There exists a weakly mixing billiard in a polygon

Jon Chaika

University of Utah

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Joint with Giovanni Forni

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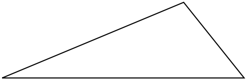

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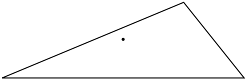

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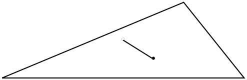

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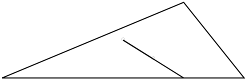

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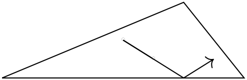

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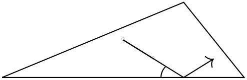

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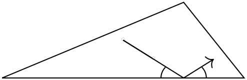

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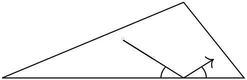

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$$X_Q := Q \times S^1 / \sim$$

and we let F_Q^t denote the straight line flow on X_Q.
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(Kerckhoff-Masur-Smillie '86) There exists a polygon Q so that the flow on X_Q is ergodic with respect to $\mathbf{m}_Q$.

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2. F_Q^t has at most a countable number of families of homotopic periodic orbits (Boldrighini-Keane-Marchetti).

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Theorem

(Kerckhoff-Masur-Smillie) For every rational polygon Q, for almost every invariant surface $S_\theta \subset X_Q$, F_Q^t is ergodic with respect to the (2-dimensional) Lebesgue measure on $S_\theta \subset X_Q$.

We denote this measure λ_θ.

A word on the proof of Kerckhoff-Masur-Smillie's Theorem

Let $Lip(X_Q)$ be the set of 1-Lipschitz functions on X_Q.

Lemma

F_Q^t is ergodic iff for all $f \in Lip(X_Q)$ we have that there exists $T_i \to \infty$ so that

$$\lim_{i\to\infty} \int_{X_Q} \Big(\Big| \frac{1}{T_i} \int_0^{T_i} f(F^t(\theta, x))dt - \int_{X_Q} f d\mathbf{m}_Q \Big| \Big) d\mathbf{m}_Q = 0. \quad (1)$$

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Proposition

For all $\epsilon > 0$ *if* Q *satisfies that for all* $f \in Lip(X_Q)$ *there exists a* T *so that*

$$\int_{X_Q} \left(\left| \frac{1}{T} \int_0^T f(F_Q^t(\theta, x))dt - \int f d\mathbf{m}_Q \right| \right) d\mathbf{m}_Q < \epsilon$$

then the set of Q' *so that for all* $f \in Lip(X(Q'))$ *there exists* T *so that*

$$\int_{X_{Q'}} \left(\left| \frac{1}{T} \int_0^T f(F_{Q'}^t(\theta, x))dt - \int f d\mathbf{m}_{Q'} \right| \right) d\mathbf{m}_{Q'} < 2\epsilon$$

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contains an open neighborhood of Q.

By the ergodicity result of Kerckhoff-Masur-Smillie this set is dense for each fixed ϵ.

If Q is rational, for almost every θ, for every $f \in Lip(X_Q)$

$$\lim_{T\to\infty} \int_{S_\theta} \Big(\Big| \frac{1}{T} \int_0^T f(F^t(\theta, x))dt - \int_{S_\theta} f d\lambda_\theta \Big| \Big) d\lambda_\theta = 0.$$

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If G_Q contains a small rotation, for all $f \in Lip(X_Q)$ we have

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is small (for all θ).
By the Baire Category Theorem we have that a dense G_δ subset of the space of polygons satisfies (**??**).

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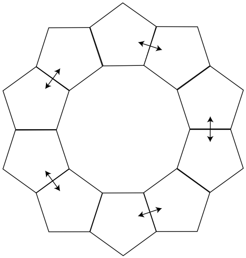
So our proof is similar to Kerckhoff-Masur-Smillie's proof:
Replace $Lip(X_Q)$ with $Lip(X_Q \times X_Q)$.
Replace the ergodicity of F_Q^t restricted to a.e. S_θ when Q is rational by the ergodicity of $F_Q^t \times F_Q^t$ restricted to a.e. $S_\theta \times S_\phi$ when Q is rational.

Theorem

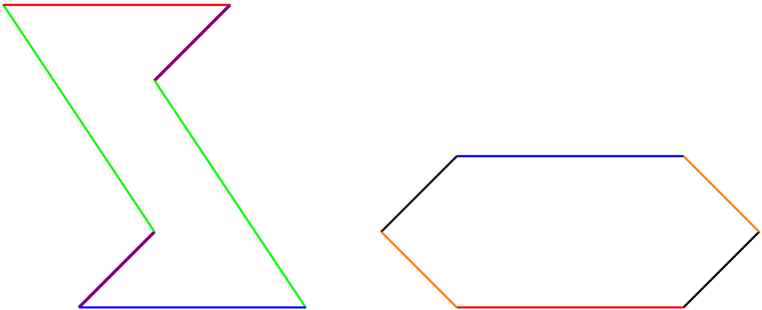
(C-Forni) For every rational Q, for almost every (θ, ϕ) we have that $F_Q^t \times F_Q^t$ is $\lambda_\theta \times \lambda_\phi$ ergodic.

Reflection

Figure: Photo Credit: Evelyn Lamb



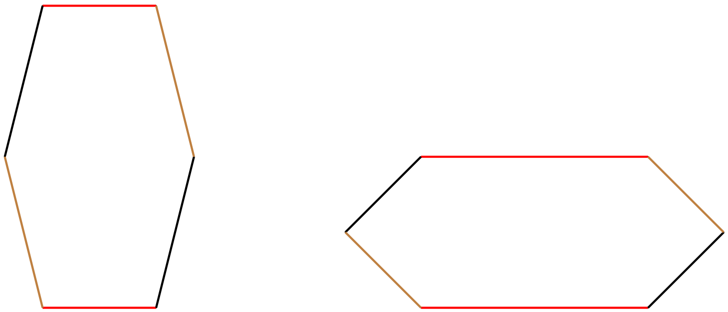
A translation surface



$SL(2, \mathbb{R})$ action

$SL(2, \mathbb{R})$ acts on translation by acting on the charts.

Figure: $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ applied to a translation surface



Let $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ and $r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.

Theorem

(C-Forni)Let M be a translation surface and F_θ^t *denote the flow in direction* θ*. For a.e.* θ, ϕ*,* $F_\theta^t \times F_\phi^t$ *is* λ_M^2 *ergodic.*

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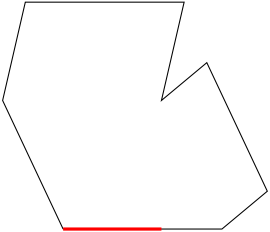
Is it uniquely ergodic?
Hubert and I showed that almost surely it is with respect to any $SL(2, \mathbb{R})$ invariant measure.

Transversals for translation surfaces



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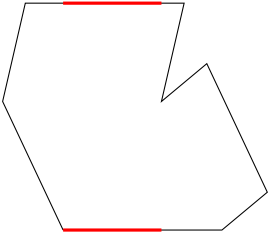

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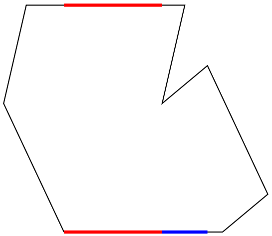

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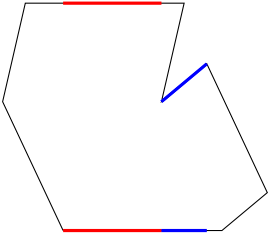

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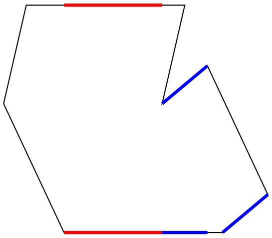

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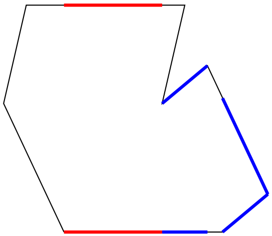

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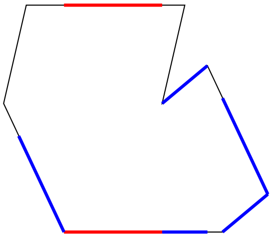

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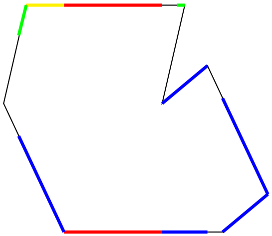

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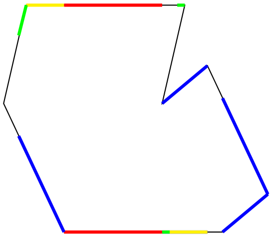

Veech Criterion: continuous case

If α is a continuous eigenvalue of F^t, J_i are sequence of transversals so that $diam(J_\ell) \to 0$ and $\vec{r_i}$ are the sequence of return time vectors to J_i then

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So if $x, F^t x \in J_\ell$ then $e^{2\pi i\alpha t} \sim 1$.

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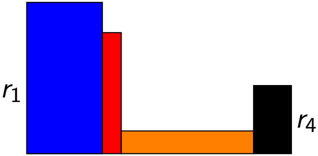
Pictures for a translation surface



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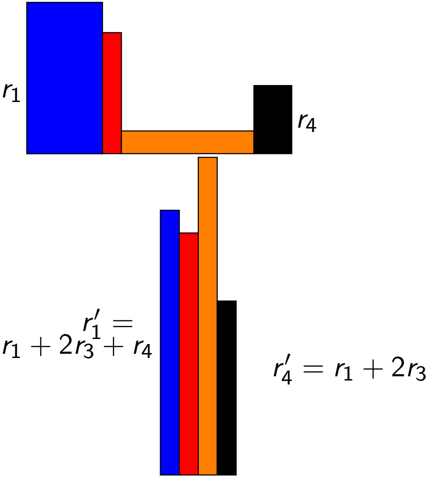

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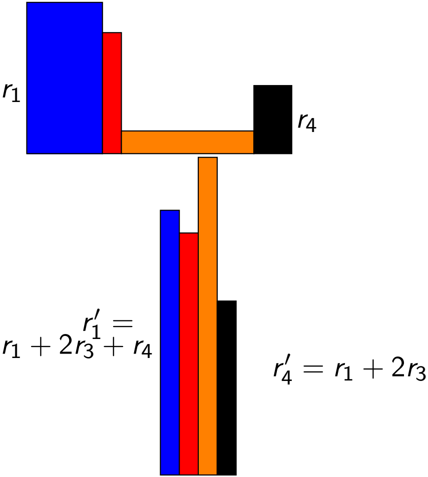
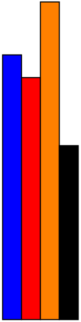
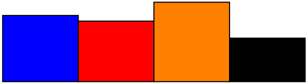


$$\vec{r'} = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix} \vec{r}$$

Renormalization



Applying $\begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$



Veech criterion final form

Transversals are given by a cocycle
$RV : \mathbb{R} \times \mathcal{H} \to SL(d, \mathbb{Z})$.
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(Veech Criterion slight lie) If the exists a compact set $\mathcal{K} \subset \mathcal{H}$ and $\epsilon > 0$ so that for arbitrarilly large L we have $\|\alpha RV(\log(L), Y)\vec{r}_1\|_{\mathbb{Z}^d} > \epsilon$ and $g_{\log(L)}Y \in \mathcal{K}$ then α is not an eigenvalue for F^t.

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Really there exists $s := s_{\mathcal{K}}$ and need $\begin{pmatrix} sL & 0 \\ 0 & \frac{1}{sL} \end{pmatrix} Y \in \mathcal{K}$ and $\begin{pmatrix} \frac{L}{s} & 0 \\ 0 & \frac{s}{L} \end{pmatrix} Y \in \mathcal{K}$ as well.

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$$\lambda(\{\theta : \exists t_\theta < \log(N) \text{ so that } \|RV(t_\theta, r_\theta Y)\vec{v}\| > N^\sigma \|v\| \text{ and } g_{t_\theta} r_\theta Y \in \mathcal{K}\}) < N^{-\rho}.$$

$\vec{v} = \alpha \vec{r}_k - \vec{n}$.

Iterating this for $N_1 = \frac{1}{\|\vec{v}\|}$, $N_2 = \frac{1}{\|RV(t_\theta, r_\theta Y)\vec{v}\|}, \ldots$ we obtain Veech's criterion.

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Proof of large deviations estimate

Proposition

(C-Eskin Lie) For any $\epsilon > 0$ *there exists* L *and* U *an open set with* $\mu_Y(U) > 1 - \epsilon$ *such that if* $Y \in U$ *and* $\vec{v}$ *is any vector then for all but an* ϵ *measure set of* θ *we have*
$(\lambda_1 - \epsilon)^L < |RV(g_L, r_\theta Y)\vec{v}| < (\lambda_1 + \epsilon)^L$.

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Because g_t expands circles, one can show that the conditional probability that $\frac{|RV(g_{t+L}, r_\theta Y)\vec{v}|}{|RV(g_t, r_\theta)\vec{v}|} < (\lambda_1-\epsilon)^L$ given $RV(g_t, r_\theta Y)$ and that $g_t r_\theta Y \in U$ is at most $C\epsilon$.

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If the measure of θ so that

$$\sum_{i=0}^{M} \chi_U(g_{Li} r_\theta Y) > M - CM\epsilon$$

we have the key estimate.

Proof of large deviations estimate

To prove this result we results of Eskin-Mirzakhani-Mohammadi:

Theorem

(Eskin-Mirzakhani-Mohammadi) We say Y is T, ϵ bad if

$$|\frac{1}{T\sigma}\int_0^T \int_0^\sigma \chi_U(g_t r_\theta Y) d\theta dt - \mu_Y(U)| > \epsilon.$$

The T, ϵ bad set is contained in the union of neighborhoods of finitely many affine ($SL_2(\mathbb{R})$-invariant) submanifolds. Moreover for fixed ϵ, σ the μ_Y-measure of these neighborhoods goes to zero as T goes to infinity.

Theorem

(Eskin-Mirzakhani-Mohammadi) Let $\mathcal{M}$ be any affine submanifold contained in supp(μ). Then there exists an SO_2 invariant function f, constants $c, b, \sigma, t_0 \in \mathbb{R}, c < 1$ such that

1. $f(x) = \infty$ *iff* $x \in \mathcal{M}$. *Also f is bounded on compact subsets of* $\mathcal{H}_1(\alpha) \setminus \mathcal{M}$. *Also* $\overline{\{x : f(x) \leq N\}}$ *is compact for any* N.
2. $\frac{1}{2\pi}\int_0^{2\pi} f(g_t r_\theta x) d\theta \leq cf(x) + b$ *for all* $t > t_0$.
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We now state an anachronistic corollary:

Corollary

(Athreya) For almost every θ and all large enough T the set of i such that $g_{iT} r_\theta Y$ is in the T, ϵ bad set has upper density at most ϵ.

Using this corollary, our first theorem of Eskin-Mirzakhani-Mohammadi and the expansion of circles by g_t we obtain that for all by an exponentially small in M set of θ, there exists C so that

$$\sum_{i=0}^{M} \chi_U(g_{Li} r_\theta Y) > M - CM\epsilon.$$

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C is independent of ϵ.