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Tuesday, October 25, 2022 9:38 PM

Pointwise Ergodic Theorem along Subsequence

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Notation

Let (X, Σ, μ) be a non-atomic probability space, and T be a measure-preserving transformation on (X, Σ, μ). We will call the quadruple (X, Σ, μ, T) a dynamical system.

We define the Cesàro averages along a subsequence (a_n) of integers as follows:

$$A_{n\in[N]} f(T^{a_n}x) = \frac{1}{N} \sum_{n\in[N]} f(T^{a_n}x). \tag{0.1}$$

Pointwise Ergodic Theorem: If $(a_n) = (n)$, $f \in L^1$ then $A_{n\in[N]} f(T^n x)$ converges a.e.

Question: Is it possible to generalize the Pointwise Ergodic Theorem along any sequence (a_n)?

Known Results

- Krengel(1971) proved that $\exists$ an increasing sequence (a_n) such that in any aperiodic system (X, Σ, μ, T), there exists $f \in L^1$ such that $A_{n\in[N]} f(T^{a_n}x)$ fail to converge for almost every x.
- Bellow showed in 1983 that the sequence (a_n) can be taken to be any *lacunary sequence*.
- Bourgain proved in 1988 that for any system (X, Σ, μ, T) and for any function $f \in L^2$, the averages along (n^2) i.e. $A_{n\in[N]} f(T^{n^2}x)$ converges a.e.

The following theorem is from (Jones and Wierdl 1994).

0.1 Theorem ▶ Jones-Wierdl

If a sequence (a_n) satisfies $\frac{a_{n+1}}{a_n} \geq 1 + \frac{1}{(\log n)^{\frac{1}{2}-\epsilon}}$ for some $\epsilon > 0$, then in any aperiodic system (X, Σ, μ, T), we can find a function $f \in L^2$ such that the averages $A_{n\in[N]} f(T^{a_n}x)$ fail to converge a.e.

Example: An example of such sequence is $2^{\frac{n}{(\log n)^{1/2-\epsilon}}}$.

0.2 Theorem ▶ Bourgain

There exists a sequence (a_n) satisfying $\frac{a_{n+1}}{a_n} \geq 1 + \frac{1}{(\log n)^{1+\epsilon}}$ for some $\epsilon > 0$ such that in any system (X, Σ, μ, T), the averages $A_{n\in[N]} f(T^{a_n}x)$ converge a.e. for every $f \in L^2$.

Open Problems

0.3 Problem ▶ Problem 1

Which of the above bound is sharp?

0.4 Problem ▶ Problem 2

Is it possible to improve Bourgain's range for L^p function when $p > 2$?

Idea of the proof

Given $\frac{a_{n+1}}{a_n} \geq 1 + \frac{1}{(\log n)^{\frac{1}{2}-\epsilon}}$. We want to show that $A_{n\in[N]} f(x + ra_n)$ fail to converge a.e. for some $f \in L^2$.

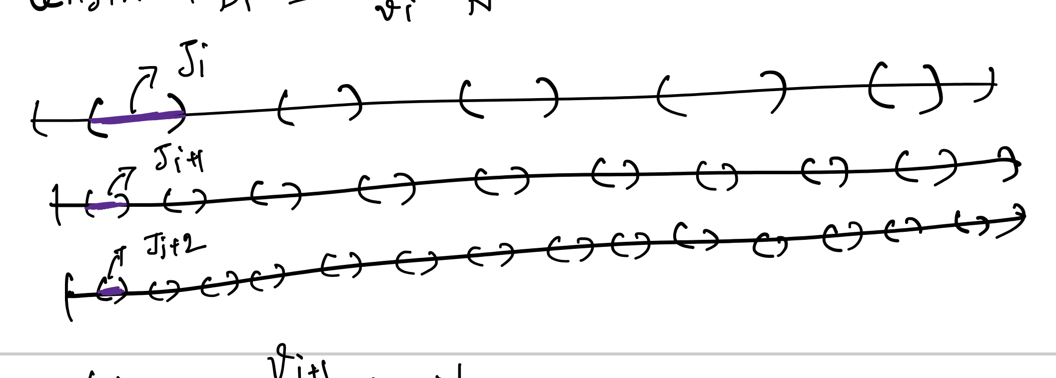
0.5 Lemma

Suppose $(v_1, v_2, \ldots, v_m) \in \mathbb{R}^m$ satisfies $\frac{v_{i+1}}{v_i} \geq 2N$ for $i \in [m-1]$. Let $1 \leq e_1, e_2, \ldots, e_m \leq N$. Then we can find an irrational number r such that $rv_i (\text{mod } 1) \in (\frac{e_i - 1}{N}, \frac{e_i}{N})$ $\forall i \in [m]$.

proof: Define $B_i := \left\{ r : rv_i \in \left(\frac{e_i-1}{N}, \frac{e_i}{N}\right) \right\}$

$$= \left(\frac{1}{v_i}\cdot\frac{e_i-1}{N}, \frac{1}{v_i}\cdot\frac{e_i}{N}\right) \cup \frac{1}{v_i} + \left(\frac{1}{v_i}\cdot\frac{e_i-1}{N}, \frac{1}{v_i}\cdot\frac{e_i}{N}\right) \cup \frac{2}{v_i} + \left(\frac{1}{v_i}\cdot\frac{e_i-1}{N}, \frac{1}{v_i}\cdot\frac{e_i}{N}\right) \cup \cdots$$

Length of $B_i = \frac{1}{v_i}\cdot\frac{1}{N}$. Period of $B_i = \frac{1}{v_i}$.



Given $\frac{v_{i+1}}{v_i} \geq 2N$

$\Rightarrow \frac{1}{v_i}\cdot\frac{1}{N} \geq \frac{2}{v_{i+1}}$

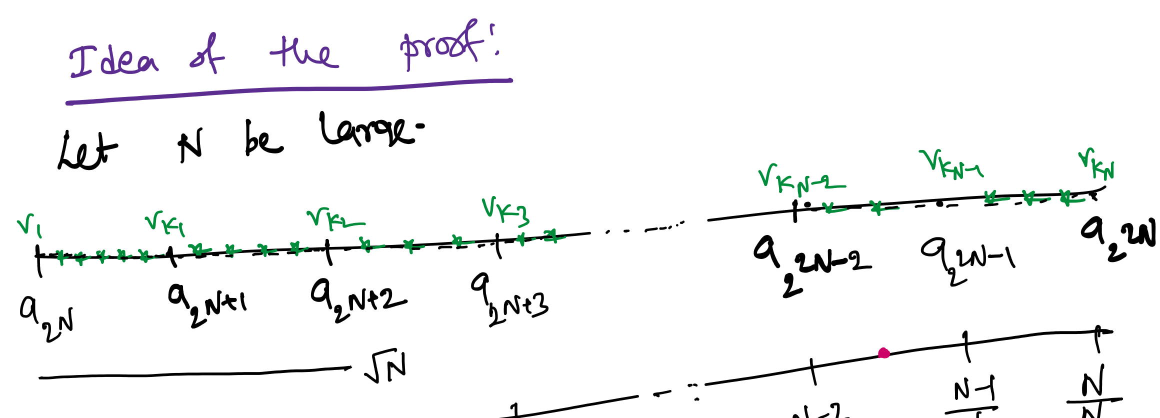
$\Rightarrow$ length of $J_i \geq 2\cdot$ Period of B_{i+1}

$\Rightarrow$ J_i must contain J_{i+1} fully. $\forall i \in [m-1]$.

Hence $\exists$ J_m s.t. $J_m \subseteq \bigcap_{i\in[m]} B_i$. ∎

Idea of the proof:

Let N be large.



(v_i) is a thin subsequence of (a_i) s.t. $\frac{v_{i+1}}{v_i} \geq 2N$.

More precisely, $v_1 = a_{2^N}$, $v_2 = a_{2^N+t}$, $v_3 = a_{2^N+2t}$, $v_4 = a_{2^N+3t}$ . . . . .

where $t \approx N^{1/2-\varepsilon}$.

Now by applying the above lemma, choose r s.t.

$rv_i \in [0, \frac{1}{N}]$ $\forall$ $1 \leq i \leq K_1$

$rv_i \in [\frac{1}{N}, \frac{2}{N}]$ $\forall$ $K_1 < i \leq K_2$

$\vdots$

$rv_i \in [\frac{N-1}{N}, 1]$ $\forall$ $K_{N-1} < i \leq K_N$.

Now define the function f as follows:

$f(x) = \sqrt{N}\, 1_{[0,\frac{2}{N}]}(x)$. $[\Rightarrow \|f\|_2^2 = 2]$

Let x be an arbitrary point in $\mathbb{T}$.

Suppose $x \in [\frac{N-2}{N}, \frac{N-1}{N}]$.

Observe that $\forall y \in [\frac{2}{N}, \frac{3}{N}]$, $x+y \in [0, \frac{2}{N}]$.

Hence,

$$\frac{1}{2^{N+3}} \sum_{n\in[2^{N+3}]} f(x+ra_n)$$

$$\geq \frac{1}{2^{N+3}} \sum_{n\in[2^{N+2}, 2^{N+3}]} f(x+ra_n)$$

$$\geq \frac{1}{2^{N+3}} \sum_{i\in[K_2,K_3]} f(x+rv_i)$$

$$= \frac{1}{2^{N+3}}\cdot\sqrt{N}\cdot(K_3-K_1)$$

$$= \frac{1}{2^{N+3}}\cdot\sqrt{N}\cdot\frac{2^{N+2}}{t} \quad (\text{Since } v_i \text{ has density } \tfrac{1}{t})$$

$$\simeq \frac{1}{2}\cdot\frac{N^{1/2}}{N^{1/2-\varepsilon}} = \frac{1}{2}N^{\varepsilon}$$

Thus we are able to construct a function f, $\|f\|_2 = \sqrt{2}$ s.t. $\sup_N A_{n\in[N]} f(x+ra_n) \geq N^{\varepsilon}$ for all $x \in \mathbb{T}$.

Since, N is arbitrary, the averages cannot satisfy a weak (2-2) maximal inequality. This, by Sawyer's theorem, finishes the proof.