An Elliptic Curve Prime Race Problem

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joint with Yifeng Huang and Chi Hoi (Kyle) Yip

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Fix prime $\ell \neq p$, the undergraduates plot some graphs on the race of a_p for $a_p \equiv r_1 \mod \ell$ and $a_p \equiv r_{-1} \mod \ell$, for some $0 \leq r_1, r_{-1} \leq \ell - 1$.

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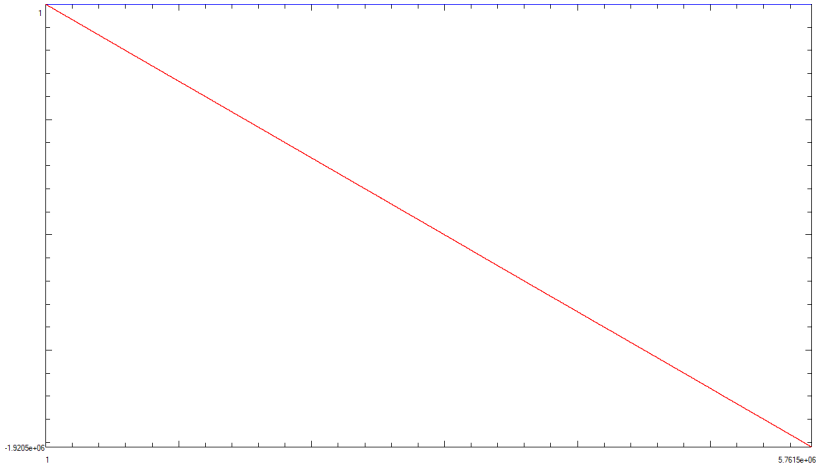
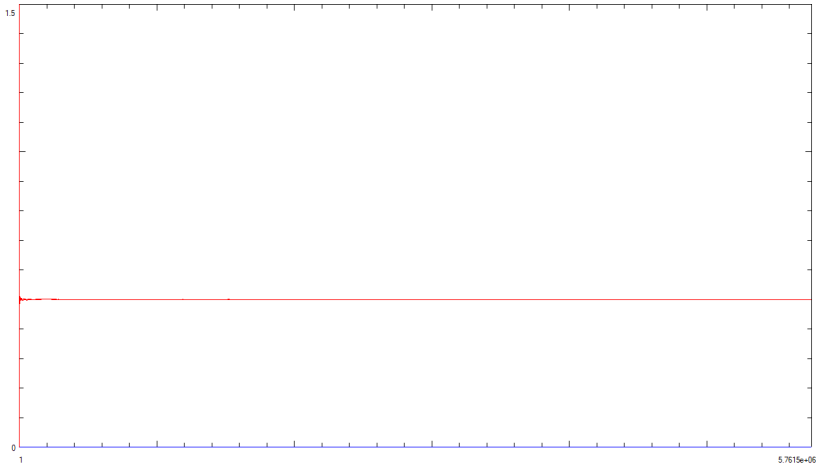
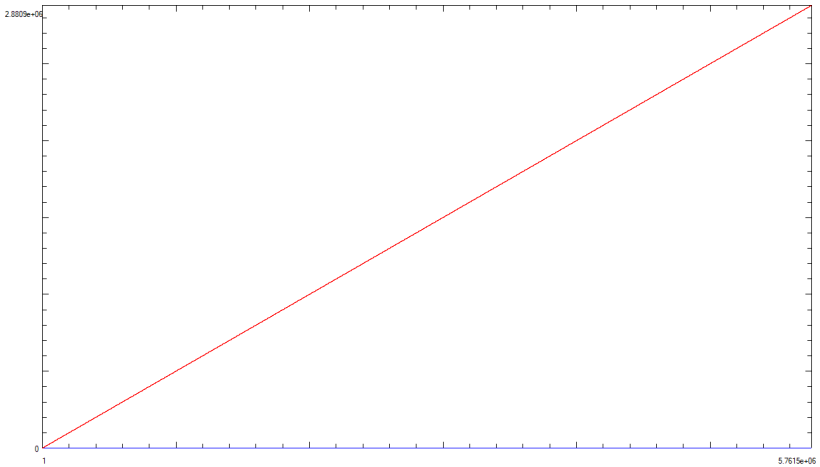
Graphs of elliptic curves LMFDB label 37.b1 (credits to Tighe McAsey) on next slide.

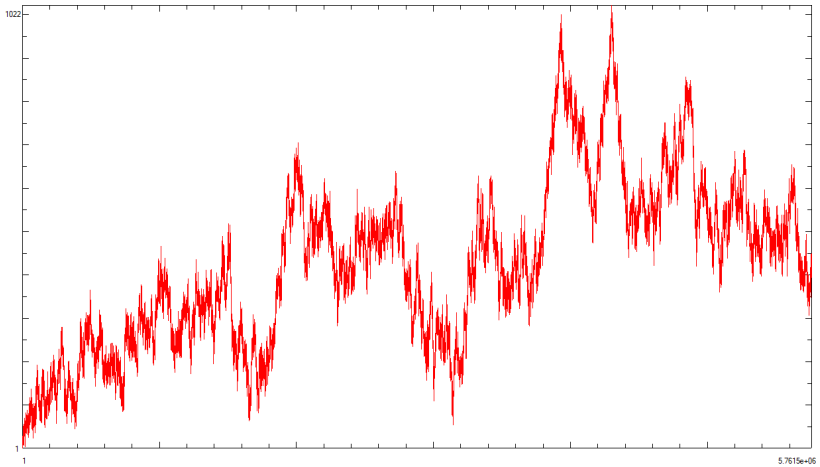
$$change = \begin{cases} +1 & \text{if } a_p \equiv r_1 \pmod{\ell} \\ -1 & \text{if } a_p \equiv r_{-1} \pmod{\ell} \end{cases}$$





LHS: $\ell = 2$, $r_1 = 1$, $r_{-1} = 0$; RHS: ratio





LHS: $\ell = 3$, $r_1 = 0$, $r_{-1} = 1$; RHS $\ell = 3$, $r_1 = 0$, $r_{-1} = 2$

Natural question: Explain the slope of the graph.

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Fact 1

Let p be a prime of good reduction for $E/\mathbb{Q}$, $\ell \neq p$ be prime and $E[\ell]$ be the ℓ-torsion. We have the representation

$$\overline{\rho}_\ell : \mathrm{Gal}(\mathbb{Q}(E[\ell])/\mathbb{Q}) \to \mathrm{Aut}(E[\ell]) \cong \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z}).$$

Then

$$\mathrm{tr}(\overline{\rho}_\ell(\mathrm{Frob}_p)) = a_p(E) \pmod{\ell}$$

where Frob_p is the Frobenius substitution.

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Takeaway

By Chebotarev density theorem, the distribution of $a_p(E) \mod \ell$ is governed by the distribution of Frob_p in $\mathrm{Gal}(\mathbb{Q}(E[\ell])/\mathbb{Q})$ and the trace map $\mathrm{Gal}(\mathbb{Q}(E[\ell])/\mathbb{Q}) \to \mathbb{Z}/\ell\mathbb{Z}$.

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Let $E/\mathbb{Q}$ be an elliptic curve in the reduced Weierstrass form

$$y^2 = x^3 + ax + b = f(x),$$

where $a, b \in \mathbb{Q}$, with discriminant $\Delta = -(4a^3 + 27b^2)$ and α_1, α_2 and α_3 be the roots of $f(x)$. Let

$$S_0 = \{p \text{ prime} : a_p(E) \equiv 0 \mod 2\}$$
$$S_1 = \{p \text{ prime} : a_p(E) \equiv 1 \mod 2\}.$$

Let $L = \mathbb{Q}(E[2])$, $G = \mathrm{Gal}(L/\mathbb{Q})$ and δ be any density of set of primes (say natural density).

Case 1: Suppose $L = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ with $[L : \mathbb{Q}] = 6$. Then $G \cong S_3$. Up to conjugacy, we have

$$\begin{array}{rcl} \mathrm{Gal}(L/\mathbb{Q}) & \longrightarrow & \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{class size} \\ (1)(2)(3) & \longmapsto & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\mathrm{tr}} 0 \mod 2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \\ (12)(3) & \longmapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\mathrm{tr}} 0 \mod 2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3 \\ (123) & \longmapsto & \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{\mathrm{tr}} 1 \mod 2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2 \end{array}$$

Hence, $\delta(S_0) = \frac{1+3}{6} = \frac{2}{3}$.

Case 2: $f(x)$ splits completely in $\mathbb{Q}$, i.e. $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}$. Then $L = \mathbb{Q}$ and G is trivial. Hence, $\delta(S_0) = 1$.

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Case 3: $\alpha_1 \in \mathbb{Q}$ but $\alpha_2, \alpha_3 \notin \mathbb{Q}$. Then $L = \mathbb{Q}(\alpha_2)$ and $G \cong C_2$. Hence, $\delta(S_0) = 1$.

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Case 3: $\alpha_1 \in \mathbb{Q}$ but $\alpha_2, \alpha_3 \notin \mathbb{Q}$. Then $L = \mathbb{Q}(\alpha_2)$ and $G \cong C_2$. Hence, $\delta(S_0) = 1$.
Case 4: Suppose $L = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ with $[L : \mathbb{Q}] = 3$. Then $G \cong C_3$. Fix a basis $\{P_1, P_2\}$ for $E[2]$ and one can see $P_3 = P_1 + P_2$. We have

$$\begin{aligned} \mathrm{Gal}(L/\mathbb{Q}) &\longrightarrow \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \\ id &\longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{tr}} 0 \mod 2 \\ (123) &\longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{tr}} 1 \mod 2 \\ (132) &\longmapsto \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\text{tr}} 1 \mod 2 \end{aligned}$$

Hence, $\delta(S_0) = \frac{1}{3}$.

Theorem 2

Let $E/\mathbb{Q}$ be an elliptic curve in the reduced Weierstrass form

$$y^2 = x^3 + ax + b,$$

where $a, b \in \mathbb{Q}$, with discriminant $\Delta = -(4a^3 + 27b^2)$. Let $f(x) = x^3 + ax + b$ and α_1, α_2 and α_3 be the roots of $f(x)$. Let $S_0 = \{p \text{ prime} : a_p(E) \equiv 0 \mod 2\}$ and $S_1 = \{p \text{ prime} : a_p(E) \equiv 1 \mod 2\}$. Let δ be any density of set of primes (say natural density). We have the following cases:

1. *If f is reducible over $\mathbb{Q}$, then $\delta(S_0) = 1$ and $\delta(S_1) = 0$.*
2. *If f is irreducible over $\mathbb{Q}$ and $\sqrt{\Delta} \in \mathbb{Q}$, then $\delta(S_0) = 1/3$ and $\delta(S_1) = 2/3$.*
3. *If f is irreducible over $\mathbb{Q}$ and $\sqrt{\Delta} \notin \mathbb{Q}$, then $\delta(S_0) = 2/3$ and $\delta(S_1) = 1/3$.*

Further question to be asked

Classical prime number race: the prime race between 1 mod 4 and 3 mod 4

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Our question: After suitable weightings, the a_p race between r_1 mod ℓ and r_{-1} mod ℓ.

Warning: In literature, prime race of an elliptic curve refers to the a_p race between positives and negatives.

Framework

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Let L/K be a normal extension of number fields. Let $G = \mathrm{Gal}(L/K)$ be the corresponding Galois group and let C be a conjugacy class of G. Let $\mathfrak{p}$ be a prime of K which is unramified in L. Define the Frobenius substitution attached to $\mathfrak{p}$ be the conjugacy class

$$\mathrm{Frob}_{\mathfrak{p}} = \{\mathrm{Frob}_{\mathfrak{P}} : \mathfrak{P} \text{ lying over } \mathfrak{p}\}.$$

Define the prime counting functions

$$\pi(x; K) = \#\{N\mathfrak{p} \leq x : \mathfrak{p} \text{ (unramified) prime of } K\}$$
$$\pi(x; C) = \#\{N\mathfrak{p} \leq x : \mathrm{Frob}_{\mathfrak{p}} = C \text{ and } \mathfrak{p} \text{ (unramified) prime of } K\}$$

(We follow the treatment in Ng's thesis) Assume GRH throughout.
Notation:

$$E(x; C) := \frac{\log x}{\sqrt{x}} \left(\frac{|G|}{|C|} \pi(x; C) - \pi(x; K) \right)$$

$$\psi(x, \chi) := \sum_{N\mathfrak{p}^m \leq x,\ \mathfrak{p} \text{ unramified}} \chi(\mathrm{Frob}_{\mathfrak{p}}^m) \log(N\mathfrak{p})$$

$$sq^{-1}(C) := \bigcup_{i=1}^{t} C_i, \quad \text{where } C_i^2 \subset C$$

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Lemma 3 (Ng, 2000)

$$E(x; C) = \left(1 - \frac{|sq^{-1}(C)|}{|C|}\right) - \sum_{\chi \neq \chi_0} \overline{\chi(C_1)} \frac{\psi(x, \chi)}{\sqrt{x}} + O\left(\frac{1}{x}\right)$$

To race between two conjugacy classes,

$$\begin{aligned} E_{C_1-C_2}(x) :=& E(x; C_1) - E(x; C_2) \\ =& c(C_2) - c(C_1) - \sum_{\chi \neq \chi_0} (\overline{\chi(C_1)} - \overline{\chi(C_2)}) \sum_{0<|\gamma_\chi| \leq X} \frac{x^{i\gamma_\chi}}{\frac{1}{2} + i\gamma_\chi} \\ &+ O\left(\frac{\sqrt{x}\log^2 X}{X} + \frac{1}{\log x}\right) \end{aligned} \tag{1}$$

where $c(C_i) = c_{sq}(C_i) + c_{\frac{1}{2}}(C_i)$ with

$$c_{sq}(C) = -1 + \frac{|sq^{-1}(C)|}{|C|}$$

$$c_{\frac{1}{2}}(C) = 2\sum_{\chi \neq 1} \overline{\chi}(C)\eta_\chi$$

and $\eta_\chi = \mathrm{ord}_{s=\frac{1}{2}} L(s, \chi)$.

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and $\eta_\chi = \mathrm{ord}_{s=\frac{1}{2}} L(s, \chi)$.
The term $c(C_i)$ is the bias factor. There is bias factor c_{sq} in classical Chebyshev bias but not $c_{\frac{1}{2}}$.

S_3 Mod 2 Example

We work with the case $\mathrm{Gal}(L/K) = S_3$.
Let $G = \mathrm{Gal}(L/\mathbb{Q}) = S_3$. There are three conjugacy classes in this group, namely

$$C_1 = \{(1)\}, \quad C_2 = \{(12), (13), (23)\}, \quad \text{and} \quad C_3 = \{(123), (132)\}.$$

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Straight forward computation:

$$C_1^2 = C_1, \quad C_2^2 = C_1 \quad \text{and} \quad C_3^2 = C_3$$

and hence

$$-1 + \frac{|sq^{-1}(C_1)|}{|C_1|} = 3; \quad -1 + \frac{|sq^{-1}(C_2)|}{|C_2|} = -1,$$
$$-1 + \frac{|sq^{-1}(C_3)|}{|C_3|} = 0$$

This is not quite what we want!

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We can also calculate the bias terms with union of conjugacy classes. Let $C_{2,0} = C_1 \cup C_2$ and $C_{2,1} = C_3$. Then, one can calculate

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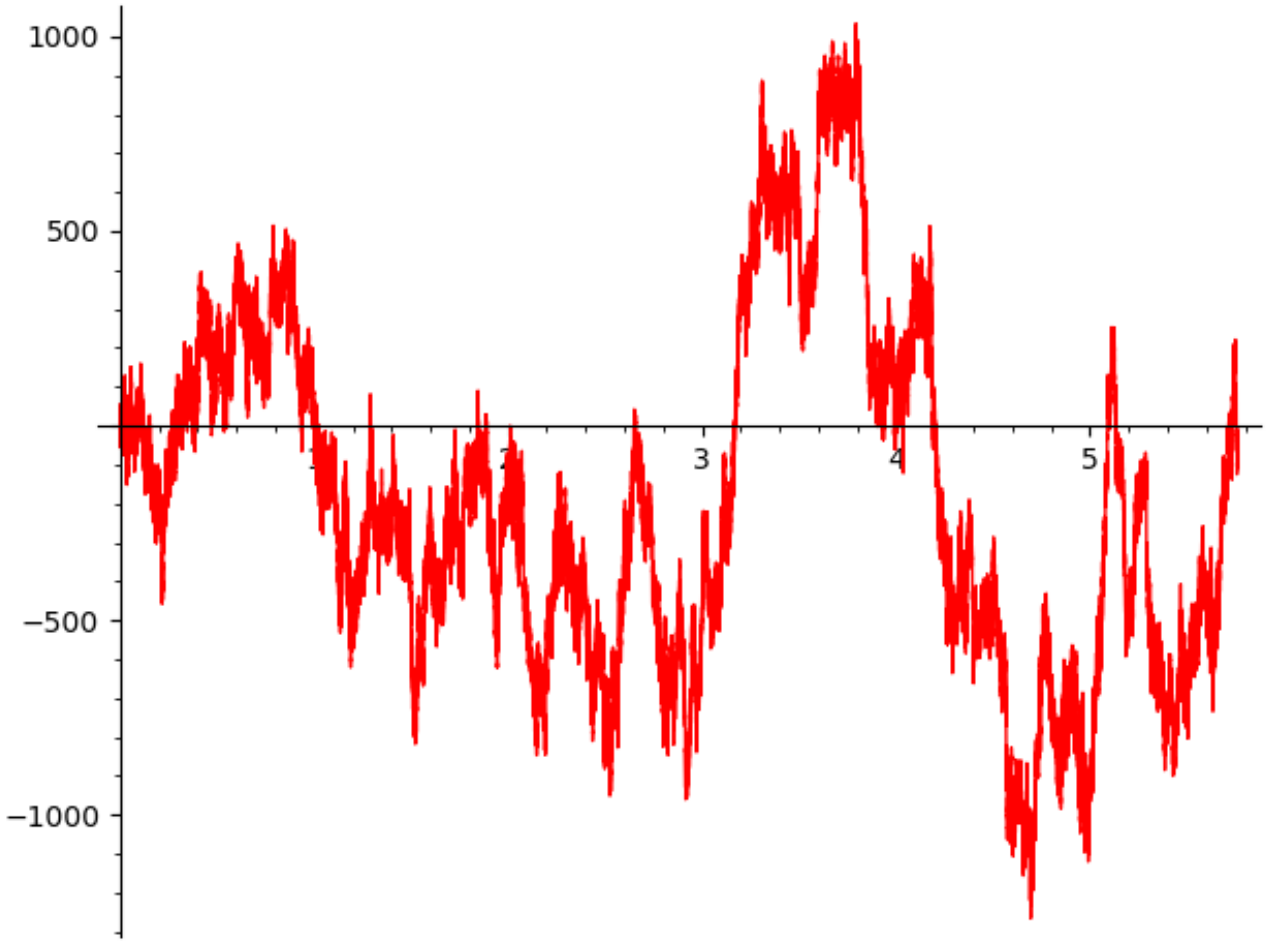
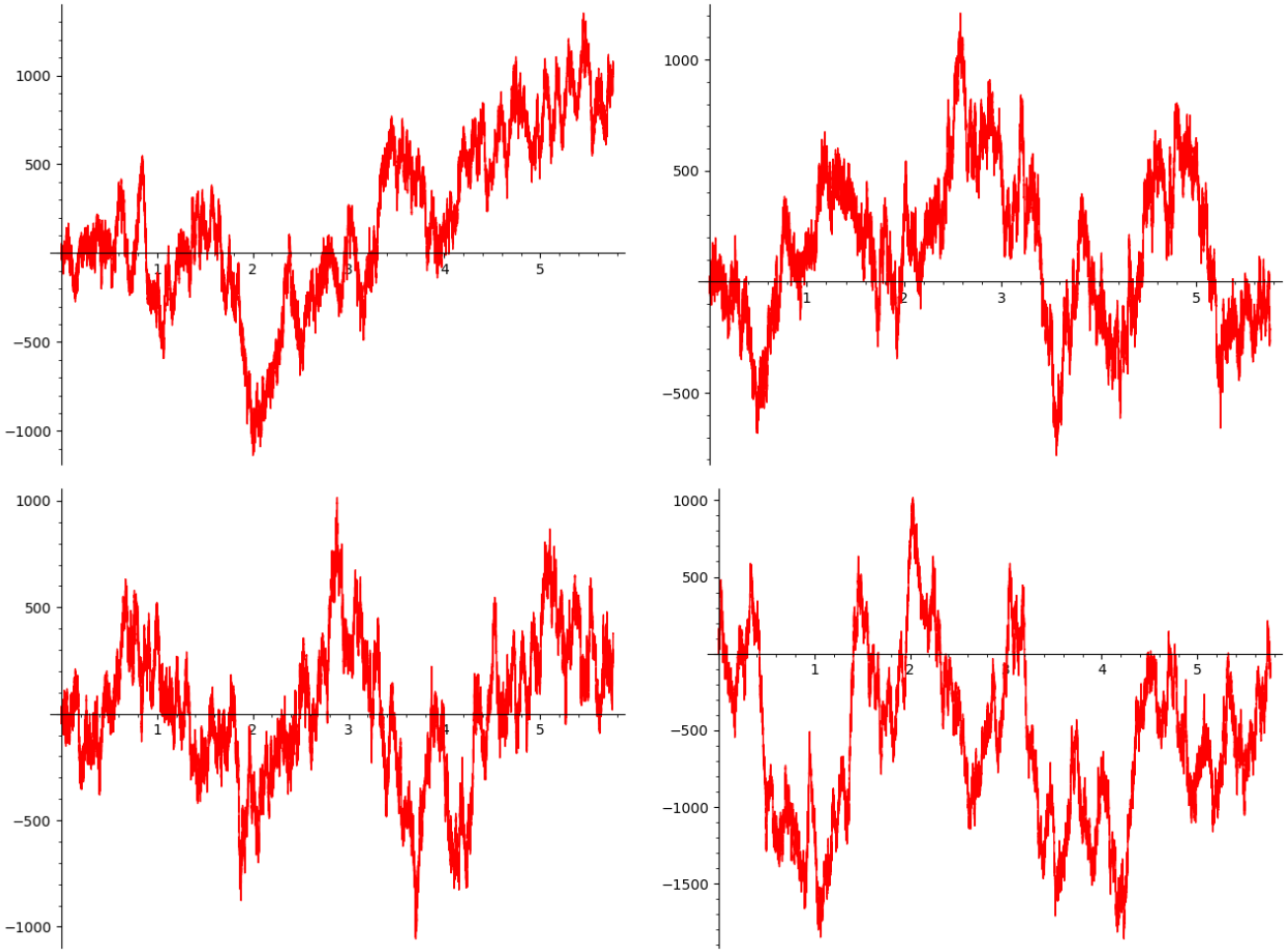
We then have to estimate bias from $c_{\frac{1}{2}}$ term. In S_3, we do not expect there is any bias from $c_{\frac{1}{2}}$.

37.b1 mod 2 race



LMFDB 37.b1; $\ell = 2$, $r_1 = 0$, $r_{-1} = 1$ with suitable weightings.

more mod 2 race



UL: 11.a1, UR: 57.a1, LL: 1862.a1, LR: 19047851.a1;
$\ell = 2$, $r_1 = 0$, $r_{-1} = 1$ with suitable weightings.

Story again

Story begins when we want to find more $G = S_3$ examples. It happens when we plot mod 3 for elliptic curves $E/\mathbb{Q}$, which has torsion subgroup $\mathbb{Z}/3\mathbb{Z}$.

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Not interesting: $a_p \equiv p + 1 \mod 3$. In fact, we will have a plot C_{+1} vs C_{-1}, where

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Question: Is it all such race between C_{+1} vs C_{-1} in S_3 is just prime race?

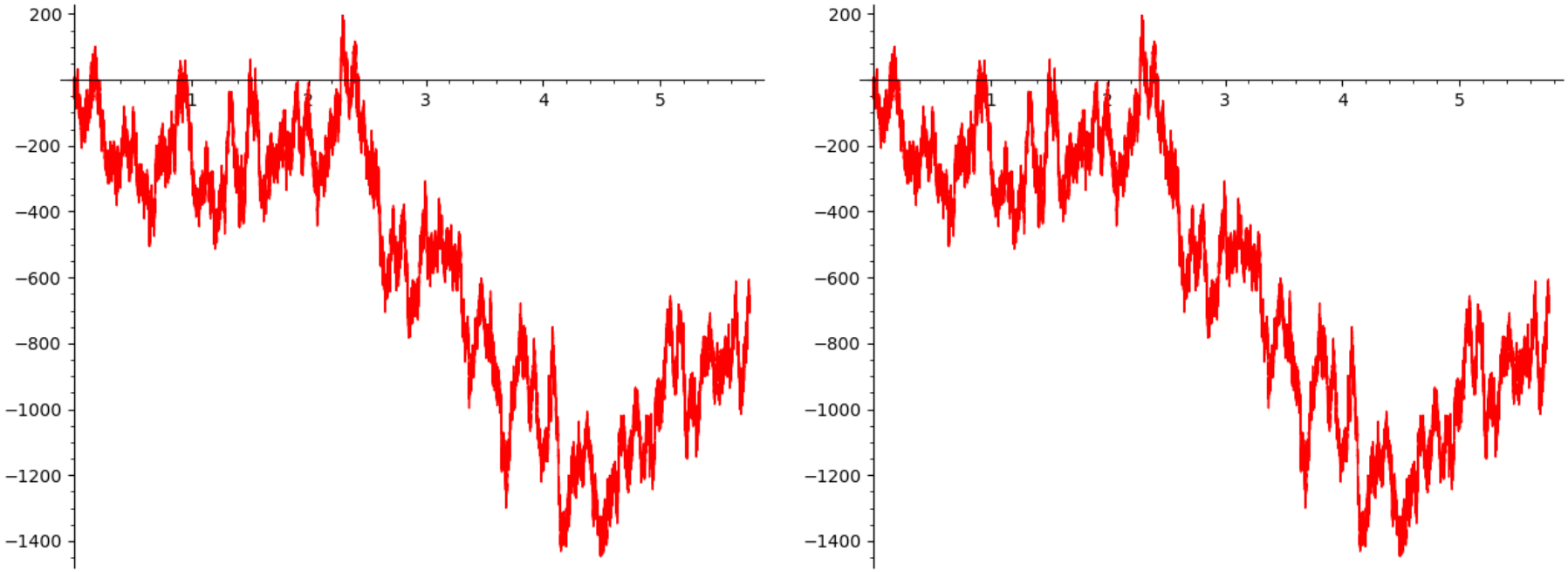
Theorem 4

Let $G = \mathrm{Gal}(L/\mathbb{Q}) = S_3$. Let d be the discriminant of any of the three cubic subfields of L over $\mathbb{Q}$. Denote

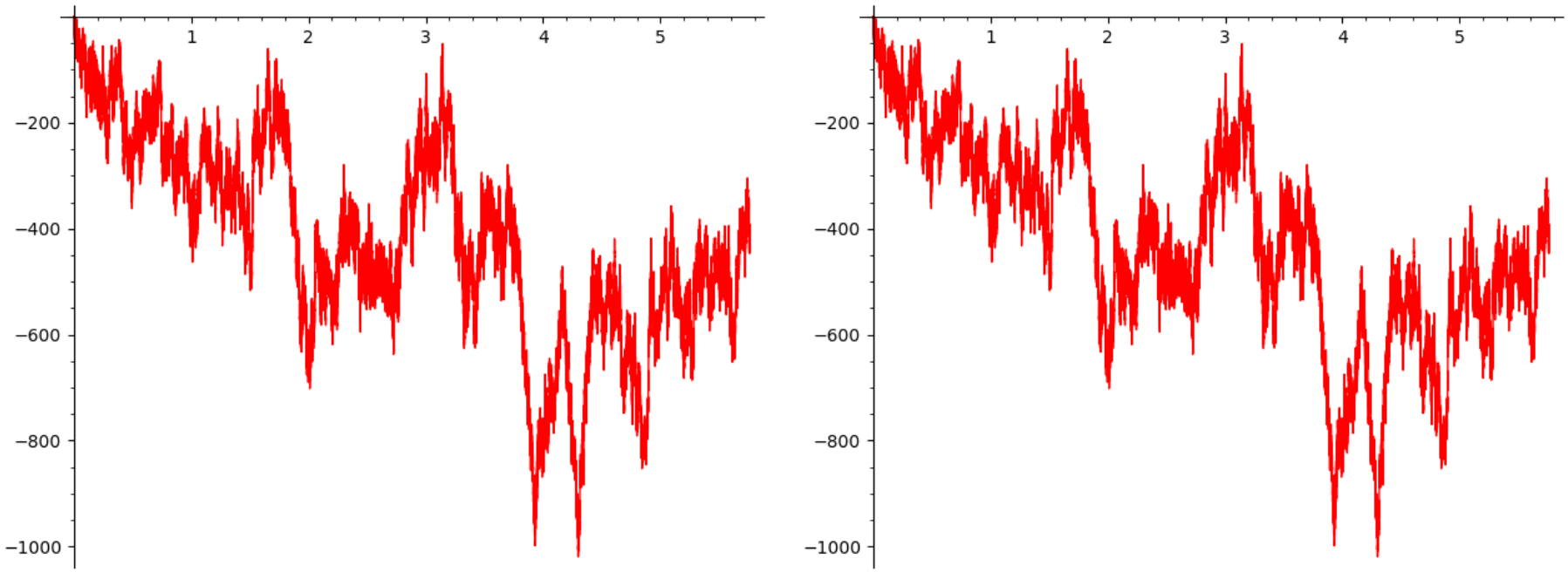
$$C_{+1} := C_1 \cup C_3 = \{(1), (123), (132)\}$$
$$C_{-1} := C_2 = \{(12), (13), (23)\}.$$

the union of conjugacy classes of S_3. The race between C_{+1} vs C_{-1} are equivalent to the prime race between quadratic residue vs quadratic non-residue modulo $|d|$.

plots



LHS: Galois race of defining polynomial $f(x) = x^3 - x^2 + 1$, $D = -23$;
RHS: residue vs non-residue prime mod 23.
They are the same.



LHS: Galois race of defining polynomial $f(x) = x^3 - x^2 - 3x - 3$, $D = -300$;
RHS: residue vs non-residue prime mod 300.
They are the same.

Thank you!