A nonsmooth approach to Einstein's theory of gravity

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In metric(-measure) geometry with positive signature, there are theories of

• sectional curvature bounds based on triangle comparison (Aleksandrov. . .)

• pointed Gromov-Hausdorff limits of manifolds under lower Ricci and upper dimensional bounds (Fukaya, Gromov, Cheeger-Colding, . . .)

• Ricci lower bounds via displacement convexity of entropy (Bakry-Emery, Lott-Sturm-Villani, Ambrosio-Gigli-Savare, . . .)

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Can something similar be done in Lorentzian geometry?

• tidal forces (Kunzinger-Sämann '18)

• convergence of spaces (Müeller 22+, Minguzzi-Suhr 22+)

• Einstein equation (M. 20, Mondino-Suhr 18+, Cavalletti-Mondino 20+, Braun 22+, ...)

Definition (Time-separation function)

On a set M of events, a *time-separation function* refers to $\ell : M \times M \longrightarrow \{-\infty\} \cup [0, \infty)$ satisfying the reverse triangle inequality and antisymmetry: $\forall x, y, z \in M$

$$\ell(x, y) \geq \ell(x, z) + \ell(z, y) \tag{1}$$

$$\min\{\ell(x, y), \ell(y, x)\} > -\infty \Leftrightarrow x = y. \tag{2}$$

Remark: (1) + (2) $\Rightarrow \ell(x, x) = 0$; (2) gives the arrow of time

Example (Minkowski space)

$M = R^{1,3}$ with $\ell(x, y) = L(y - x)$ where

$$L(v) = \begin{cases} |g(v, v)|^{1/2} & \text{if } v \text{ is future-directed} \\ -\infty & \text{else.} \end{cases}$$

Notice $L(v)$ is *concave* (as is $L(v)^q$ for any $0 < q \leq 1$ if $(-\infty)^q := -\infty$).

Example (Causal spaces)

A time-separation function induces $M^2_\leq = \{(x,y) \in M^2 \mid \ell(x,y) \geq 0\}$ a partial order and $M^2_\ll = \{(x,y) \in M^2 \mid \ell(x,y) > 0\}$ a preorder. The triple $(M, \leq, \ll)$ is a special example of what Kronheimer and Penrose '67 call a *causal space*.

Definition (Causal and timelike futures)

We say y lies in the *causal future* of x and write $x \leq y$ if $\ell(x,y) \geq 0$; we say y lies in the *timelike future* of x and write $x \ll y$ if $\ell(x,y) > 0$. Also

$$J^+(x) = \{y \in M \mid \ell(x,y) \geq 0\} \qquad J^+(X) = \cup_{x \in X} J(x)$$
$$J^-(y) = \{x \in M \mid \ell(x,y) \geq 0\} \qquad J^-(Y) = \cup_{y \in Y} J(y)$$
$$J(x,y) = J^+(x) \cap J^-(y) \qquad J(X,Y) = J^+(X) \cap J^-(Y)$$

and similarly $I^\pm(z)$ and $I(X,Y)$ but with strict inequalities $\ell > 0$.

Definition (Causal and timelike paths)

A path $s \mapsto \sigma(s) \in M$ is called *causal* if and only if $\ell(\sigma(s), \sigma(t)) \geq 0$ for all $s \leq t$, and *timelike* if and only if $\ell(\sigma(s), \sigma(t)) > 0$ for all $s < t$.

Definition (Lorentzian length of a causal path)

The (*negative*) ℓ-length of a causal path $\sigma : [a, b] \longrightarrow M$ is defined by

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The (*negative*) ℓ-length of a causal path $\sigma : [a, b] \longrightarrow M$ is defined by

$$L_{-\ell}(\sigma) := \sup_{k \in \mathbf{N}} \sup_{a=t_0 \leq t_1 \leq \cdots \leq t_k = b} -\sum_{i=1}^{k} \ell(\sigma(t_{i-1}), \sigma(t_i))$$
$$\geq -\ell(\sigma(a), \sigma(b))$$

by the triangle inequality.

Definition (ℓ-path)

A path $\sigma : [0,1] \longrightarrow M$ is called an ℓ-path if and only if

$$\ell(\sigma(s), \sigma(t)) = (t - s)\ell(\sigma(0), \sigma(1)) > 0 \qquad \forall 0 \leq s < t \leq 1.$$

We denote the set of ℓ-paths by $\mathrm{TPath}^{\ell}(M)$.

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Definition

We call (M,ℓ) a *timelike ℓ-path space* if each timelike related pair of events $x \ll y$ are connected by an ℓ-path.

- Kunzinger and Sämann's *(regular) globally hyperbolic Lorentzian length spaces* provide a rich class of examples of timelike ℓ-path spaces
- to achieve this, they need a (metrizable) topology

a variation on Kunzinger & Sämann (hereafter K-S)

Definition (Metric spacetime)

A metric space (M, d) equipped with its metric topology and a time-separation function ℓ is called a *metric spacetime*

Definition (Causal curve)

A nonconstant causal *path* is called a causal *curve* if it is d-Lipschitz.

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Definition (Non-totally imprisoning)

A metric spacetime (M, d, ℓ) is *non-totally imprisoning* if each compact $K \subset M$ admits a bound $B < \infty$ such that all causal curves σ in K (i.e. $\sigma : I \subset \mathbb{R} \longrightarrow K$ with $\sigma(I) \subset K$) have d-length $L_d(\sigma) \leq B$.

Definition (Globally hyperbolic)

A metric spacetime (M, d, ℓ) is *globally hyperbolic* if it is non-totally imprisoning and the causal diamond $J(x, y)$ is compact for each $x, y \in M$.

Definition (Timelike curve-connected; Lorentzian geodesic space)

A metric spacetime is *timelike curve-connected* iff each $x \ll y$ are connected by a timelike curve; it is a *Lorentzian geodesic space* iff each $x < y$ are connected by a causal curve σ with $L_{-\ell}(\sigma) = -\ell(\sigma(0), \sigma(1))$.

Without global hyperbolicity, K-S's definition of a *Lorentzian length space (LLS)* is involved. With global hyperbolicity it can be defined via:

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Theorem ((M. 23+) Characterizing Lorentzian length spaces)

Assuming globally hyperbolicity, a metric spacetime (M, d, ℓ) is an LLS iff it is (a) a timelike curve-connected (b) Lorentzian geodesic space; (c) $I^{\pm}(x)$ both nonempty $\forall x \in M$; (d) ℓ is lower semicontinuous and (e) $\ell_+ = \max\{\ell, 0\}$ is continuous.

- In such spaces, K-S showed that metric topology coincides with the order topology induced by ℓ; this implies gh LLS's are independent of d!

- Burtscher & Garcia-Hevelling 21+ characterize global hyperbolicity of an LLS via existence of Cauchy time functions (and surfaces)

- Unfortunately, its not clear that all ℓ-paths are continuous!

Definition (Regular(ly localizable))

An LLS is *regular* (or *regularly localizable*) if for any $L_{-\ell}$-minimizing causal curve, $L_{-\ell}(\sigma|_{[a,b]}) = 0$ with $\sigma|_{[a,b]}$ non-constant implies $L_{-\ell}(\sigma) = 0$.

Lemma (M. 23+)

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Lemma (M. 23+)

In a globally hyperbolic regular LLS, each ℓ-path is continuous.

Corollary (Relation of ℓ-paths to $L_{-\ell}$-extremizers)

In a globally hyperbolic regularly localizable Lorentzian length space:
(a) Every ℓ-path becomes a d-Lipschitz $L_{-\ell}$-minimizing curve after a continuous increasing (not necessarily Lipschitz) reparameterization.
(b) K-S: Conversely, every $L_{-\ell}$-minimizing curve with timelike separated endpoints becomes an ℓ-path after a similar reparameterization.

(a) resolves an awkward gap in the literature.

Proof of lemma:

• For convenience, we deal only with metric spacetimes (M, d, ℓ) which are closed Lorentzian geodesic subsets of globally hyperbolic regular Lorentzian length spaces (g.h.r. LLS).

Now that timelike geodesics exist:

• given a triple $x \ll z \ll y$ of timelike related events, we can compare the Lorentzian length of a bisector to that of the Minkowski triangle with the same Lorentzian sidelengths

• and similarly for generalized bisectors (i.e. ratios other than 1 : 1)

• K-S define T-sec$(M, d, \ell) \geq 0$ if our generalized bisector is longer (and T-sec$(M, d, \ell) \leq 0$ if it is shorter) for all such timelike triangles

• they define $\pm$ T-sec$(M, d, \ell) \geq k \in \mathbb{R}$ analogously by comparing to timelike triangles in constant curvature Lorentzian spaces

• they also give causal sectional curvature bounds and show such bounds prevent branching of ℓ-geodesics:

Definition (timelike nonbranching)

(M, ℓ) *timelike nonbranching* if for all $\tilde{\sigma}, \sigma \in \mathrm{TPath}^{\ell}$ with $\sigma|_{[\frac{1}{3},\frac{2}{3}]} = \tilde{\sigma}|_{[\frac{1}{3},\frac{2}{3}]}$ then $\tilde{\sigma} = \sigma$;

• Alexander-Bishop '08 shows consistency of these definitions with smooth timelike sectional curvature bounds on Lorentzian manifolds

• Minguzzi-Suhr '22+ show stability of a similar bound

• Beran-Ohanyan-Rott-Solis '22+: T-sec$(M, d, \ell) \geq 0$ and existence of a timelike line implies geometric splitting of (M, d, ℓ)

To pass from sectional to Ricci curvature / Einstein eq requires averaging:

Definition (Optimal transport distance between measures)

• Given metric spaces $(M^{\pm}, d^{\pm})$, let $\mathcal{P}(M)$ denote the Borel probability measures on M and $\mathcal{P}_c(M)$ those with compact support.

• *Push-forward:* given $G : M^- \longrightarrow M^+$ Borel and $\mu^- \in \mathcal{P}(M^-)$, define $\mu^+ = G_\# \mu^- \in \mathcal{P}(M^+)$ by $\mu^+(B) = \mu^-(G^{-1}(B))$ for all $B \subset M^+$.

• Letting $\pi^\mp(x^-, x^+) = x^\mp$ denote the projection from $M^- \times M^+$ onto its left and right factors, set $\Gamma(\mu^-, \mu^+) = \{\gamma \in \mathcal{P}(M^- \times M^+) \mid \pi^\pm_\# \gamma = \mu^\pm\}$.

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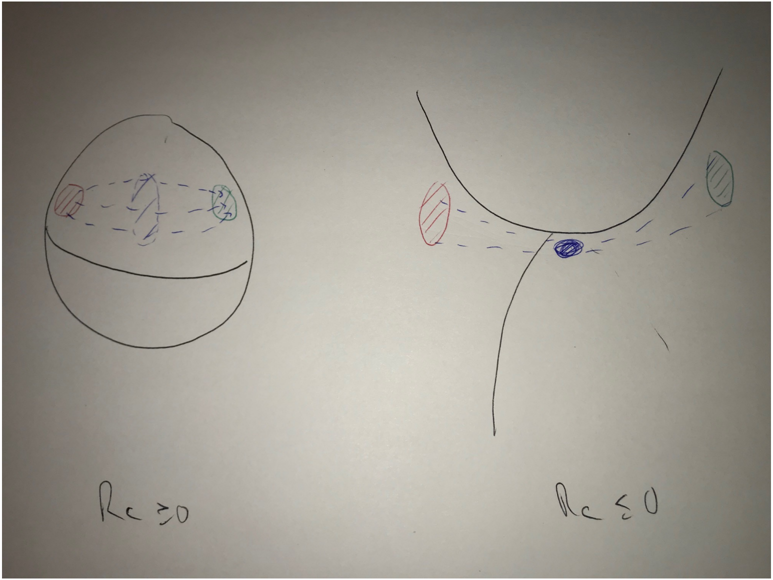
- Given $p \in [1, \infty)$ and $M = M^\pm$, the *p-Kantorovich-Rubinstein-Wasserstein distance d_p* between $\mu^\pm \in \mathcal{P}(M)$ defined by

$$d_p(\mu^-, \mu^+) := \inf_{\gamma \in \Gamma(\mu^+, \mu^-)} \left(\int_{M^2} d(x,y)^p d\gamma(x,y) \right)^{1/p} \quad (3)$$

is well-known to metrize convergence against functions growing no faster than $d(x, \cdot)^p$ provided (M, d) is *Polish* (i.e. complete and separable), in which case the inf is attained.
- If (M, d) is a geodesic space so is $(\mathcal{P}_c(M), d_p)$.


Rc ≥ 0
Rc ≤ 0

Definition (Causal and timelike measures)

In a Polish g.h.r LLS (M, d, ℓ), given $\mu, \nu \in \mathcal{P}(M)$ and $q \in (0, 1]$ set

$$\Gamma_{\leq}(\mu,\nu) := \{\gamma \in \Gamma(\mu,\nu) \mid \gamma[M^2_{\leq}] = 1\} = \{\textit{causal measures}\}$$

$$\Gamma_{\ll}(\mu,\nu) := \{ \quad \text{''} \quad \mid \gamma[M^2_{\ll}] = 1\} = \{\textit{timelike measures}\}$$

Lemma (Lift time-separation from events to measures)

$$\ell_q(\mu,\nu) := \max_{\gamma \in \Gamma_{\leq}(\mu,\nu)} \left(\int_{M^2} \ell(x,y)^q d\gamma(x,y) \right)^{1/q} \tag{4}$$

makes $(\mathcal{P}_c(M), \ell_q)$ into a timelike ℓ_q-path space. Not all such ℓ_q-paths are d_1-continuous;

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makes $(\mathcal{P}_c(M), \ell_q)$ into a timelike ℓ_q-path space. Not all such ℓ_q-paths are d_1-continuous; one will be if (μ,ν) is timelike q-dualizable:

Definition (timelike q-dualizability)

Let $\Gamma^q = \Gamma^q(\mu,\nu)$ denote the set of maximizers. Then

- (μ,ν) are *timelike q-dualizable* if $\Gamma^q_{\ll} := \Gamma^q \cap \Gamma_{\ll}(\mu,\nu)$ is non-empty and $\exists u \oplus v \in L^1(\mu \times \nu)$ which dominates ℓ^q on $\mathrm{spt}(\mu \times \nu) \cap M^2_{\leq}$.
- (μ,ν) are *strongly* timelike q-dualizable if, in addition, $\Gamma^q \subset \Gamma_{\ll}(\mu,\nu)$

Definition (Polish / proper metric-measure spacetime)

A *metric-measure spacetime* refers to a Lorentzian geodesic closed subset (M, d, ℓ) of a g.h.r. LLS, equipped with a Borel measure $m \geq 0$, finite on bounded sets, satisfying $M = \mathrm{spt}\, m$. It's called *Polish* if complete and separable, and *proper* if all bounded subsets $X \subset M$ are compact.

Example (Smooth metric-measure spacetimes)

Any smooth, connected, Hausdorff, time-oriented, n-dimensional Lorentzian manifold (M^n, g) of signature $(+ - \ldots -)$ is second-countable (Ozeki-Nomizu '61) and its topology comes from a complete Riemannian metric $\tilde{g}$ (Geroch '68). With the distance $d_{\tilde{g}}$ and time-separation function ℓ_g induced by $\tilde{g}$ and g respectively, is a proper g.h.r. LLS provided it has no closed causal curves and causal diamonds $J(x, y)$ are compact. Letting $V \in C^\infty(M)$ and vol_g denote its Lorentzian volume, setting $dm = e^{-V} d\mathrm{vol}_g$ makes it a proper metric-measure spacetime. We call such spaces *smooth metric-measure spacetimes*.

Synthetic timelike Ricci bounds

Desiderata:

- consistency (with the analogous smooth bounds)
- stability (preservation under suitable limits)
- consequences (e.g. Hawking-type singularity theorem)

Definition (Entropy)

We define the relative *entropy* by

$$H(\mu \mid m) := \begin{cases} \int_M \rho \log \rho \, dm & \text{if } \mu \in \mathcal{P}_c^{ac}(M) \text{ and } \rho := \frac{d\mu}{dm}, \\ +\infty & \text{if } \mu \in \mathcal{P}_c(M) \setminus \mathcal{P}^{ac}(M). \end{cases}$$

- our sign convention is opposite to that of the physicists' entropy

Entropic weak timelike curvature-dimension conditions

Definition (TCD versus wTCD; e.g. $K = 0 = 1/N$)

For $(K, N, q) \in \mathbb{R} \times (0, \infty] \times (0, 1]$ write $(M, d, \ell, m) \in wTCD^e_q(K, N)$ if and only if every strongly timelike q-dualizable finite entropy pair $\mu_0, \mu_1 \in \mathcal{P}_c(M)$ admit a maximizer $\gamma \in \Gamma^q_\ll$ and corresponding ℓ_q-path $(\mu_t)_{t\in[0,1]}$ along which the entropy $t \in [0, 1] \mapsto h(t) := H(\mu_t \mid m)$ is upper-semicontinuous and distributionally solves the semiconvexity inequality

$$h''(t) \geq \frac{h'(t)^2}{N} + K \|\ell\|^2_{L^2(\gamma)}.$$

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$$h''(t) \geq \frac{h'(t)^2}{N} + K \|\ell\|_{L^2(\gamma)}^2.$$

Cavalletti-Mondino '20+ prove all limits of $TCD_q^e(K, N)$ space in a suitable (pointed measured weak) sense lie in $wTCD_q^e(K, N)$ if $N < \infty$; they also display remarkable similarities to smooth spacetimes (such as a Hawking singularity theorem)

c.f. Burtscher-Ketterer-M.-Woolgar '20 analogous sharp Riemannian injectivity radius bound; characterizes $RCD(K, N)$ spaces which attain it

Pointed measured weak convergence [Cav.-Mondino 20+]

Fixing $x_j \in M_j = \mathrm{spt}\, m_j$ where m_j is a Radon measure, we say $(M_j, d_j, \ell_j, m_j, x_j) \to_{pmGL} (M_\infty, d_\infty, \ell_\infty, m_\infty, x_\infty)$ iff all $(M_j, d_j, \ell_j, m_j, x_j)$ embed d-continuously and ℓ-isometrically into a single proper g.h.r. LLS (X, d, ℓ) and after this embedding, $d(x_j, x_\infty) \to 0$ and the measures $m_j \to m_\infty$ converge weakly against continuous compactly supported test functions: i.e.

$$\lim_{j\to\infty} \int_X \phi dm_j = \int_X \phi dm_\infty \qquad \forall \phi \in C_c(X).$$

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• although the limit of $TCD_q^e(K,N)$ spaces is only $wTCD_q^e(K,N)$, Braun '22+ shows (q-essentially) timelike nonbranching $wTCD_q^e(K,N)$ spaces are $TCD_q^e(K,N)$. Hence a limit of timelike nonbranching $wTCD_q^e(K,N)$ spaces is $wTCD_q^e(K,N)$.

• OPEN QUESTION: unlike in positive signature, it is not known whether some version of timelike nonbranchingness survives the preceding limits

Positive energy ⇔ displacement convexity of entropy

DEF (N-Bakry-Emery modified Ricci tensor; cf. Erbar-Kuwada-Sturm'15)
Given $N \neq n$ and $V \in C^\infty(M^n)$ define

$$R_{ij}^{(N,V)} := R_{ij} + \nabla_i \nabla_j V - \frac{1}{N-n}(\nabla_i V)(\nabla_j V)$$

THM (M '20 Consistency) Fix $(K, N, q) \in \mathbb{R} \times (0, \infty] \times (0, 1)$ and a smooth metric-measure spacetime (M^n, g) with $dm = e^{-V} d\mathrm{vol}_g$. Then $(M, d_{\tilde{g}}, \ell_g, m) \in (w)TCD_q^e(K, N)$ if and only if either

(a) $N = n$, $V = const$ and $R_{ij} v^i v^j \geq K$ for all unit timelike $(v, x) \in TM$,

(b) $N > n$ and $R_{ij}^{(N,V)} v^i v^j \geq K$ for all unit timelike vectors $(v, x) \in TM$.

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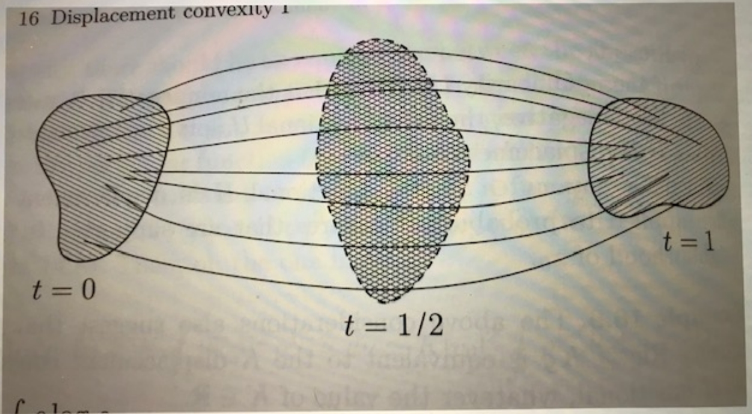
Mondino-Suhr '18+ Use entropic convexity to say also when equality holds, giving a weak (but unstable) solution concept for Einstein field equation.

Akdemir-Cavalletti-Colinet-M.-Santarcangelo '21
$CD_p(K, N) \cap \{\text{nonbranching}\}$ is independent of $p > 1$

Lazy Gas Experiment (M. 94, Villani 09)



Action minimizing paths satisfy pressureless Euler equation.

Braun 22+:

• $N = \infty$

• alternative definitions of $(w)TCD_q^{(*)}(K,N)$ based on convexity properties of a power-law entropy (instead of $H(\mu \mid m)$) along ℓ_q-paths

$$S_N(\mu) := -N \int_M (\frac{d\mu}{dm})^{1-\frac{1}{N}} dm$$

• equivalence of most of these various definitions to $TCD_q^e(K,N)$ assuming (q-essential) timelike nonbranching

Cavalletti-Mondino '22:

• asked for a synthetic formulation of the null energy condition (NEC)

• stronger physical motivation; more widely satisfied

• forms a key hypothesis in the Penrose singularity theorem for stellar collapse

Braun 22+:

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- alternative definitions of $(w)TCD_q^{(*)}(K, N)$ based on convexity properties of a power-law entropy (instead of $H(\mu \mid m)$) along ℓ_q-paths

$$S_N(\mu) := -N \int_M (\frac{d\mu}{dm})^{1-\frac{1}{N}} dm$$

- equivalence of most of these various definitions to $TCD_q^e(K, N)$ assuming (q-essential) timelike nonbranching

Cavalletti-Mondino '22:

- asked for a synthetic formulation of the null energy condition (NEC)
- stronger physical motivation; more widely satisfied
- forms a key hypothesis in the Penrose singularity theorem for stellar collapse

Unlike Riemannian, geometry, in Lorentzian geometry, smoothness need not imply a local lower bound on Ricci curvature!

Theorem (M' 23+)

Fix a smooth spacetime (M^n, g) *with signature* $(+-\cdots-)$ *and symmetric 2-tensor field* Q. *Then*

$$Q(v,v) \geq 0 \quad \forall (v,x) \in TM \text{ with } g(v,v) = 0$$

holds if and only if each compact subdomain $X \subset M^n$ *admits a timelike lower bound* $K = K_X$ *for* Q*, i.e.*

$$Q(v,v) \geq Kg(v,v) \quad \forall (v,x) \in TX \text{ with } g(v,v) > 0$$

Taking $Q = \mathrm{Ric}^{(N,V)}$ (or $Q_{ab} = 8\pi T_{ab}$ if Einstein holds) motivates

Definition (A synthetic null energy-dimension condition)

Given $(N,q) \in (0,\infty] \times (0,1)$, a metric-measure spacetime (M,d,ℓ,m) satisfies $wNC_q^{(e)}(N)$ if and only if each compact subset $X \subset M$ admits a bound $K = K_X \in \mathbb{R}$ such that $J(X,X) \in wTCD_q^{(e)}(K,N)$.

• in other words, the null energy condition is equivalent to a variable lower (semicontinuous) bound $k(x)$ on the timelike Ricci curvature

• Consistency with smooth (NC) + ($n \leq N$): follows from theorem above

• for (q-essentially) timelike nonbranching spaces $wNC_q^e(N) = NC_q^*(N)$

• Consequences: many of Cavalletti & Mondino's nice properties (timelike Bishop-Gromov and Brunn-Minkowski inequalities, needle decomposition, etc) of nonsmooth $wTCD_q^{(e)}(K, N)$ spacetimes are therefore inherited directly by $wNC_q^{(e)}(N)$ spacetimes; c.f. Braun-M. (in progress)

• OPEN: it is natural to wonder if a Penrose singularity theorem can hold in this nonsmooth setting? (c.f. Graf '20 on $g \in C^1$ spacetimes (M^n, g), Ketterer '23+ entropic convexity derivation on $g \in C^\infty$ spacetimes)

• (In)stability: on the other hand, any stability result appears hopeless unless we are will to assume some uniformity in j of the lower bound $k(\cdot)$ along the sequence $(M_j, d_j, \ell_j, m_j, x_j)$

A few references

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THANK YOU!