The recipe for moments of L-functions

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Moments of the Riemann zeta-function

Conjecture (Folklore)

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt \sim c_k T(\log T)^{k^2} \quad \textit{as } T \to \infty$$

for some (unspecified) constant c_k

- $c_1 = 1$ (Hardy and Littlewood, 1918)
- $c_2 = 1/(2\pi^2)$ (Ingham, 1926)
- No other c_k is known (i.e. proven), but we have conjectures:
- Conrey and Ghosh (1993): c_3
- Conrey and Gonek (1998): c_3 and c_4
- Keating and Snaith (1998) using random matrix theory: c_k for all complex k with $\mathrm{Re}(k) \geq -1/2$
- Diaconu, Goldfeld, and Hoffstein (2000) using multiple Dirichlet series: c_k for all positive integers k
- Conrey, Farmer, Keating, Rubinstein, and Snaith (2000) via a procedure called the *recipe*: c_k for all positive integers k
- *All the conjectured values of* c_k *agree.*

The CFKRS recipe for moments of zeta

- Let $\mathcal{M}_{A,B}(T) := \int_T^{2T} \prod_{\alpha\in A} \zeta(\tfrac{1}{2}+\alpha+it) \prod_{\beta\in B} \zeta(\tfrac{1}{2}+\beta-it)\,dt$, where the "shifts" α, β are small complex numbers.
- basic steps in recipe: Use the approximate functional equation, ignore any oscillating terms, ignore any "off-diagonal" terms
- The expectation is that these ignored terms somehow cancel

Conjecture (CFKRS, 2000)

$$\mathcal{M}_{A,B}(T) \sim \sum_{\substack{U\subseteq A, V\subseteq B\\ |U|=|V|}} \int_T^{2T} \left(\frac{t}{2\pi}\right)^{-\sum_{\alpha\in U}\alpha - \sum_{\beta\in V}\beta} \sum_{n=1}^{\infty} \frac{\tau_{A\smallsetminus U\cup V^-}(n)\tau_{B\smallsetminus V\cup U^-}(n)}{n}\,dt$$

- Notation: $\prod_{\alpha\in A}\zeta(\alpha+s) =: \sum_{n=1}^{\infty}\frac{\tau_A(n)}{n^s}$ and $U^- := \{-\alpha : \alpha\in U\}$
- We call the cardinality $|U| = |V|$ the number of *swaps*

Example: fourth moment of zeta (known)

Conjecture (CFKRS, 2000)

$$\int_T^{2T} \prod_{\alpha \in A} \zeta(\tfrac{1}{2} + \alpha + it) \prod_{\beta \in B} \zeta(\tfrac{1}{2} + \beta - it)\, dt \sim$$

$$\sum_{\substack{U \subseteq A, V \subseteq B \\ |U| = |V|}} \int_T^{2T} \left(\frac{t}{2\pi}\right)^{-\sum\limits_{\alpha \in U} \alpha - \sum\limits_{\beta \in V} \beta} \sum_{n=1}^{\infty} \frac{\tau_{A \setminus U \cup V^-}(n) \tau_{B \setminus V \cup U^-}(n)}{n}\, dt$$

If $A = \{\alpha_1, \alpha_2\}$ and $B = \{\beta_1, \beta_2\}$, then

$$\sum_{n=1}^{\infty} \frac{\tau_A(n)\tau_B(n)}{n}$$
$$= \frac{\zeta(1+\alpha_1+\beta_1)\zeta(1+\alpha_1+\beta_2)\zeta(1+\alpha_2+\beta_1)\zeta(1+\alpha_2+\beta_2)}{\zeta(2+\alpha_1+\alpha_2+\beta_1+\beta_2)}$$
$$=: \mathcal{Z}(\alpha_1, \alpha_2; \beta_1, \beta_2), \text{ say.}$$

Example: fourth moment of zeta (known)

Conjecture (CFKRS, 2000)

$$\int_T^{2T} \prod_{\alpha\in A} \zeta(\tfrac12+\alpha+it) \prod_{\beta\in B} \zeta(\tfrac12+\beta-it)\,dt \sim$$
$$\sum_{\substack{U\subseteq A, V\subseteq B\\ |U|=|V|}} \int_T^{2T} \left(\frac{t}{2\pi}\right)^{-\sum_{\alpha\in U}\alpha-\sum_{\beta\in V}\beta} \sum_{n=1}^{\infty} \frac{\tau_{A\smallsetminus U\cup V^-}(n)\tau_{B\smallsetminus V\cup U^-}(n)}{n}\,dt$$

Theorem (CFKRS, 2000)

$$\int_T^{2T} \zeta(\tfrac12+\alpha_1+it)\zeta(\tfrac12+\alpha_2+it)\zeta(\tfrac12+\beta_1-it)\zeta(\tfrac12+\beta_2-it)\,dt \sim$$
$$\int_T^{2T} \Bigg\{ \mathcal{Z}(\alpha_1,\alpha_2;\beta_1,\beta_2)$$
$$+\left(\frac{t}{2\pi}\right)^{-\alpha_1-\beta_1} \mathcal{Z}(-\beta_1,\alpha_2;-\alpha_1,\beta_2)+\left(\frac{t}{2\pi}\right)^{-\alpha_1-\beta_2} \mathcal{Z}(-\beta_2,\alpha_2;\beta_1,-\alpha_1)$$
$$+\left(\frac{t}{2\pi}\right)^{-\alpha_2-\beta_1} \mathcal{Z}(\alpha_1,-\beta_1;-\alpha_2,\beta_2)+\left(\frac{t}{2\pi}\right)^{-\alpha_2-\beta_2} \mathcal{Z}(\alpha_1,-\beta_2;\beta_1,-\alpha_2)$$
$$+\left(\frac{t}{2\pi}\right)^{-\alpha_1-\alpha_2-\beta_1-\beta_2} \mathcal{Z}(-\beta_1,-\beta_2;-\alpha_1,-\alpha_2)\Bigg\}\,dt.$$

Analogous theorem in random matrix theory

Conjecture (CFKRS, 2000)

$$\int_T^{2T} \prod_{\alpha\in A} \zeta(\tfrac{1}{2}+\alpha+it) \prod_{\beta\in B} \zeta(\tfrac{1}{2}+\beta-it)\, dt \sim$$

$$\sum_{\substack{U\subseteq A, V\subseteq B \\ |U|=|V|}} \int_T^{2T} \left(\frac{t}{2\pi}\right)^{-\sum_{\alpha\in U}\alpha - \sum_{\beta\in V}\beta} \sum_{n=1}^{\infty} \frac{\tau_{A\smallsetminus U\cup V^-}(n)\tau_{B\smallsetminus V\cup U^-}(n)}{n}\, dt$$

Theorem (CFKRS, 2000)

Let $U(N)$ be the group of $N\times N$ unitary matrices. Then integrating with respect to the Haar measure gives

$$\int_{U(N)} \prod_{\alpha\in A} \det\left(1-e^{-\alpha}M\right) \prod_{\beta\in B} \det\left(1-e^{-\beta}M^{-1}\right) dM$$

$$= \sum_{\substack{U\subseteq A, V\subseteq B \\ |U|=|V|}} \left(e^N\right)^{-\sum_{\alpha\in U}\alpha - \sum_{\beta\in V}\beta} Z(A\smallsetminus U\cup V^-, B\smallsetminus V\cup U^-),$$

where $Z(A,B) := \prod_{\alpha\in A,\beta\in B} (1-e^{-\alpha-\beta})^{-1}$.

The recipe for moments of quadratic Dirichlet L-functions

- Let $\chi_{8d}(\cdot) = (8d|\cdot)$ be the Kronecker symbol. This is a real primitive character for odd square-free d.
- We have the approximate functional equation

$$L(\tfrac{1}{2}+\alpha, \chi_{8d}) \approx \sum_n \frac{\chi_{8d}(n)}{n^{\frac{1}{2}+\alpha}} + \left(\frac{8d}{\pi}\right)^{-\alpha} \sum_n \frac{\chi_{8d}(n)}{n^{\frac{1}{2}-\alpha}}.$$

- We also have the expected value

$$\sum_{d\le D} \mu^2(2d)\chi_{8d}(n) \approx \sum_{\substack{d\le D\\(d,n)=1}} \mu^2(2d)$$

if n is an odd square, and ≈ 0 otherwise. These lead to

Conjecture (CFKRS, 2000)

$$\sum_{d\le D} \mu^2(2d) \prod_{\alpha\in A} L(\tfrac{1}{2}+\alpha, \chi_{8d})$$
$$\sim \sum_{U\subseteq A} \sum_{d\le D} \mu^2(2d) \left(\frac{8d}{\pi}\right)^{-\sum_{\alpha\in U}\alpha} \sum_{\substack{1\le n<\infty\\ n=\square\\ (n,2d)=1}} \frac{\tau_{A\setminus U\cup U^-}(n)}{\sqrt{n}}$$

Multiple Dirichlet series

- We may use Perron's formula to write (assuming convergence)

$$\sum_{d\le D}\mu^2(2d)\prod_{\alpha\in A}L(\tfrac12+\alpha,\chi_{8d})=\frac{1}{2\pi i}\int_{(2)}\frac{D^w}{w}F(A;w)\,dw,$$

  where $F(A;w):=\sum_{d=1}^{\infty}\frac{\mu^2(2d)}{d^w}\prod_{\alpha\in A}L(\tfrac12+\alpha,\chi_{8d})$.
- Expanding each L-function leads us to expect a pole at $w=1$.
- By the functional equation

$$L(\tfrac12+\alpha,\chi_{8d})\approx\left(\frac{8d}{\pi}\right)^{-\alpha}L(\tfrac12-\alpha,\chi_{8d}),$$

  we have for each $U\subseteq A$ the functional equation

$$F(A;w)\approx\left(\frac{8}{\pi}\right)^{-\sum_{\alpha\in U}\alpha}F\Big(A\smallsetminus U\cup U^-;w+\sum_{\alpha\in U}\alpha\Big).$$

- Thus we expect a pole at $w=1-\sum_{\alpha\in U}\alpha$ for each $U\subseteq A$.
- In ongoing work, Martin Čech and I show that the residue of $F(A;w)$ at $w=1-\sum_{\alpha\in U}\alpha$ gives rise to the term corresponding to U in the recipe prediction.

Dirichlet polynomial approximations

- To understand how the off-diagonal terms cancel, Conrey and Keating (2015) examined Dirichlet polynomial approximations of zeta.
- Using Perron's formula and the recipe, we expect that

$$\int_T^{2T} \sum_{m\le X} \frac{\tau_A(m)}{m^{\frac12+it}} \sum_{n\le X} \frac{\tau_B(n)}{n^{\frac12-it}}\, dt$$

$$\sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{X^{z+w}}{zw} \int_T^{2T} \prod_{\alpha\in A} \zeta(\tfrac12+\alpha+z+it) \prod_{\beta\in B} \zeta(\tfrac12+\beta+w-it) \times dt\, dw\, dz$$

$$\sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{X^{z+w}}{zw} \sum_{\substack{U\subseteq A, V\subseteq B\\ |U|=|V|}} \int_T^{2T} \left(\frac{t}{2\pi}\right)^{-\sum_{\alpha\in U}(\alpha+z)-\sum_{\beta\in V}(\beta+w)} \times \sum_{n=1}^{\infty} \frac{\tau_{A_z\smallsetminus U_z\cup V_w^-}(n)\tau_{B_w\smallsetminus V_w\cup U_z^-}(n)}{n}\, dt\, dw\, dz$$

- If $|U|=|V|=\ell$, then we have $\left(\frac{X}{t^\ell}\right)^{z+w}$ here. Thus, intuitively, the ℓ-swap terms contribute to the main term only if $X \gg T^\ell$

One-swap terms for moments of zeta

- Conrey and Keating (2015) found that the 1-swap terms for zeta are the consequence of formulas for correlations of divisor-sums
- This connection has been made rigorous by Alia Hamieh and Nathan Ng

Theorem (Hamieh and Ng, 2021)

Assume the expected asymptotic formula for correlations of divisor sums. If $X = T^\eta$ with $1 < \eta < 2$, then as $T \to \infty$,

$$\int_T^{2T} \sum_{m \le X} \frac{\tau_A(m)}{m^{\frac{1}{2}+it}} \sum_{n \le X} \frac{\tau_B(n)}{n^{\frac{1}{2}-it}} \, dt \sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{X^{z+w}}{zw} \sum_{\substack{U \subseteq A, V \subseteq B \\ 0 \le |U| = |V| \le 1}}$$

$$\times \int_T^{2T} \left(\frac{t}{2\pi}\right)^{-\sum_{\alpha \in U}(\alpha+z) - \sum_{\beta \in V}(\beta+w)} \sum_{\substack{1 \le m,n < \infty \\ m=n}} \frac{\tau_{A_z \setminus U_z \cup V_w^-}(m) \tau_{B_w \setminus V_w \cup U_z^-}(n)}{\sqrt{mn}} dt\, dw\, dz$$

- The mentioned "expected asymptotic formula" results from applying the delta method of Duke, Friedlander, and Iwaniec (1994) to the sum $\sum_{\substack{1 \le m,n < \infty \\ m-n=h}} \tau_A(m)\tau_B(n) f(m,n)$ for a suitable f, and ignoring error terms.

One-swap terms for twisted moments of zeta

- Conrey and Keating (2015) found that the 1-swap terms for zeta are the consequence of formulas for correlations of divisor-sums
- This connection has been made rigorous by Alia Hamieh and Nathan Ng

Theorem (in progress)

Assume the expected asymptotic formula for correlations of divisor sums. If $X = T^\eta$ with $1 < \eta < 2$, then as $T \to \infty$,

$$\int_T^{2T} \left(\frac{M}{N}\right)^{it} \sum_{m \le X} \frac{\tau_A(m)}{m^{\frac{1}{2}+it}} \sum_{n \le X} \frac{\tau_B(n)}{n^{\frac{1}{2}-it}}\, dt \sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{X^{z+w}}{zw} \sum_{\substack{U \subseteq A, V \subseteq B \\ 0 \le |U| = |V| \le 1}}$$

$$\times \int_T^{2T} \left(\frac{t}{2\pi}\right)^{-\sum\limits_{\alpha \in U}(\alpha+z) - \sum\limits_{\beta \in V}(\beta+w)} \sum_{\substack{1 \le m,n < \infty \\ mN = nM}} \frac{\tau_{A_z \smallsetminus U_z \cup V_w^-}(m)\, \tau_{B_w \smallsetminus V_w \cup U_z^-}(n)}{\sqrt{mn}}\, dt\, dw\, dz$$

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Caroline Turnage-Butterbaugh and I have proven, on GLH, that the 1-swap terms predicted by the recipe are correct for the family of Dirichlet L-functions of primitive characters (averaged over the conductor).

Theorem (B. and Turnage-Butterbaugh, 2022)

Assume the Generalized Lindelöf Hypothesis. If $X = Q^\eta$ with $1 < \eta < 2$, then as $Q \to \infty$,

$$\sum_{Q<q<2Q} \sideset{}{^*}\sum_{\chi \bmod q} \sum_{m \le X} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} \sum_{n \le X} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}}$$

$$\sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{X^{z+w}}{zw} \sum_{\substack{U \subseteq A, V \subseteq B \\ 0 \le |U| = |V| \le 1}} \sum_{Q<q<2Q} \sideset{}{^*}\sum_{\chi \bmod q} \left(\frac{q}{\pi}\right)^{-\sum_{\alpha \in U}(\alpha+z) - \sum_{\beta \in V}(\beta+w)}$$

$$\times \sum_{\substack{1 \le m,n < \infty \\ m=n \\ (mn,q)=1}} \frac{\tau_{A_z \smallsetminus U_z \cup V_w^-}(m)\tau_{B_w \smallsetminus V_w \cup U_z^-}(n)}{\sqrt{mn}} \, dw \, dz$$

The cornerstone of our proof is the asymptotic large sieve developed by Conrey, Iwaniec, and Soundararajan (2011).

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Assume the Generalized Lindelöf Hypothesis. If $X = Q^\eta$ with $1 < \eta < 2$, then as $Q \to \infty$,

$$\sum_{Q<q<2Q} \sideset{}{^*}\sum_{\chi \bmod q} \overline{\chi}(M)\chi(N) \sum_{m\le X} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} \sum_{n\le X} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}}$$

$$\sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{X^{z+w}}{zw} \sum_{\substack{U\subseteq A, V\subseteq B\\ 0\le |U|=|V|\le 1}} \sum_{Q<q<2Q} \sideset{}{^*}\sum_{\chi \bmod q} \left(\frac{q}{\pi}\right)^{-\sum_{\alpha\in U}(\alpha+z)-\sum_{\beta\in V}(\beta+w)}$$

$$\times \sum_{\substack{1\le m,n<\infty\\ mN=nM\\ (mnMN,q)=1}} \frac{\tau_{A_z \smallsetminus U_z \cup V_w^-}(m)\tau_{B_w \smallsetminus V_w \cup U_z^-}(n)}{\sqrt{mn}}\, dw\, dz$$

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Theorem (Conrey and Rodgers, 2020)

Assume the Generalized Lindelöf Hypothesis. If $X = D^\eta$ with $1 < \eta < 2$, then as $D \to \infty$,

$$\sum_{D<d<2D} \mu^2(2d) \sum_{n \le X} \frac{\tau_A(n)(8d|n)}{\sqrt{n}} \sim \frac{1}{2\pi i} \int_{(\varepsilon)} \frac{X^z}{z}$$

$$\times \sum_{\substack{U \subseteq A \\ 0 \le |U| \le 1}} \sum_{D<d<2D} \mu^2(2d) \left(\frac{8d}{\pi}\right)^{-\sum_{\alpha \in U}(\alpha+z)} \sum_{\substack{1 \le n < \infty \\ n = \square \\ (n,2d)=1}} \frac{\tau_{A_z \smallsetminus U_z \cup U_z^-}(n)}{\sqrt{n}}\, dz$$

Here, $(8d|n)$ is the Kronecker symbol. The cornerstone of their proof is a technique developed by Soundararajan (2000) that uses the Poisson summation formula to transform smoothed sums of $(8d|n)$ over d into sums of Gauss-type sums.

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Assume the Generalized Lindelöf Hypothesis. If $X = D^\eta$ with $1 < \eta < 2$, then as $D \to \infty$,

$$\sum_{D<d<2D} \mu^2(2d)\, (8d|M) \sum_{n\le X} \frac{\tau_A(n)(8d|n)}{\sqrt{n}} \sim \frac{1}{2\pi i}\int_{(\varepsilon)} \frac{X^z}{z}$$

$$\times \sum_{\substack{U\subseteq A\\ 0\le|U|\le 1}} \sum_{D<d<2D} \mu^2(2d) \left(\frac{8d}{\pi}\right)^{-\sum\limits_{\alpha\in U}(\alpha+z)} \sum_{\substack{1\le n<\infty\\ nM=\square\\ (nM,2d)=1}} \frac{\tau_{A_z\smallsetminus U_z\cup U_z^-}(n)}{\sqrt{n}}\, dz$$

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Brian Conrey and Alessandro Fazzari have proven, on GLH, that the 1-swap terms predicted by the recipe are correct for the family of L-functions associated with Hecke eigencuspforms $\mathcal{H}_k$ of weight k, with $k \equiv 0 \pmod 4$. For $f \in \mathcal{H}_k$, let $\lambda_f(n)$ denote the nth Hecke eigenvalue of f.

Theorem (Conrey and Fazzari)

Assume the Generalized Lindelöf Hypothesis. Let $A = \{\alpha_1, \ldots, \alpha_r\}$. If $X = (k^2)^\eta$ with $1 < \eta < 2$, then as $k \to \infty$,

$$\sum_{f \in \mathcal{H}_k} \frac{1}{\omega_f} \sum_{m \le X} \frac{1}{\sqrt{m}} \sum_{m_1 \cdots m_r = m} \frac{\lambda_f(m_1) \cdots \lambda_f(m_r)}{m_1^{\alpha_1} \cdots m_r^{\alpha_r}}$$

$$\sim \frac{1}{2\pi i} \int_{(\varepsilon)} \frac{X^z}{z} \sum_{\substack{V \subseteq A \\ 0 \le |V| \le 1}} \sum_{f \in \mathcal{H}_k} \frac{1}{\omega_f} \left(\frac{k^2}{4\pi^2} \right)^{-\sum_{\alpha \in V} (\alpha + z)} G(A_z \smallsetminus V_z \cup V_z^-) \, dz.$$

Here, ω_f is the natural weight that arises from the Petersson norm, and

$$G(A) := \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sum_{m_1 \cdots m_r = m} \frac{\prod_p \frac{2}{\pi} \int_0^\pi U_{\mathrm{ord}_p(m_1)}(\cos\theta) \cdots U_{\mathrm{ord}_p(m_r)}(\cos\theta) \sin^2\theta \, d\theta}{m_1^{\alpha_1} \cdots m_r^{\alpha_r}}.$$

1-swap terms for twisted moments of $L_f(s)$, $f \in \mathcal{H}_k$

Brian Conrey and Alessandro Fazzari have proven, on GLH, that the 1-swap terms predicted by the recipe are correct for the family of L-functions associated with Hecke eigencuspforms $\mathcal{H}_k$ of weight k, with $k \equiv 0 \pmod 4$. For $f \in \mathcal{H}_k$, let $\lambda_f(n)$ denote the nth Hecke eigenvalue of f.

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Assume the Generalized Lindelöf Hypothesis. Let $A = \{\alpha_1, \ldots, \alpha_r\}$. *If* $X = (k^2)^\eta$ *with* $1 < \eta < 2$, *then as* $k \to \infty$,

$$\sum_{f \in \mathcal{H}_k} \frac{1}{\omega_f} \lambda_f(M) \sum_{m \le X} \frac{1}{\sqrt{m}} \sum_{m_1 \cdots m_r = m} \frac{\lambda_f(m_1) \cdots \lambda_f(m_r)}{m_1^{\alpha_1} \cdots m_r^{\alpha_r}}$$

$$\sim \frac{1}{2\pi i} \int_{(\varepsilon)} \frac{X^z}{z} \sum_{\substack{V \subseteq A \\ 0 \le |V| \le 1}} \sum_{f \in \mathcal{H}_k} \frac{1}{\omega_f} \left(\frac{k^2}{4\pi^2} \right)^{-\sum_{\alpha \in V} (\alpha + z)} G_M(A_z \smallsetminus V_z \cup V_z^-) \, dz.$$

Here, ω_f is the natural weight that arises from the Petersson norm, and

$$G_M(A) := \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sum_{m_1 \cdots m_r = m} \frac{\prod_p \frac{2}{\pi} \int_0^\pi U_{\mathrm{ord}_p(m_1)}(\cos\theta) \cdots U_{\mathrm{ord}_p(m_r)}(\cos\theta) U_{\mathrm{ord}_p(M)}(\cos\theta) \sin^2\theta \, d\theta}{m_1^{\alpha_1} \cdots m_r^{\alpha_r}}.$$

Higher swap terms: The Conrey-Keating heuristic

- Inspired by ideas of Bogomolny and Keating (1995), Conrey and Keating (2019) have developed a heuristic that shows how we may be able to split moments of zeta to get the ℓ-swap terms for general ℓ.
- In ongoing work, Brian Conrey and I refine the heuristic and adapt it to other families of L-functions.
- Starting observation: if we partition $A = A_1 \cup \cdots \cup A_\ell$, then $\tau_A = \tau_{A_1} * \cdots * \tau_{A_\ell}$, and similarly for B. Thus

$$\int_T^{2T} \sum_{m \le X} \frac{\tau_A(m)}{m^{\frac{1}{2}+it}} \sum_{n \le X} \frac{\tau_B(n)}{n^{\frac{1}{2}-it}}\,dt$$

$$= \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \frac{X^{\xi+\eta}}{\xi\eta} \int_T^{2T} \sum_{m=1}^{\infty} \frac{\tau_A(m)}{m^{\frac{1}{2}+\xi+it}} \sum_{n=1}^{\infty} \frac{\tau_B(n)}{n^{\frac{1}{2}+\eta-it}}\,dt\,d\xi\,d\eta$$

$$= \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \frac{X^{\xi+\eta}}{\xi\eta} \int_T^{2T} \prod_{j=1}^{\ell} \left\{ \sum_{m=1}^{\infty} \frac{\tau_{A_j}(m)}{m^{\frac{1}{2}+\xi+it}} \sum_{n=1}^{\infty} \frac{\tau_{B_j}(n)}{n^{\frac{1}{2}+\eta-it}} \right\} dt\,d\xi\,d\eta$$

- Instead of using this, we introduce some twisting and replace each factor by its average with respect to *only* the 1-swap terms

Higher swap terms: The Conrey-Keating heuristic

- By Perron's formula and writing τ_A, τ_B as Dirichlet convolutions (from previous slide):

$$\int_T^{2T} \sum_{m\le X} \frac{\tau_A(m)}{m^{\frac12+it}} \sum_{n\le X} \frac{\tau_B(n)}{n^{\frac12-it}}\, dt$$

$$= \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \frac{X^{\xi+\eta}}{\xi\eta} \int_T^{2T} \prod_{j=1}^{\ell} \left\{ \sum_{m=1}^{\infty} \frac{\tau_{A_j}(m)}{m^{\frac12+\xi+it}} \sum_{n=1}^{\infty} \frac{\tau_{B_j}(n)}{n^{\frac12+\eta-it}} \right\} dt\, d\xi\, d\eta$$

- We consider instead

$$\mathcal{S}_\ell := \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \frac{X^{\xi+\eta}}{\xi\eta}$$

$$\times \int_T^{2T} \frac{1}{(\ell!)^2} \sum_{\substack{M_1\cdots M_\ell = N_1 \cdots N_\ell \\ (M_j,N_j)=1\ \forall j}} \prod_{j=1}^{\ell} \left\langle \left(\frac{M_j}{N_j}\right)^{it} \sum_{m=1}^{\infty} \frac{\tau_{A_j}(m)}{m^{\frac12+\xi+it}} \sum_{n=1}^{\infty} \frac{\tau_{B_j}(n)}{n^{\frac12+\eta-it}} \right\rangle_1 \times dt\, d\xi\, d\eta$$

- Here, the symbols $\langle\ \rangle_1$ mean that we are replacing the integrand of the twisted moment by its average with respect to *only* the 1-swap terms.

Higher swap terms: The Conrey-Keating heuristic

$$\mathcal{S}_\ell := \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \frac{X^{\xi+\eta}}{\xi\eta} \int_T^{2T} \frac{1}{(\ell!)^2} \sum_{\substack{M_1\cdots M_\ell = N_1 \cdots N_\ell \\ (M_j,N_j)=1 \ \forall j}} \prod_{j=1}^{\ell} \Bigg\{ \sum_{\substack{U\subseteq A_j, V\subseteq B_j \\ |U|=|V|=1}}$$

$$\times \left(\frac{t}{2\pi}\right)^{-\sum\limits_{\alpha\in U}(\alpha+\xi) - \sum\limits_{\beta\in V}(\beta+\eta)} \sum_{\substack{1\le m,n<\infty \\ mN_j = nM_j}} \frac{\tau_{(A_j)_\xi \smallsetminus U_\xi \cup V_\eta^-}(m)\tau_{(B_j)_\eta \smallsetminus V_\eta \cup U_\xi^-}(n)}{\sqrt{mn}} \Bigg\}$$

$$\times\, dt\, d\xi\, d\eta$$

- We write the sum over U, V as a multiple contour integral along small circles
- We then evaluate the sum over $M_1, \ldots, M_\ell, N_1, \ldots, N_\ell$ using the "unitary identity"

$$\sum_{\substack{M_1\cdots M_\ell = N_1 \cdots N_\ell \\ (M_j,N_j)=1 \ \forall j}} \prod_{j=1}^{\ell} \left(\sum_{\substack{1\le m,n<\infty \\ mN_j = nM_j}} \frac{\tau_{A_j}(m)\tau_{B_j}(n)}{m^s n^s} \right) = \sum_{\substack{1\le m,n<\infty \\ m=n}} \frac{\tau_A(m)\tau_B(n)}{m^s n^s}$$

- This leads to the prediction that $\mathcal{S}_\ell$ is asymptotic to a certain "Vandermonde integral"

Vandermonde integral prediction for zeta (B. and Conrey)

$$
\begin{aligned}
\mathcal{S}_\ell \sim \frac{1}{(\ell!)^2(2\pi i)^2} \int_{(2)}\int_{(2)} \frac{X^{\xi+\eta}}{\xi\eta} \int_T^{2T} \frac{1}{(2\pi i)^{2\ell}} \oint_{|z_1|=\varepsilon} \cdots \oint_{|z_\ell|=\varepsilon} \\
\times \oint_{|w_1|=\varepsilon} \cdots \oint_{|w_\ell|=\varepsilon} \left(\frac{t}{2\pi}\right)^{\sum_{j=1}^{\ell}(z_j+w_j-\xi-\eta)} \\
\times \prod_{\substack{\alpha\in A\\ \beta\in B}} \zeta(1+\alpha+\beta+\xi+\eta) \prod_{\substack{1\le j\le\ell\\ \alpha\in A}} \zeta(1+\alpha+z_j) \prod_{\substack{1\le j\le\ell\\ \beta\in B}} \zeta(1+\beta+w_j) \\
\times \prod_{\substack{1\le j\le\ell\\ \alpha\in A}} (1/\zeta)(1+\alpha+\xi+\eta-w_j) \prod_{\substack{1\le j\le\ell\\ \beta\in B}} (1/\zeta)(1+\beta+\xi+\eta-z_j) \\
\times \prod_{\substack{1\le i,j\le\ell\\ i\ne j}} (1/\zeta)(1-z_i+z_j) \prod_{\substack{1\le i,j\le\ell\\ i\ne j}} (1/\zeta)(1-w_i+w_j) \\
\times \prod_{1\le i,j\le\ell} \zeta(1+z_i+w_j-\xi-\eta)\zeta(1-z_i-w_j+\xi+\eta) \\
\times \mathcal{A}(A,B,Z,W,\xi+\eta)\, dw_\ell\cdots dw_1\, dz_\ell\cdots dz_1\, d\xi\, d\eta
\end{aligned}
$$

- Here, $\mathcal{A}$ is an absolutely convergent Euler product that has an explicit definition.
- The right-hand side may be evaluated to give exactly the ℓ-swap terms from the recipe prediction for moments of zeta.

Vandermonde integral theorem in RMT (Rodgers and Soundararajan)

Let $U(N)$ denote the group of $N \times N$ unitary matrices, with Haar measure. Then

$$\int_{U(N)} \prod_{\alpha \in A} \det\left(1 - e^{-\alpha} g\right) \prod_{\beta \in B} \det\left(1 - e^{-\beta} g^{-1}\right) dg = \sum_{\ell=0}^{\min\{|A|,|B|\}} J_\ell,$$

where J_ℓ is defined by

$$\begin{aligned} J_\ell = &\frac{1}{(\ell!)^2 (2\pi i)^{2\ell+1}} \oint_{|\xi|=1} \mathfrak{z}(\xi) \oint_{|z_1|=\varepsilon} \cdots \oint_{|z_\ell|=\varepsilon} \oint_{|w_1|=\varepsilon} \cdots \oint_{|w_\ell|=\varepsilon} \\ &\times \left(e^N\right)^{\sum_{j=1}^{\ell}(z_j + w_j - \xi)} \prod_{\substack{\alpha \in A \\ \beta \in B}} \mathfrak{z}(\alpha + \beta) \prod_{\substack{1 \le j \le \ell \\ \alpha \in A}} \mathfrak{z}(\alpha + z_j) \prod_{\substack{1 \le j \le \ell \\ \beta \in B}} \mathfrak{z}(\beta + w_j) \\ &\times \prod_{\substack{1 \le j \le \ell \\ \alpha \in A}} (1/\mathfrak{z})(\alpha + \xi - w_j) \prod_{\substack{1 \le j \le \ell \\ \beta \in B}} (1/\mathfrak{z})(\beta + \xi - z_j) \\ &\times \prod_{\substack{1 \le i,j \le \ell \\ i \neq j}} (1/\mathfrak{z})(z_i - z_j) \prod_{\substack{1 \le i,j \le \ell \\ i \neq j}} (1/\mathfrak{z})(w_i - w_j) \\ &\times \prod_{1 \le i,j \le \ell} \mathfrak{z}(z_i + w_j - \xi)\mathfrak{z}(-z_i - w_j + \xi) \\ &\times dw_\ell \cdots dw_1 \, dz_\ell \cdots dz_1 \, d\xi, \end{aligned}$$

with $\mathfrak{z}(x)$ defined by $\mathfrak{z}(x) := (1 - e^{-x})^{-1}$.