DISCRETE MOMENTS

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Setup

• $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$ for $\Re s = \sigma > 1$.

• The zeros in the critical strip $\{s \in \mathbb{C} : 0 \leq \Re s \leq 1\}$ are called nontrivial zeros.

Notation. A generic one is denoted by $\rho = \beta + i\gamma$, its multiplicity is denoted by $m(\rho)$.

• Riemann Hypothesis (RH): $\Re(\rho) = \beta = \frac{1}{2}$ for all ρ.

Let T be large and $N(T)$ count the nontrivial zeros ρ such that $0 \leq \Re\rho \leq 1$ and $0 < \Im\rho \leq T$.

Riemann-von Mangoldt Formula

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{1}{\pi} \arg \zeta\Big(\frac{1}{2} + iT\Big) + \frac{7}{8} + O\Big(\frac{1}{T}\Big)$$

For an "interesting" function $f(s)$, one wants to estimate averages

$$\sum_{1 \leq \gamma \leq T} f(\rho).$$

Depending on the function f, this may require further knowledge on the ρ, for example, their vertical or horizontal distribution.

Example. We already have $\zeta(\rho) = 0$. So what can be said about the moments

$$\sum_{1<\gamma\leq T} |\zeta'(\rho)|^{2k} \quad ?$$

- **Milinovich & Ng, 2014** On RH, for $k \geq 0$,

$$\sum_{1<\gamma\leq T} |\zeta'(\rho)|^{2k} \gg_k (\log T)^{k(k+2)}.$$

- **Kirila, 2020** On RH, for $k \geq 0$, $\displaystyle\sum_{1<\gamma\leq T} |\zeta'(\rho)|^{2k} \ll_k (\log T)^{k(k+2)}$.
- We thus have on RH

$$\sum_{1<\gamma\leq T} |\zeta'(\rho)|^{2k} \sim C_k (\log T)^{k(k+2)}.$$

- **Gonek, 1984** An asymptotic for the case $k = 1$ on RH.
  Ng, 2004 Upper and lower bounds for the case $k = 2$ on RH.

Moments of Dirichlet polynomials

Example. Let

$$A(s) = \sum_{n \leq N} a_n n^{-s}$$

be a Dirichlet polynomial. Compute

$$\sum_{1 \leq \gamma \leq T} A(\rho)^k.$$

Many times in the usual setting, one wants to compute

$$\int_1^T A\Big(\frac{1}{2} + it\Big)^k \, dt$$

or the so-called joint moments involving another polynomial.

Moments in the usual setting

Due to the identity

$$u\overline{v} = \frac{1}{4}\left(|u+v|^2 - |u-v|^2 + i|u+iv|^2 - i|u-iv|^2\right),$$

we can focus on computing second moments, or even moments in general, instead of joint moments.

Montgomery-Vaughan Mean Value Theorem

$$\int_0^T \left| \sum_{n \le N} \frac{a_n}{n^{\sigma+it}} \right|^2 = T \sum_{n \le N} \frac{|a_n|^2}{n^{2\sigma}} + O\left(\sum_{n \le N} \frac{n|a_n|^2}{n^{2\sigma}} \right)$$

In practice, N needs to be chosen sufficiently small.

Discrete Moments, History

Theorem (Gonek, 1984)

Let $\mu, \nu \in \mathbb{Z}_{\geq 0}$. *For* $\frac{2\pi a}{\log(T/(2\pi))} \leq \frac{1}{2}$,

$$\sum_{0<\gamma\leq T} \zeta^{(\mu)}\Big(\rho + i\frac{2\pi a}{\log(T/(2\pi))}\Big)\zeta^{(\nu)}\Big(1-\rho - i\frac{2\pi a}{\log(T/(2\pi))}\Big) \sim C(\mu,\nu,a)\frac{N(T)}{\mu+\nu+1}(\log T)^{\mu+\nu+1}.$$

This is a discrete analogue of a result of Ingham:

$$\int_1^T \zeta^{(\mu)}\Big(\frac{1}{2}+it\Big)\overline{\zeta^{(\nu)}\Big(\frac{1}{2}+it\Big)}\,dt \sim \frac{T}{\mu+\nu+1}(\log T)^{\mu+\nu+1}$$

Theorem (Gonek, 1984)

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The result is proved by computing the contour integral

$$\frac{1}{2\pi i}\int_{\partial R} \frac{\zeta'}{\zeta}(s)\zeta^{(\mu)}(s+i\delta)\zeta^{(\nu)}(1-s-i\delta)\,ds$$

for $|\delta|\leq \frac{1}{2}$. R is suitably chosen to be a nice rectangle with vertices $\alpha + i, \alpha + iT, 1-\alpha+iT, 1-\alpha+i$ with $0 < \alpha < \frac{1}{2}$.

An application

For the zeroth derivative, Gonek's result gives

$$\sum_{1<\gamma\leq T}\left|\zeta\Big(\frac{1}{2}+i\gamma+i\frac{2\pi a}{\log(T/(2\pi))}\Big)\right|^2\sim\Big(1-\frac{\sin^2(\pi a)}{(\pi a)^2}\Big)N(T)\log T.$$

This can be looked at in view of the classical Hardy-Littlewood result:

$$\int_1^T\left|\zeta\Big(\frac{1}{2}+it\Big)\right|^2 dt\sim T\log T$$

An application

Corollary (Mueller)

Let

$$\lambda = \limsup_{n\to\infty} (\gamma_n - \gamma_{n-1}) \frac{\log \gamma_n}{2\pi}.$$

On RH, we have $\lambda \geq 1.9$.

Proof.

Cover the interval $\left[\frac{1}{2} + i, \frac{1}{2} + iT\right]$ by symmetric intervals centered around each zero $\frac{1}{2} + i\gamma$. They will be of length at least $\frac{2\pi\lambda}{\log(T/(2\pi))}$. That is, for $\ell \geq \lambda$,

$$\sum_{1<\gamma\leq T} \int_{\gamma-\ell/2}^{\gamma+\ell/2} \left|\zeta\Big(\frac{1}{2} + i\gamma + i\frac{2\pi a}{\log(T/(2\pi))}\Big)\right|^2 da > \int_1^T \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^2 dt\,.$$

By substituting Gonek's result and Hardy-Littlewood's result,

$$\sum_{1<\gamma\leq T}\int_{\gamma-\ell/2}^{\gamma+\ell/2}\left|\zeta\Big(\frac{1}{2}+i\gamma+i\frac{2\pi a}{\log(T/(2\pi))}\Big)\right|^2 da$$

$$\sim N(T)\int_{-\ell/2}^{\ell/2}\Big(1-\frac{\sin^2(\pi a)}{(\pi a)^2}\Big)\, da > T\log T$$

For the near optimal choice of $\ell = 1.9$, the above inequality is valid, so λ is at least 1.9.

Analogue to Selberg's central limit theorem

Interested in moments of

$$\log|\zeta(\rho+z)| \quad \text{for small and nonzero } z.$$

- $\log|\zeta(\rho+z)| = M(\rho,z) + \Re \sum_{p\le T^{\theta(T)}} \frac{1}{p^{\frac{1}{2}+i\gamma+i\Im z}} +$ some other terms

  where

  $$M(\rho,z) = m(\rho+i\Im z)\Big\{\log\Big(\frac{(\Re z)\log T}{4}\Big) - \frac{(\Re z)\log T}{4}\Big\}.$$

- Conditionally on RH and the pair correlation conjecture, we can obtain asymptotics for moments

  $$\sum_{1\le\gamma\le T}\big(\log|\zeta(\rho+z)| - M(\rho,z)\big)^{2k}$$

  by computing moments of the above Dirichlet polynomial.

Calculate

$$\sum_{0<\gamma\le T}\left(\Re\sum_{p\le X^2}\frac{1}{p^{1/2+i(\gamma+\Im z)}}\right)^k$$
$$=\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{1\le\gamma\le T}\left(\sum_{p\le X^2}\frac{1}{p^{1/2+i(\gamma+\Im z)}}\right)^j\left(\overline{\sum_{p\le X^2}\frac{1}{p^{1/2+i(\gamma+\Im z)}}}\right)^{k-j}.$$

Here

$$\sum_{1\le\gamma\le T}\left(\sum_{p\le X^2}\frac{1}{p^{1/2+i(\gamma+\Im z)}}\right)^j\left(\overline{\sum_{p\le X^2}\frac{1}{p^{1/2+i(\gamma+\Im z)}}}\right)^{k-j}$$
$$=\sum_{p_i,q_j\le X^2}\left(\frac{q_1\dots q_{k-j}}{p_1\dots p_j}\right)^{i\Im z}\frac{1}{\sqrt{p_1\dots p_j q_1\dots q_{k-j}}}\sum_{1\le\gamma\le T}\left(\frac{q_1\dots q_{k-j}}{p_1\dots p_j}\right)^{i\gamma}.$$

Landau-Gonek formula Assume RH. For $x > 1$,

$$\sum_{1 \leq \gamma \leq T} x^{i\gamma} = -\frac{T}{2\pi} \frac{\Lambda(x)}{\sqrt{x}} + \mathcal{E}(x, T).$$

- For $x = 1$, the sum above is $N(T)$.
- For $x < 1$, we can replace x by $\dfrac{1}{x}$ and take conjugates.
- Note that we have a main term for prime power x.
- Usually we are especially interested in the case where x is a prime.

• In general, to be used in computing discrete moments of Dirichlet polynomials, a general formula can be obtained from the Landau-Gonek formula.

Lemma

Let $N \leq T$.

$$\sum_{0<\gamma\leq T}\Big|\sum_{n\leq N} a_n n^{-i\gamma}\Big|^2$$

$$= N(T)\sum_{n\leq N}|a_n|^2 - \frac{T}{2\pi}\sum_{m,n\leq N} a_n\overline{a_m}\frac{\Lambda(m/n)}{\sqrt{m/n}} - \frac{T}{2\pi}\sum_{m,n\leq N} a_m\overline{a_n}\frac{\Lambda(n/m)}{\sqrt{n/m}}$$

$$+ O\Bigg(N\log T\log\log T\sum_{n<m\leq N}|a_n a_m|\sqrt{\frac{m}{n}}\Bigg) + O\Bigg(N\log^2 T\sum_{n\leq N}|a_n|^2\Bigg).$$

Recall

$$\sum_{0<\gamma\le T}\Big(\Re\sum_{p\le X^2}\frac{1}{p^{1/2+i(\gamma+\Im z)}}\Big)^k$$

$$=\sum_{p_i,q_j\le X^2}\Big(\frac{q_1\dots q_{k-j}}{p_1\dots p_j}\Big)^{i\Im z}\frac{1}{\sqrt{p_1\dots p_j q_1\dots q_{k-j}}}\sum_{0<\gamma\le T}\Big(\frac{q_1\dots q_{k-j}}{p_1\dots p_j}\Big)^{i\gamma}$$

Let $N\le T^{\frac{1}{8k}}$. By using the Landau-Gonek formula, for even k,

$$\sum_{0<\gamma\le T}\Big(\Re\sum_{p\le N}\frac{1}{p^{1/2+i(\gamma+\Im z)}}\Big)^k\sim\frac{k!}{2^k(k/2)!}N(T)(\log\log N)^{\frac{k}{2}}.$$

For odd k,

$$\sum_{0<\gamma\le T}\Big(\Re\sum_{p\le N}\frac{1}{p^{1/2+i(\gamma+\Im z)}}\Big)^k$$

$$\sim-\frac{(k+1)!}{2^{k+1}((k+1)/2)!\,\pi}\frac{\sin((\Im z)\log N)-\sin((\Im z)\log 2)}{v}T(\log\log N)^{\frac{k-1}{2}}.$$

For even moments, when we choose k suitably, we then deduce

$$\sum_{0<\gamma\leq T} \big(\log|\zeta(\rho+z)| - M(\rho,z)\big)^k \sim \frac{k!}{2^k(k/2)!} N(T)(\log\log T)^{\frac{k}{2}}.$$

NOTE. This is following the footsteps of Selberg's central limit theorem.

C., 2020

Assume RH and Montgomery's Pair Correlation Conjecture. Let $z = u + iv$ be nonzero with $0 < u \ll \frac{1}{\log T}$ and $|v| \ll \frac{1}{\log T}$. Then

$$\frac{1}{N(T)} \#\left\{0 < \gamma \leq T : \frac{\log|\zeta(\rho+z)| - M(\rho,z)}{\sqrt{\frac{1}{2}\log\log T}} \in [a,b]\right\}$$
$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx + O\left(\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right),$$

where

$$M(\rho,z) = m(\rho+iv)\Big(\log\Big(\frac{eu\log T}{4}\Big) - \frac{u\log T}{4}\Big).$$

Thank you!