Moments of large families of Dirichlet L-functions

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Moments of the Riemann zeta function

The Riemann zeta function

For $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

- It can be analytically continued to the whole complex plane except a simple pole at $s = 1$.
- Moments of the Riemann zeta function is defined to be

$$I_{2k}(T) = \frac{1}{T} \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt.$$

Moments of the Riemann zeta function

$k = 1$

Hardy and Littlewood (1918) showed that

$$I_2(T) \sim a_1 \log T.$$

$k = 2$

Ingham (1926) showed that

$$I_4(T) \sim 2a_2 \frac{(\log T)^4}{4!}.$$

$k \geq 3$

Asymptotic formulae are not proven unconditionally. However, we have a good conjecture.

$$I_{2k}(T) \sim g_k a_k \frac{(\log T)^{k^2}}{k^2!},$$

where

- a_k is easy to understand from $\sum_{n \leq T} \frac{d_k^2(n)}{n}$, i.e.

$$\sum_{n \leq T} \frac{d_k^2(n)}{n} \sim a_k (\log T)^{k^2}$$

- g_k is some constant that remains unsolved.

Conjecture

- Conrey and Ghosh (1998): $g_3 = 42$.
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Conjecture from Random matrix theory

Based on a heuristic that the distribution of the zeros $\zeta(s)$ behaves like distribution of eigenvalues of matrices in circular unitary ensembles, Keating and Snaith (2000) conjectured for any positive integers k,

$$I_{2k}(T) \sim \frac{g_k a_k}{k^2!}(\log T)^{k^2},$$

where

$$g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}.$$

Note: $g_1 = 1$, $g_2 = 2$, $g_3 = 42$, and $g_4 = 24024$.

Conjecture

Conrey, Farmer, Keating, Rubinstein, Snaith (2005) gave a more precise conjecture of $I_{2k}(T)$ including an asymptotic expansion for lower order terms through the shifted moments. Their recipe also applies to moments of other families of L-functions.

Diaconu, Goldfeld and Hoffstein (2003) gave an alternative approach, based on multiple Dirichlet series, to give the same conjectures.

Conrey and Keating (2017) conjectured the $2k^{th}$ moments and shifted moments of the Riemann zeta function using long Dirichlet polynomial and divisor correlations.

Why is it difficult to compute high moments ($k > 2$) ?

For $\mathrm{Re}(s) > 1$,

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}.$$

Approximate functional equation

$$\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} \approx \sum\sum_{mn \ll T^k} \frac{d_k(m)d_k(n)}{(mn)^{1/2}} \left(\frac{m}{n}\right)^{it}$$

$$\frac{1}{T}\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} \, dt \sim \underbrace{\sum_{n \le T^{k/2}} \frac{d_k^2(n)}{n}}_{\text{diagonal terms } m=n} + \underbrace{\text{off-diagonal terms}}_{m \neq n}$$

- When $k = 1, 2$ the main term solely comes from the diagonal terms.
- When $k \geq 3$, the Dirichlet polynomial is too long. The off-diagonal terms also give main contribution. Moreover, it is related to shifted convolution sums of the form $\sum_{n \leq x} d_k(n) d_k(n+f)$, which is another difficult problem.
- The off-diagonal terms are also a main difficulty for high moments of other families of L-functions.

Moments of Dirichlet L-functions

Dirichlet L-functions

For $\mathrm{Re}(s) > 1$,

$$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where χ is a primitive Dirichlet character modulo q.

- The family of all primitive Dirichlet L-functions of modulus q is analogous in some ways to the Riemann zeta function in t-aspects.
- In fact, this family is associated to unitary ensemble.
- The moments of this family behave similarly to the moments of the $\zeta(1/2 + it)$.

Notation

- $\sum^*$ is the sum over all primitive characters mod q.
- $\phi^*(q)$ is number of primitive characters mod q.
- Moments of $L(s, \chi)$ is defined to be

$$M_{2k}(q) = \frac{1}{\phi^*(q)} \sum_{\chi \pmod q}^* |L(1/2, \chi)|^{2k}.$$

Moments of $L(s, \chi)$

$k = 1$: $M_2(q) \sim \log q$.

$k = 2$: Heath-Brown (1981), Soundararjan (2007), Young (2010) showed that

$$M_4(q) \sim 2b_2 \frac{(\log q)^4}{4!}.$$

$k \geq 3$: Unknown. It is conjectured that

$$M_{2k}(q) \sim g_k b_k \frac{(\log q)^{k^2}}{k^2!},$$

where $g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}$ (same as the constant in the asymptotic formula of Riemann zeta function case).

$$\sum_{\substack{n \leq q \\ (n,q)=1}} \frac{d_k^2(n)}{n} = b_k (\log q)^2 \qquad \text{(diagonal term)}$$

Bounds for moments of Dirichlet L-functions

- $\sum^*_{\chi \pmod q} |L(1/2,\chi)|^{2k} \gg q(\log q)^{k^2}$
  - For rational $k \geq 1$, Rudnick and Soundararajan, 2005
  - For real $k \geq 1$, Radziwiłł and Soundararajan, 2012
  - For rational $0 < k < 1$, C. and Li, 2013
  - For real $0 < k < 1$, Heap and Soundararajan, 2020 and Gao (2021).
- $\sum^*_{\chi \pmod q} |L(1/2,\chi)|^{2k} \ll q(\log q)^{k^2}$ under GRH (Soundararajan, 2009 and Harper, 2013).

  With the modification of Heap, Radziwiłł, and Soundararajan's work (2019), we can obtain the bound without GRH when $0 < k \leq 2$

For positive integers k, approximate functional equation gives that

$$L\left(\frac{1}{2},\chi\right)^k \approx \sum_{m \ll q^{\frac{k}{2}}} \frac{d_k(m)\chi(m)}{\sqrt{m}} + g_\chi \sum_{m \ll q^{\frac{k}{2}}} \frac{d_k(m)\overline{\chi(m)}}{\sqrt{m}}.$$

Using large sieve inequality

$$\sum_{\chi \bmod q} \left| \sum_{n \le N} a_n \chi(n) \right|^2 \ll (q+N) \sum_{n \le N} |a_n|^2,$$

we have

$$\begin{aligned} \sum_{\chi \bmod q}^{*} \left| L\left(\frac{1}{2},\chi\right) \right|^{2k} &\ll \sum_{\chi \bmod q} \left| \sum_{m \ll q^{k/2}} \frac{d_k(m)\chi(m)}{\sqrt{m}} \right|^2 \\ &\ll (q+q^{k/2}) \sum_{m \ll q^{k/2}} \frac{d_k^2(m)}{m} \ll q(\log q)^{k^2} \end{aligned}$$

if $k \le 2$.

By using large sieve inequality, Huxley (1970) showed that for $k = 3, 4$,

$$\sum_{q \sim Q} \sideset{}{^*}\sum_{\chi \pmod q} |L(1/2, \chi)|^{2k} \ll Q^2 (\log Q)^{k^2}.$$

The large sieve inequality:

$$\sum_{r \sim R} \sideset{}{^*}\sum_{\chi \pmod r} |\sum_{n \le N} a(n)\chi(n)|^2 \ll (R^2 + N) \sum_{n \le N} |a_n|^2.$$

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The large sieve inequality:

$$\sum_{r \sim R} \sideset{}{^*}\sum_{\chi \pmod r} |\sum_{n \leq N} a(n)\chi(n)|^2 \ll (R^2 + N) \sum_{n \leq N} |a_n|^2.$$

By approximate functional equation, the moment is bounded by

$$\ll \sum_{q \leq Q} \sideset{}{^*}\sum_{\chi \bmod q} \left| \sum_{m \ll Q^{\frac{k}{2}}} \frac{d_k(m)\chi(m)}{\sqrt{m}} \right|^2 \ll (Q^2 + Q^{\frac{k}{2}}) \sum_{m \ll Q^{\frac{k}{2}}} \frac{d_k^2(m)}{m}$$
$$\ll Q^2 (\log Q)^{k^2}$$

for $k \leq 4$.

	$\sum^{*}_{\chi \pmod q}$	$\sum_{q\sim Q} \sum^{*}_{\chi \pmod q}$
conductor	q	q
family	$\sim q$	$\sim Q^2$

- If you have only $\sum^{*}_{\chi \pmod q}$, the length can go up to $q^{k/2}$, where $k \leq 2$ to obtain a good bound without GRH.
- With two average $\sum_{q\sim Q} \sum^{*}_{\chi \pmod q}$, the length can go up to $q^{k/2}$, where $k \leq 4$.

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- With two average $\sum_{q\sim Q} \sum^*_{\chi \pmod q}$, the length can go up to $q^{k/2}$, where $k \leq 4$.

Question: Is there an asymptotic formula of the sixth and the eighth moment for the larger family of Dirichlet L-functions, i.e.

$$\sum_{q\sim Q} \sum^*_{\chi \pmod q} |L(1/2,\chi)|^{2k} \quad ?$$

for $k = 3$ and 4.

The sixth moment of Dirichlet L-functions

From the recipe of Conrey, Farmer, Keating, Rubinstein and Snaith, it is conjectured that

$$\sum_{\chi \pmod q}^{*} |L(1/2, \chi)|^6 \sim 42 a_3 \prod_{p|q} \frac{\left(1 - \frac{1}{p}\right)^5}{\left(1 + \frac{4}{p} + \frac{1}{p^2}\right)} \phi^*(q) \frac{(\log q)^9}{9!}.$$

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Theorem (Conrey, Iwaniec and Soundararajan)

$$\sum_{q \sim Q} \sum_{\chi \pmod q}^{*} \int_{-\infty}^{\infty} \left| L(\frac{1}{2}+it,\chi) \right|^6 \left| \Gamma\left(\frac{1/2+it}{2}\right) \right|^6 dt$$

$$\sim 42 a_3 \sum_{q \sim Q} \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^5}{\left(1+\frac{4}{p}+\frac{1}{p^2}\right)} \phi^*(q) \frac{(\log q)^9}{9!} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{1/2+it}{2}\right) \right|^6 dt$$

- They also state a more precise technical result which gives the asymptotic for the sixth moment including shifts with a power saving error term of size $Q^{2-1/10+\epsilon}$.
- The contribution of the main term comes from both diagonal and off-diagonal terms.
- The integration over t is fairly short due to the rapid decay of the Γ function along vertical lines.
- Deriving an analogous result without the average over t is challenging due to certain "unbalanced" sums.

The sixth moment without the t-average

Theorem (C., X. Li, K. Matomäki, and M. Radziwiłł (2022+))

$$\sum_{q\sim Q} \sideset{}{^*}\sum_{\chi \pmod q} \left| L(\tfrac{1}{2},\chi)\right|^6 \sim 42 a_3 \sum_{q\sim Q} \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^5}{\left(1+\frac{4}{p}+\frac{1}{p^2}\right)} \phi^*(q) \frac{(\log q)^9}{9!}$$

$$\sim 42\ \widetilde{c}_3 Q^2 \frac{(\log Q)^9}{9!}.$$

Note: We can also state a more precise technical result which gives the asymptotic for the sixth moment including shifts with a power saving error term of size $Q^{2-3/236+\epsilon}$.

The eighth moment of Dirichlet L-functions

From the recipe of Conrey, Farmer, Keating, Rubinstein and Snaith, it is conjectured that

$$\sum_{\chi \pmod q}^{*} |L(1/2,\chi)|^8 \sim 24024 a_4 \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^7}{\left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right)} \phi^*(q) \frac{(\log q)^{16}}{16!}.$$

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On GRH, C. and Li (2014) derived asymptotic formula for the eighth moment of a large family of Dirichlet L-functions with extra average over t, i.e.

$$\mathcal{M}_8(Q) = \sum_{q \sim Q} \sum_{\chi \pmod q}^{*} \int_{-\infty}^{\infty} \left| L(\frac{1}{2} + it, \chi) \right|^8 \left| \Gamma\left(\frac{1/2 + it}{2}\right) \right|^8 dt.$$

Theorem (C., Li, Matomäki, and Radziwiłł (2022+))

We have $\mathcal{M}_8(Q)$ *is*

$$\sim 24024\, a_4 \sum_{q\sim Q} \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^7}{\left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right)} \phi^*(q) \frac{(\log q)^{16}}{16!}$$

$$\times \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{1/2+it}{2}\right)\right|^8 \, dt$$

$$\sim 24024\, \widetilde{a}_4 Q^2 \frac{(\log Q)^{16}}{16!} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{1/2+it}{2}\right)\right|^8 \, dt.$$

Note: We cannot get power saving error terms. Our error term is of size $Q^2(\log Q)^{15+\epsilon}$.

Sketch of the proof

Initial setup for the sixth and eighth moment with t-average

For $k = 3, 4$, we consider

$$\sum_q \Psi\left(\frac{q}{Q}\right) \sideset{}{^*}\sum_{\chi \pmod q} \int_{\infty}^{\infty} \left| L(\frac{1}{2} + it, \chi) \right|^{2k} \left| \Gamma\left(\frac{1}{2} + it\right) \right|^{2k} dt$$

where Ψ is a smooth function compactly supported in [1, 2].

Appoximate functional equation

$$\left| L(\frac{1}{2} + it, \chi) \right|^{2k} \approx \sum_{\substack{m,n \\ mn \leq Q^k}} \frac{d_k(n) d_k(m)}{\sqrt{mn}} \chi(m) \overline{\chi}(n) \left(\frac{m}{n}\right)^{it}.$$

After integrating over t, we have

$$\int_{\infty}^{\infty} \left| L(\frac{1}{2}+it,\chi) \right|^{2k} \left| \Gamma\left(\frac{1}{2}+it\right) \right|^{2k} dt$$
$$\approx \sum_{m,n \le Q^{\frac{k}{2}}} \frac{d_k(n)d_k(m)}{\sqrt{mn}} \chi(m)\overline{\chi}(n).$$

After the orthogonality relation of Dirichlet characters, we roughly need to understand the sum of the form

$$\sum_{d,r}\sum \Psi\left(\frac{dr}{Q}\right)\mu(d)\phi(r) \sum_{\substack{m,n \le Q^{k/2} \\ m \equiv n \mod r}}\sum \frac{d_k(n)d_k(m)}{\sqrt{mn}}$$

$$k=3 : Q^{3/2} \quad , \qquad k=4 : Q^2$$

- The main contribution will come from when d is small. So for our simple model, we consider only $d = 1$ and $r = q$.
- **Without the integration over** t, the main contribution will comes from $mn \ll Q^k$.

$$Q \sum_{q} \Psi\left(\frac{q}{Q}\right) \sum_{\substack{mn \leq Q^k \\ m \equiv n \mod q}} \sum \frac{d_k(n) d_k(m)}{\sqrt{mn}}.$$

We need to consider unbalanced sums when one variable is large and another one is small, e.g. $m = Q^{5/2}$ and $n = Q^{1/2}$.

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We need to consider unbalanced sums when one variable is large and another one is small, e.g. $m = Q^{5/2}$ and $n = Q^{1/2}$.

- The diagonal term $m = n$ is easy to understand.
- For the off-diagonal term, write $m - n = hq$, where $h \neq 0$. So $h \asymp \frac{|m-n|}{Q}$.

- Conrey, Iwaniec and Soundararajan use the complementary divisor trick, which is replacing the congruence condition modulo q with a congruence condition modulo h (reduce the size of the conductor).
- The original size of conductor is $q \asymp Q$, and $h \asymp \frac{|m-n|}{Q}$.

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	the sixth moment	**the eighth moment**
critical range	$m \asymp n \asymp Q^{3/2}$	$m \asymp n \asymp Q^2$
size of h	$\ll Q^{1/2}$	$\ll Q$
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Note: if we do not have integration over t, the size of h can be much larger than Q. (e.g. h can be Q^2 for the sixth moment and of size Q^3 for the eighth moment. (too big!))

Truncation for the eighth moment

We would like to show that

$$\sum_{q \asymp Q} \sideset{}{^*}\sum_{\chi \pmod q} \sum_{Q^{2-\epsilon} \le m,n \le Q^2} \frac{d_4(n) d_4(m)}{\sqrt{mn}} \chi(m)\overline{\chi}(n) \ll \epsilon Q^2 \log^{16} Q.$$

- This would allows us to consider the shorter sum with $m, n \le Q^{2-\epsilon}$.
- $\epsilon = \frac{1}{(\log Q)^{1-\epsilon_0}}$, where ϵ_0 is a small positive number.
- The complementary divisor $h \ll Q^{1-\epsilon}$.

The large sieve inequality:

$$\sum_{r\sim R}\ \sideset{}{^*}\sum_{\chi \pmod r} \Big|\sum_{n\le N} a(n)\chi(n)\Big|^2 \ll (R^2+N)\sum_{n\le N}|a_n|^2.$$

By the Large Sieve inequality,

$$\begin{aligned}
&\sum_{q\asymp Q}\ \sideset{}{^*}\sum_{\chi \pmod q}\left|\sum_{Q^{2-\epsilon}\le n\le Q^2}\frac{d_4(n)}{\sqrt{n}}\chi(n)\right|^2\\
&\ll (Q^2+Q^2)\left(\sum_{Q^{2-\epsilon}\le n\le Q^2}\frac{d_4(n)^2}{n}\right)\\
&\ll Q^2\left((\log Q^2)^{16}-(\log Q^{2-\epsilon})^{16}\right)\\
&\ll \epsilon Q^2\log^{16} Q.
\end{aligned}$$

Off-diagonal terms after the complementary divisor trick

The congruence condition modulo h is expressed using orthogonality relation of characters mod h

$$m \equiv n \mod h \qquad \Longrightarrow \qquad \frac{1}{\phi(h)} \sum_{\chi \text{ mod } h} \chi(m)\overline{\chi(n)}.$$

Roughly, we study

$$Q \sum_{h} \frac{1}{\phi(h)} \sum_{\chi \mod h} \sum\sum_{\substack{m,n \leq Q^{2-\epsilon} \\ |m-n| \asymp hQ}} \frac{d_k(m)d_k(n)\chi(m)\overline{\chi(n)}}{\sqrt{mn}}$$

- The principal character mod h contributes to the main term.
- The rest gives the error term.

Error terms (simple model) – ignore $|m-n| \asymp hQ$

Recall that $N \ll Q^{2-\epsilon}$.

Before the complementary divisor trick, error terms (non-principal characters) will be

$$\sum_{q \sim Q} \sideset{}{^*}\sum_{\chi \bmod q} \left| \sum_{n \leq N} \frac{d_k(n)\chi(n)}{\sqrt{n}} \right|^2 \ll (Q^2 + N) \sum_{n \leq N} \frac{d_k(n)^2}{n} \asymp Q^2 (\log Q)^{k^2}$$

which is as large as the main term.

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After the complementary divisor trick, error terms will be

$$\sum_{h \sim \frac{N}{Q}} \sideset{}{^*}\sum_{\chi \bmod h} \left| \sum_{n \le N} \frac{d_k(n)\chi(n)}{\sqrt{n}} \right|^2 \ll (Q^{2-2\epsilon} + N) \sum_{n \le N} \frac{d_k(n)^2}{n} \ll Q^{2-\epsilon_0}.$$

Error terms for the eighth moment: include $|m-n| \asymp hQ$

Assume $h \asymp H$.

- When H is small, $|m-n| \asymp HQ$ is also small. Hence, for fixed n, the sums over m is restricted to an interval much shorter than Q^2.
- Technically, in my joint work with Li (2014), GRH was used to find cancellation in the sums over m, n restricted to a narrow region.

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How to remove GRH condition

1. An idea from Matomäki and Radziwiłł's work "Multiplicative functions in short intervals."

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How to remove GRH condition

1. An idea from Matomäki and Radziwiłł's work "Multiplicative functions in short intervals."
2. Understand Mellin transform better and then apply functional equation and hybrid large sieve.

By Mellin transform, the short interval condition $|m-n| \asymp HQ$ may be expressed as

$$\frac{HQ}{N} \int \left(\frac{m}{n}\right)^{it} g\left(\frac{HQt}{N}\right) dt$$

for some Schwartz class function g.

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Roughly we want to study

$$\frac{Q}{H}\frac{HQ}{N} \sum_{h \asymp H} \sideset{}{^*}\sum_{\chi \bmod h} \int \left| \sum_{m \asymp N} \frac{d_4(m)\chi(m)}{m^{1/2+it}} \right|^2 g\left(\frac{HQt}{N}\right) dt$$

$$= \frac{Q^2}{N} \sum_{h \asymp H} \sideset{}{^*}\sum_{\chi \bmod h} \int \left| \sum_{m} \frac{d_4(m)\chi(m)}{m^{1/2+it}} w\left(\frac{m}{N}\right) \right|^2 g\left(\frac{t}{T}\right) dt,$$

writing $T = \frac{N}{HQ}$.

The hybrid large sieve gives

$$\ll \frac{Q^2}{N}\left(H^2T + N\right)\sum_{m\sim N}\frac{d_4(n)^2}{n} \asymp Q^2(\log Q)^{16},$$

which is too big.

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$$\ll \frac{Q^2}{N}\left(H^2T+N\right)\sum_{m\sim N}\frac{d_4(n)^2}{n}\asymp Q^2(\log Q)^{16},$$

which is too big.

$$\sum_{m}\frac{d_4(m)\chi(m)}{m^{1/2+it}}w\left(\frac{m}{N}\right)\sim L^4(1/2+it,\chi),$$

and we apply the functional equation for $L^4(1/2+it)$ (Voronoi summation).

$$\frac{Q^2}{N}\sum_{h\asymp H}\ \sideset{}{^*}\sum_{\chi \bmod h}\int\left|\sum_{m\ll\frac{(TH)^4}{N}}\frac{d_4(m)\bar{\chi}(m)}{m^{1/2-it}}\right|^2 g\left(\frac{t}{T}\right)dt.$$

Recall that $N = Q^{2-\epsilon}$.

$$\frac{(TH)^4}{N} = \frac{1}{N}\left(\frac{N}{Q}\right)^4 \asymp Q^{2-3\epsilon},$$

and this is shorter than the original sum!

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By the hybrid large sieve, we have the above is bounded by

$$\ll \frac{Q^2}{Q^{2-\epsilon}}\left(H^2T + Q^{2-3\epsilon}\right)\sum_{m \ll \frac{(TH)^4}{N}} \frac{d_4^2(n)}{n}$$

$$= Q^{\epsilon}(Q^{2-2\epsilon} + Q^{2-3\epsilon})(\log Q)^{16}$$

$$\ll Q^{2-\epsilon_0}.$$

The sixth moment – without average over t

Recall that we consider

$$\sum_q \Psi\left(\frac{q}{Q}\right)\phi(q) \sum\sum_{\substack{mn\leq Q^3 \\ m\equiv n \mod q}} \frac{d_3(n)d_3(m)}{\sqrt{mn}}$$

We do dyadic sum for $m \sim M, n \sim N$. There are three ranges of M and N to consider.

1. $M, N \ll Q^{2-\delta}$ [Main contribution]
   Use CIS's work. The complementary divisor reduces the size of the conductor. $h \asymp \frac{|M-N|}{Q} \ll Q^{1-\delta}$.
2. M or $N \gg Q^{2-\delta}$. WLOG, $M \geq N$. [Error terms]

$$M \gg Q^{2-\delta} \quad \text{and} \quad N \ll Q^{1+\delta}$$

Few words about the unbalanced sums

2.1 $M \gg Q^{5/2+\delta_0}$ (very unbalanced – easier)

Apply Voronoi summation for $d_3(m)$, and then bound the dual summation. The bound is

$$\ll \frac{Q^{\frac{9}{2}+\epsilon}}{M} \ll Q^{2-\epsilon_1}.$$

Few words about the unbalanced sums

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Apply Voronoi summation for $d_3(m)$, and then bound the dual summation. The bound is

$$\ll \frac{Q^{\frac{9}{2}+\epsilon}}{M} \ll Q^{2-\epsilon_1}.$$

2.2 $Q^{2-\delta} \ll M \ll Q^{5/2+\delta_0}$ (difficult)

- $d_3(m) = 1 \star 1 \star 1(m) = \sum_{efg=m} 1.$
- WLOG, $e \sim E$, $f \sim F$, and $g \sim G$, where $E \geq F \geq G$.
- The congruence condition is $efg \equiv n \mod q$

We apply Poisson summation to the sum over e and obtain

$$\frac{\sqrt{E}}{\sqrt{FGN}} \sum_q \Psi\left(\frac{q}{Q}\right) \sum_{n \sim N} d_3(n) \sum_{f \sim F} \sum_{g \sim G} \sum_{e \ll \frac{Q}{E}} \mathrm{e}\left(\frac{n e \overline{fg}}{q}\right) U(e, f, g)$$

- Reciprocity: $\mathrm{e}\left(\frac{ne\overline{fg}}{q}\right) = \mathrm{e}\left(-\frac{ne\overline{q}}{fg}\right)\mathrm{e}\left(\frac{ne}{qfg}\right)$, where $\mathrm{e}(x) = e^{2\pi i}$.

$$\frac{ne}{qfg} \ll N\frac{Q}{E}\frac{1}{QFG} \ll \frac{N}{EFG} \ll \frac{1}{Q}$$

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- Poisson summation in $q \longrightarrow$ the sum over Kloosterman sum. In particular, we need to understand the sum of the form

$$\frac{Q\sqrt{E}}{\sqrt{FGN}} \sum_{n\sim N} \sum_{g\sim G} \sum_{e\sim \frac{Q}{E}} \sum_{r\sim \frac{FG}{Q}} a_n b_{r,e,g} \times \sum_{f\sim F} \frac{1}{fg} \mathcal{H}\left(\frac{4\pi\sqrt{nre}}{fg}\right) \underbrace{S(n, -re; fg)}_{\text{Kloosterman sum}}$$

- Kuznetsov's formula

Kuznetsov's formula

Kloosterman sum:

$$S(m,n;c) = \sum_{\substack{a \bmod c \\ (a,c)=1}} \mathrm{e}\left(\frac{am+\bar{a}n}{c}\right).$$

Roughly, Kuznetsov's formula is of the form

$$\sum_c \frac{1}{c} S(m,n;c)\psi\left(\frac{mn}{c}\right) = \sum_{\substack{\lambda_j-\text{eigenvalue} \\ \text{of the Laplacian}}} \overline{\rho_j(m)}\rho_j(n)\tilde{\psi}(\lambda_j)$$
$$+ \text{ manageable terms},$$

where

- $\tilde{\psi}$ is a kind of transform of the function ψ with compact support.
- $\rho_j(n)$ is the n^{th} Fourier coefficient of the eigen-cusp form associated with λ_j.

Eigenvalue is of the form

$$\lambda_j = \frac{1}{4} + k_j^2.$$

It is conjectured that $\lambda_j \geq \frac{1}{4}$.

We call $\lambda_j < \frac{1}{4}$ as an **exceptional eigenvalue**.

The sum over exceptional eigenvalues is the hardest!