# Zero-free regions for $\zeta(s)$

Andrew Yang

University of New South Wales, Canberra

CPNTS 2024

# The classical zero-free region

Let $\zeta(s)$ denote the Riemann zeta-function, $s = \sigma + it$.

Complex zeroes of $\zeta(s)$ lie in the critical strip $0 < \sigma < 1$.

**Theorem (de la Vallée Poussin, Hadamard c. 1899)**

*There are no zeroes of $\zeta(s)$ in the region*

$$\sigma \geq 1 - \frac{A}{\log t}, \qquad (t \geq 2)$$

*for an absolute constant $A > 0$.*

# The constant $A$

- de la Vallée Poussin (1899) $A = 1/30.4679$
- Whesphal (1938) $A = 1/17.537$
- Rosser and SchoenFeld (1962) $A = 1/17.51631$
- Stechkin (1970) $A = 1/9.65$
- Rosser and Schoenfeld (1975) $A = 1/9.547897$
- Kondrat'ev (1977) $A = 1/9.547$
- Ford (2002) $A = 1/8.463$
- Kadiri (2005) $A = 1/5.69693$
- Jang and Kwon (2014) $A = 1/5.68371$
- Mossinghoff and Trudgian (2015) $A = 1/5.573412$
- Mossinghoff, Trudgian and Y. (2024) $A = 1/5.558691$

# Proof sketch (1/2)

Suppose there is a zero $\rho_0 = \beta_0 + it$ with $\beta_0 > 1/2$.
Choose a zero-detector, e.g.

$$-\Re\frac{\zeta'}{\zeta}(s) = \Re\sum_{n\geq 1}\Lambda(n)n^{-s} = -\Re\sum_{\rho}\left(\frac{1}{s-\rho}\right) + O(\log t)$$

Choose a non-negative trigonometric polynomial, e.g.

$$3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \geq 0.$$

Combine:

$$\begin{aligned} 0 &\leq \sum_{n\geq 1}\Lambda(n)n^{-\sigma}(3 + 4\cos(t\log n) + \cos(2t\log n)) \\ &= -3\frac{\zeta'}{\zeta}(\sigma) - 4\Re\frac{\zeta'}{\zeta}(\sigma+it) - \Re\frac{\zeta'}{\zeta}(\sigma+2it). \end{aligned}$$

# Proof sketch (2/2)

Assuming that $\sigma > \Re\rho$,

$$\Re\frac{1}{s-\rho} \geq 0.$$

Drop all the terms except the one corresponding to $\rho = \rho_0$!

$$0 \leq \frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta_0} + O(\log t).$$

Choosing $\sigma$ appropriately:

$$\beta_0 \leq 1 - \frac{c}{\log t}.$$

# Dropping all the terms (but one)

Doing better than $\Re(s-\rho)^{-1} \geq 0$ requires some knowledge about the distribution of zeroes within the critical strip.

Levinson (1970) showed that if zeroes are $O(1)$-spaced close to $\sigma = 1$, then there are no zeroes of $\zeta(s)$ in the region

$$\sigma \geq 1 - \frac{A}{\log \log t}.$$

# Observation 1: induction of zeroes

If there is a zero $\rho$ very close to $\sigma = 1$, then there must be many zeroes close to $\rho$.

## Theorem (Montgomery)

*If* $\zeta(\beta + i\gamma) = 0$ *then for* $\delta \in [1-\beta, 1]$, *we have*

$$\delta^2 \int_0^1 \int_0^\infty \frac{n(\gamma, w, h) + n(2\gamma, w, h)}{(h+\delta)^5 \exp(w/\delta)} dhdw \gg (1-\beta)^{-1}$$

*where* $n(t, w, h)$ *is the number of zeroes with*

$$|\gamma - t| \leq h, \qquad 1 - w \leq \beta \leq 1.$$

# Observation 1: induction of zeroes

Alternatively, if there are many zeroes close to $\rho_0 = \beta_0 + it$, then there is a good zero-free region at height $t$.

$$\Re \sum_{\rho : |\rho - \rho_0| \leq \delta} \frac{1}{s - \rho} \approx \frac{\#\{\rho : |\rho - \rho_0| \leq \delta\}}{\sigma - \beta_0}$$

# Observation 2: long-range interaction of zeroes

Good zero-free region at height $2t + O(1) \implies$ good zero-free region at height $t$.

E.g. if all zeroes $\rho = \beta + i\gamma$ with $|\gamma - 2t| \leq 1$ satisfy $1 - \beta > A$, then

$$\sum_{\rho} \Re \frac{1}{\sigma + 2it - \rho} \geq \sum_{\rho : |\gamma - 2t| \leq 1} \frac{\sigma - \beta}{(\sigma - \beta)^2 + (2t - \gamma)^2} \gg \log t.$$

# Observation 2: long-range interaction of zeroes

Bad zero-free region at height $2t + O(1/\log t) \implies$ good zero-free region at height $t$.

Take a single zero $\rho$ at height $2t + O(1/\log t)$ with $\sigma - \beta = O(1/\log t)$. Then

$$\Re \frac{1}{\sigma + 2it - \rho} \geq \frac{\sigma - \beta}{(\sigma - \beta)^2 + (2t - \gamma)^2} \gg \log t.$$

Summary and thank you.