM (Conrey - F., 2022)

Assume GLH and let $k \in \mathbb{N}$ be an even integer. For any given positive integer r, let $A = \{\alpha_1, \ldots, \alpha_r\}$ be a set of shifts of size $\ll (\log k)^{-1}$, $X > 0$ and $l \in \mathbb{N}$ two parameters and ψ a smooth function supported in $(0,1)$. Then, as $k \to \infty$, we have

$$\sum_{f\in H_k}^{h} \lambda(l) \sum_{n=1}^{\infty} \psi\left(\frac{n}{X}\right) \frac{1}{\sqrt{n}} \sum_{n_1\cdots n_r = n} \frac{\lambda(n_1)\cdots\lambda(n_r)}{n_1^{\alpha_1}\cdots n_r^{\alpha_r}}$$

$$= \frac{1}{2\pi i}\int_{(\varepsilon)} \tilde{\psi}(z) X^z \sum_{\substack{V\subset A\\ 0\le |V|\le 1}} i^{|V|k} \sum_{f\in H_k}^{h} \left(\frac{k^2}{4\pi^2}\right)^{-\sum_{\alpha\in V}(\alpha+z)} G_l(A_z \smallsetminus V_z \cup V_z^-)dz$$

$$+ O_\varepsilon\left((Xlk)^{3r\varepsilon}\frac{\sqrt{Xl^3}}{k^2}\right),$$

for every $\varepsilon > 0$, where $V^- := \{-v : v \in V\}$, $V_z := \{v + z : v \in V\}$,

$$G_l(A) := \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sum_{m_1\cdots m_r = m} \frac{\prod_p \frac{2}{\pi}\int_0^\pi \left(\prod_{i=1}^r U_{\mathrm{ord}_p(m_i)}(\cos\theta)\right) U_{\mathrm{ord}_p(l)}(\cos\theta)\sin^2\theta\, d\theta}{m_1^{\alpha_1}\cdots m_r^{\alpha_r}}$$

and $U_m(\cos\theta) = \sin((m+1)\theta)/\sin\theta$ denotes the Chebyshev polynomials.

ex For $l=1$, we have

$\mathcal{E} \lesssim \frac{\sqrt{X}}{k^2} = o(1)$ if $X < k^4$

M.T. $\asymp 1$ if $X > k^2$

~ x ~ x ~ x ~ x ~ x ~ x ~ x ~

Main ideas

① Harmonic detector $\to$ Petersson formula

② Arithmetical manipulations

③ $M_k(X) = ⓪ + Kl$ with ⓪ = 0-swap

(4) Kl = Poles + remaining integral

Estermann function

1-swap gymnastiek

GLH

(1) Petersson formula and some arithmetical manipulations

$$\sum_{f\in H_k}^{h} \lambda(m)\lambda(n) = \delta_{m,n} + 2\pi i^{-k} \sum_{c} \frac{S(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

0-swap

$$\sum_{a=1}^{c}{}' e\left(\frac{am+\bar{a}n}{c}\right)$$

Kloosterman sum

$$\frac{1}{2\pi i}\int_{(\frac{1}{2}+\varepsilon)} \frac{(2\pi)^{2w}\Gamma\left(\frac{k-1}{2}+w\right)}{\Gamma\left(\frac{k+1}{2}-w\right)}\left(\frac{4\pi\sqrt{mn}}{c}\right)^{2w} dw$$

J-Bessel

Mellin

$$\sum_{f\in H_k}^{h}\sum_{n\le X}\frac{\lambda_A(n)}{\sqrt{n}} \overset{\downarrow}{=} \frac{1}{2\pi i}\int_{(2)}\frac{X^z}{z}\sum_{f\in H_k}^{h}\underbrace{\prod_{\alpha\in A} L_f\left(\tfrac{1}{2}+\alpha+z\right)}\,dz$$

$$\left(\sum_{n_1,\dots,n_r}\frac{\lambda(n_1)\cdots\lambda(n_r)}{n_1^{s+\alpha_1}\cdots n_r^{s+\alpha_r}} = \sum_{m=1}^{\infty}\lambda(m)F_A\left(m,\tfrac{1}{2}+z\right) \overset{\text{VERY ROUGHLY}}{\approx} \prod_{i<j}\zeta(1+\alpha_i+\alpha_j)\sum_{m=1}^{\infty}\lambda(m)\frac{\tau_A(m)}{m^{1/2+z}}\right)$$

Chebyshev polynomials

Hecke

$$\overset{\downarrow}{=} \frac{1}{2\pi i}\int_{(2)}\frac{X^z}{z}\sum_{m=1}^{\infty}F_A\left(m,\tfrac{1}{2}+z\right)\sum_{f\in H_k}^{h}\lambda(m)\,dz$$

Petersson

$$\overset{\downarrow}{=} D_A(X) + Kl_A(X)$$

from δ

→ 0-swap!

(2) The Kloosterman part

$$Kl_A(X) = \frac{2\pi}{2\pi i}\int_{(2)}\frac{X^z}{z}\sum_{m=1}^{\infty}F_A\left(m,\tfrac{1}{2}+z\right)\sum_{c=1}^{\infty}\frac{S(1,m;c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{m}}{c}\right)dz$$

$$= \frac{1}{2\pi i}\int_{(2)}\frac{X^z}{z}\frac{1}{2\pi i}\int_{(\frac{1}{2}+\varepsilon)}\frac{(2\pi)^{1-2w}\Gamma\left(\frac{k-1}{2}+w\right)}{\Gamma\left(\frac{k+1}{2}-w\right)}\times$$

$$\times\sum_{c=1}^{\infty}\frac{1}{c^{1-2w}}\sum_{a=1}^{c}{}' e\left(\frac{\bar{a}}{c}\right)\underbrace{\sum_{m=1}^{\infty}\frac{F_A\left(m,\frac{1}{2}+z\right)e\left(\frac{am}{c}\right)}{m^{w}}}_{\text{Estermann function!!}}dz\,dw$$

"$\sum_{\chi}\tau(\chi)\chi(a)\chi(m)$"

Estermann function!!

poles for $x = x_0$ at $\boxed{\omega + z = \frac{1}{2} - \alpha}$
entire otherwise

$$= \sum_{\alpha \in A} \mathcal{P}_\alpha + o(I)$$

1-swaps!

small on GLH