Consecutive sums of two squares in arithmetic progressions.

(joint with Noam Kimmel)

$a \equiv \square + \square \bmod q$

Throughout, fix q modulus and assume $(a,q)=1$.

Q: Are there infinitely many ~~primes~~ $\square + \square$ $p \equiv a \bmod q$?

A: Yes. ~~(Dirichlet)~~ Landau, Iwaniec

Stronger: equidistribution

$$\cancel{\pi}(x;a,q) := \#\{\cancel{p}\,E_n \le x : \cancel{p}\,E_n \equiv a \bmod q\}$$

$$\sigma(x;a,q)$$

$$\sim \frac{\pi(x)}{\varphi(q)} \qquad \asymp_{a,q} \sigma(x)$$

Q: $\cancel{\pi}\,\sigma(x;(a_1,a_2),q) := \#\{p \le x : p \equiv a_1 \bmod q,\ p_{next} \equiv a_2 \bmod q\}$

$\cancel{\pi}\,\sigma(x;(a_1,a_2),q) \longrightarrow \infty$ as $x \to \infty$?

Q: $\cancel{\pi}\,\sigma(x;(a_1,\dots,a_r),q) := \#\{p_n \le x : p_{n+i-1} \equiv a_i \bmod q \ \ \forall\, 1 \le i \le r\}$

$\longrightarrow \infty$?

Conj: $\sim \dfrac{\pi(x)}{\varphi(q)^r}$. $\qquad \asymp_{\vec{a},q} \sigma(x)$

Conj: Due to second-order terms, repeated values are less common.

Easy Lemma: If $\varphi(q)=2$ and $a_1 \not\equiv a_2 \bmod q$, then $\pi(x;(a_1,a_2),q) \longrightarrow \infty$.

Thm (Shiu, Banks-Freiberg-Turnage-Butterbaugh): $\pi(x;(a,\dots,a),q) \longrightarrow \infty$ for any length.

Thm (Maynard): $\pi(x;(a,\dots,a),q) \gg \pi(x)$.

Open Qn: Show $\pi(x;\vec{a},q) \longrightarrow \infty$ in any other case.

Sums of two squares:

$$\mathbb{E} = 1, 2, 4, 5, 8, 9, 10, 13, \dots$$
$$= \{z \in \mathbb{N} : z = x^2+y^2,\ x,y \in \mathbb{N}\}$$

Thm (Fermat): $n \in \mathbb{E} \iff n = \prod_p p^{\nu_p}$, ν_p even whenever $p \equiv 3 \bmod 4$.

$$\mathbb{E} = (E_n) \text{ where } E_n < E_{n+1}$$

Thm (Kimmel, K.): $\forall\, a,b,c \bmod q$,
$$\sigma(x;(a,b,c),q) \longrightarrow \infty \quad \text{as} \quad x \longrightarrow \infty.$$

Thm (Kimmel, K.): $\forall\, a,b \bmod q$,
$$\sigma(x;(a,\dots,a,b,\dots,b),q) \longrightarrow \infty$$ for any lengths of a's and b's.

In progress: $\sigma(x;(a,\dots,a,b,\dots,b);q) \gg \sigma(x)$.

For triples:

Thm (Hooley): $\forall\, h,k \in \mathbb{N}$,

$$\sum_{n\in\mathbb{N}} \mathbb{1}_E(n)\,\mathbb{1}_E(n+h)\,\mathbb{1}_E(n+k) \longrightarrow \infty.$$

Note: techniques from quadratic forms are very helpful for sums of two squares.

Eg: inf'ly many n w/ $n-1, n, n+1 \in E$.

Pf: If $n-1, n, n+1$ is an example

$$\underset{(n-1)(n+1)}{\overset{\parallel}{n^2-1}},\ n^2,\ n^2+1 \quad \text{is another}$$

$8, 9, 10$ works.

Spoiler: We can estimate certain weighted correlations of $\square+\square$.

Back to primes:

Ideas of proof (BFT/M):

Maynard: For $m\in\mathbb{N}$, let k be big enough. For each "admissible" k-tuple

$L_1(n) = qn + a_1, \dots, L_k(n) = qn + a_k$,

$\exists$ ∞'ly many n s.t. at least m of the values $L_1(n), \dots, L_k(n)$ are all prime.

Trick to get BFT:

① Choose $\{L_i(n) = qn + a_i\}$ admissible, with $a_i \equiv a \bmod q$

and $a_1 < \cdots < a_k$.

② Define

$$S = \{t \in \mathbb{N} : t \neq a_i \ \forall i,\ a_1 < t < a_k\}$$

$\mathcal{P} = \{q_t : t \in S\}$ of distinct primes $q_t \neq q$

s.t. $t \not\equiv a_i \bmod q$.

$$Q = \prod_t q_t.$$

③ CRT: $\exists$ $A \bmod Q$ s.t.

$$qA + t \equiv 0 \bmod q_t \quad \forall t.$$

④ $\{\tilde{L}_i(n) = qQn + qA + a_i\}$ is admissible

and $qQn + qA + t$ is never prime for $t \in S$.

$\rightsquigarrow$ Output of Maynard's thm for $\tilde{L}_i(n)$ are consecutive.

D + D: $q_t \equiv 3 \bmod 4$

$Q = \prod_t q_t^2$

$qA + t \equiv q_t \bmod q_t^2$.

$\equiv a \quad \equiv a \quad \equiv a \quad \equiv a \quad \equiv a$

$\equiv a \quad \equiv a \quad \equiv a$

$\equiv a \quad \equiv b \quad \equiv a \quad \equiv b \quad \equiv a$

$\equiv a \quad \equiv a$

ask for a pink prime and a red prime.

Idea: Divide the tuple into baskets and look for a prime in each basket.

Dream:

$\underbrace{\text{basket} \equiv a \bmod q}$ $\underbrace{\text{basket} \equiv b \bmod q}$

$L_1(n) \cdots\cdots \quad L_k(n) \quad L_{k+1}(n) \cdots \quad L_{2k}(n)$

inf'ly often $\exists$ a prime in each basket.

How to find primes in baskets:

Maynard's original idea:

$$S := \sum_{n \sim N} \underbrace{\left(\sum_{i=1}^{k} \mathbb{1}_p(L_i(n)) - 1 \right)}_{>0 \text{ for some } n \sim N} w(n)$$

Find $w(n) \geq 0$ s.t. $S > 0$.

$\Rightarrow \exists n$ s.t. $\sum_{i=1}^{k} \mathbb{1}_p(L_i(n)) > 1$.

Need to estimate $\sum_n w(n)$, $\sum_n \mathbb{1}_p(L_i(n))\, w(n)$.

2^{nd} moment argument: Divide a Bk-tuple into B equal baskets.

$$S' = \sum_{n \sim N} \left(\sum_{i=1}^{Bk} \mathbb{1}_p(L_i(n)) - 1 - \sum_{\ell=1}^{B} \sum_{i,j \in B_\ell} \mathbb{1}_p(L_i(n))\, \mathbb{1}_p(L_j(n)) \right) w(n)$$

> 0

for some $w(n) \geq 0$.

Here we also need to understand

$$\sum_{n\sim N} \mathbb{1}_p(L_i(n))\,\mathbb{1}_p(L_j(n))\,w(n)$$

upper bounds lose a factor of 4 (or ≈ 4)

<u>Banks – Freiberg – Maynard</u>: One can find primes in two different baskets if you start with ≥ 5 baskets.

<u>Merikoski</u>: ≥ 4

<u>McGrath</u> (2020): For $\square + \square$, you can divide a tuple into 8 equal baskets and find inf'ly often a $\square + \square$ in each basket.