Lethbridge Number Theory and Combinatorics Seminar

THE SIZE FUNCTION FOR IMAGINARY CYCLIC SEXTIC FIELDS

Ha Tran, Peng Tian, and Amy Feaver

Concordia University of Edmonton, East China University of Science and Technology, Gordon College (Wenham, MA)

November 28, 2023

Content

Notations

- Let F be a number field with discriminant Δ and the ring of integers O_F.

Notations

- Let F be a number field with discriminant Δ and the ring of integers O_F.
- Let $\sigma_1, \ldots, \sigma_{r_1}, \sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}$ be $r_1 + r_2$ embeddings of F.

Notations

- Let F be a number field with discriminant Δ and the ring of integers O_F.
- Let $\sigma_1, \ldots, \sigma_{r_1}, \sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}$ be $r_1 + r_2$ embeddings of F.
- Denote by $\Phi = (\sigma_1, ..., \sigma_{r_1+r_2})$. Then

$$\Phi : F \hookrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \text{ takes } x \in F \text{ to } (\sigma_i(x))_i.$$

Lattices and ideal lattices

- A lattice is a discrete additive subgroup of an Euclidean space.
  Ex: $\mathbb{Z}^m \subset \mathbb{R}^m$.

Lattices and ideal lattices

- A lattice is a discrete additive subgroup of an Euclidean space.
  Ex: $\mathbb{Z}^m \subset \mathbb{R}^m$.

Ex: Let $F = \mathbb{Q}(\sqrt{5})$. Then $O_F = \mathbb{Z} \oplus (1+\sqrt{5})/2\mathbb{Z}$.

Lattices and ideal lattices

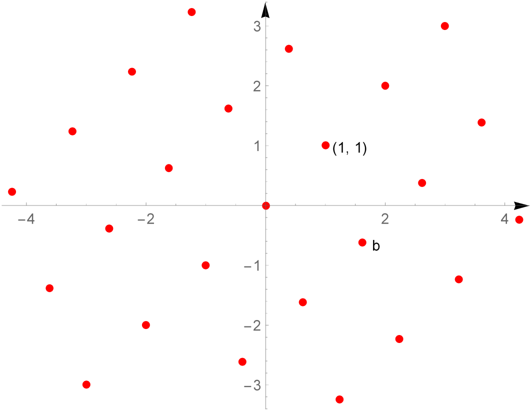
- A lattice is a discrete additive subgroup of an Euclidean space.
  Ex: $\mathbb{Z}^m \subset \mathbb{R}^m$.

Ex: Let $F = \mathbb{Q}(\sqrt{5})$. Then $O_F = \mathbb{Z} \oplus (1+\sqrt{5})/2\mathbb{Z}$. What $\Phi(O_F)$ looks like?

Lattices and ideal lattices

- A lattice is a discrete additive subgroup of an Euclidean space.
  Ex: $\mathbb{Z}^m \subset \mathbb{R}^m$.

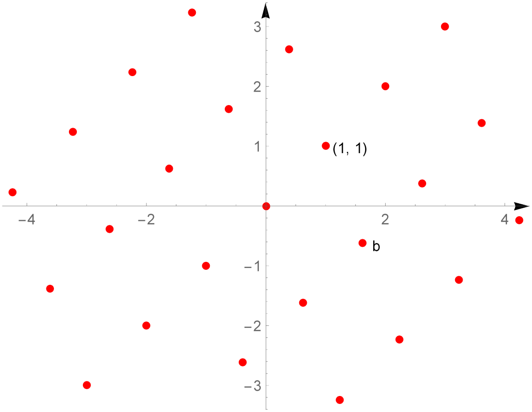
Ex: Let $F = \mathbb{Q}(\sqrt{5})$. Then $O_F = \mathbb{Z} \oplus (1+\sqrt{5})/2\mathbb{Z}$. What $\Phi(O_F)$ looks like?



Lattices and ideal lattices

- A lattice is a discrete additive subgroup of an Euclidean space.
  Ex: $\mathbb{Z}^m \subset \mathbb{R}^m$.

Ex: Let $F = \mathbb{Q}(\sqrt{5})$. Then $O_F = \mathbb{Z} \oplus (1+\sqrt{5})/2\mathbb{Z}$. Then $\Phi(O_F) = \Phi(1)\mathbb{Z} \oplus \Phi((1+\sqrt{5})/2)\mathbb{Z}$ is a lattice in $\mathbb{R}^2$.



Lattices and ideal lattices

- A lattice is a discrete additive subgroup of an Euclidean space.
  Ex: $\mathbb{Z}^m \subset \mathbb{R}^m$.

Proposition

Let I be a factional ideal of F. Then $\Phi(I)$ is a lattice in $\mathbb{R}^n$.

Ideal lattices

Definition (Ideal lattices)

An ideal lattice is a lattice (I, q), where

- I is a (fractional) O_F-ideal and
- $q : I \times I \longrightarrow \mathbb{R}$ is a non-degenerate symmetric bilinear form st
  $q(\lambda x, y) = q(x, \bar{\lambda} y)$ (Hermitian property)
  for all $x, y \in I$ and for all $\lambda \in O_F$.

Ideal lattices

Definition (Ideal lattices)

An ideal lattice is a lattice (I, q), where

- I is a (fractional) O_F-ideal and
- $q : I \times I \longrightarrow \mathbb{R}$ is a non-degenerate symmetric bilinear form st
  $q(\lambda x, y) = q(x, \bar{\lambda} y)$ (Hermitian property)
  for all $x, y \in I$ and for all $\lambda \in O_F$.

Let I be a factional ideal of F and let $u = (u_i)_i \in (\mathbb{R}_{>0})^n$.

Ideal lattices

Definition (Ideal lattices)

An ideal lattice is a lattice (I, q), where

- I is a (fractional) O_F-ideal and
- $q : I \times I \longrightarrow \mathbb{R}$ is a non-degenerate symmetric bilinear form st
  $q(\lambda x, y) = q(x, \bar{\lambda} y)$ (Hermitian property)
  for all $x, y \in I$ and for all $\lambda \in O_F$.

Let I be a factional ideal of F and let $u = (u_i)_i \in (\mathbb{R}_{>0})^n$.
Define $q_u(x, y) = \langle u\Phi(x), u\Phi(y)\rangle$ for any $x, y \in I$.

$$\|x\|_u^2 = q_u(x, x) = \|u\Phi(x)\|^2 = \sum_{i=1}^{n} u_i^2 [\sigma_i(x)]^2.$$

Ideal lattices

Definition (Ideal lattices)

An ideal lattice is a lattice (I, q), where

- I is a (fractional) O_F-ideal and
- $q : I \times I \longrightarrow \mathbb{R}$ is a non-degenerate symmetric bilinear form st
  $q(\lambda x, y) = q(x, \bar{\lambda} y)$ (Hermitian property)
  for all $x, y \in I$ and for all $\lambda \in O_F$.

Let I be a factional ideal of F and let $u = (u_i)_i \in (\mathbb{R}_{>0})^n$.
Define $q_u(x, y) = \langle u\Phi(x), u\Phi(y)\rangle$ for any $x, y \in I$.

$$\|x\|_u^2 = q_u(x, x) = \|u\Phi(x)\|^2 = \sum_{i=1}^{n} u_i^2 [\sigma_i(x)]^2.$$

Then (I, q_u) is an ideal lattice.

The size function for lattices

Let L be a lattice of $\mathbb{R}^n$.

$$k^0(L) := \sum_{x \in L} e^{-\pi \|x\|^2}, \qquad h^0(L) = \log(k^0(L)).$$

The size function for a number field

Similarly, h^0 is defined for the ideal lattice (I, q_u).

$$k^0(I, q_u) = \sum_{x \in I} e^{-\pi \|x\|_u^2}, \qquad h^0(I, q_u) = \log(k^0(I, q_u)).$$

The size function for a number field

Similarly, h^0 is defined for the ideal lattice (I, q_u).

$$k^0(I, q_u) = \sum_{x \in I} e^{-\pi \|x\|_u^2}, \qquad h^0(I, q_u) = \log(k^0(I, q_u)).$$

Definition

- The pair $D = (I, u)$ is called an Arakelov divisor of F.

The size function for a number field

Similarly, h^0 is defined for the ideal lattice (I, q_u).

$$k^0(I, q_u) = \sum_{x \in I} e^{-\pi \|x\|_u^2}, \qquad h^0(I, q_u) = \log(k^0(I, q_u)).$$

Definition

- The pair $D = (I, u)$ is called an Arakelov divisor of F.
- (I, q_u) is also called the ideal lattice associated to D.

The size function for a number field

Similarly, h^0 is defined for the ideal lattice (I, q_u).

$$k^0(I, q_u) = \sum_{x \in I} e^{-\pi \|x\|_u^2}, \qquad h^0(I, q_u) = \log(k^0(I, q_u)).$$

Definition

- The pair $D = (I, u)$ is called an Arakelov divisor of F.
- (I, q_u) is also called the ideal lattice associated to D.
- $h^0(D) := h^0(I, q_u)$.

Analogies

Algebraic curve

- Divisor D.
- Principal divisor.
- Picard group.
- Canonical divisor κ.
- dimension $\ell(D)$.
- Riemann–Roch theorem: $\ell(D) - \ell(\kappa - D) = \deg(D) - (g - 1)$.

Number field F [a]

[a] van der Geer and Schoof (1999)

Analogies

Algebraic curve

- Divisor D.
- Principal divisor.
- Picard group.
- Canonical divisor κ.
- dimension $\ell(D)$.
- Riemann–Roch theorem: $\ell(D) - \ell(\kappa - D) = \deg(D) - (g - 1)$.

Number field F [a]

- Arakelov divisor $D = (I, u)$.
- Principal Arakelov divisor.
- Arakelov class group Pic_F^0.
- The inverse different.
- size function of F: $h^0(D)$
- Riemann–Roch theorem: $h^0(D) - h^0(\kappa - D) = \deg(D) - \frac{1}{2}\log|\Delta|$.

[a] van der Geer and Schoof (1999)

The Arakelov class group Pic_F^0

- Arakelov divisor $D = (I, u)$ where I is a fractional ideal of F and $u = (u_i) \in (\mathbb{R}_{>0})^{r_1+r_2}$.

The Arakelov class group Pic_F^0

- Arakelov divisor $D = (I, u)$ where I is a fractional ideal of F and $u = (u_i) \in (\mathbb{R}_{>0})^{r_1+r_2}$.
- $\deg(D) := -\log(N(I)\prod_i u_i)$.

The Arakelov class group Pic_F^0

- Arakelov divisor $D = (I, u)$ where I is a fractional ideal of F and $u = (u_i) \in (\mathbb{R}_{>0})^{r_1+r_2}$.
- $\deg(D) := -\log(N(I)\prod_i u_i)$.
- The set of all Arakelov divisors of degree 0 form a group, denoted by Div_F^0.

The Arakelov class group Pic_F^0

- Arakelov divisor $D = (I, u)$ where I is a fractional ideal of F and $u = (u_i) \in (\mathbb{R}_{>0})^{r_1+r_2}$.
- $\deg(D) := -\log(N(I)\prod_i u_i)$.
- The set of all Arakelov divisors of degree 0 form a group, denoted by Div_F^0.
- A principal Arakelov divisor has the form (I, u) where $I = x^{-1}O_F$ and $u = |\Phi(x)| = (|\sigma_i(x)|)_i$ and $x \in F^\times$.

The Arakelov class group Pic_F^0

- Arakelov divisor $D = (I, u)$ where I is a fractional ideal of F and $u = (u_i) \in (\mathbb{R}_{>0})^{r_1+r_2}$.
- $\deg(D) := -\log(N(I)\prod_i u_i)$.
- The set of all Arakelov divisors of degree 0 form a group, denoted by Div_F^0.
- A principal Arakelov divisor has the form (I, u) where $I = x^{-1}O_F$ and $u = |\Phi(x)| = (|\sigma_i(x)|)_i$ and $x \in F^\times$.
- The Arakelov class group Pic_F^0 is the quotient of Div_F^0 by its subgroup of principal divisors.

The structure of Pic_F^0

$O_F^\times$: the unit group of O_F.

$\mathcal{H} = \{(x_i) \in \mathbb{R}^{r_1} \times C^{r_2} : (x_1 + \cdots + x_{r_1}) + 2(x_{r_1+1} + \cdots + x_{r_1+r_2}) = 0\}$.

$\Lambda_F := \{(\log|\sigma_i(x)|)_i : x \in O_F^\times\} \subseteq \mathcal{H}$ - the log unit lattice of F.

Proposition

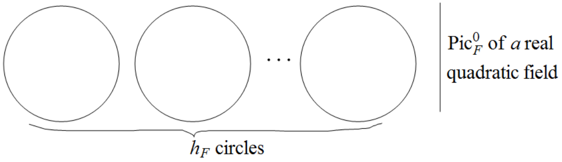
$\mathrm{Pic}_F^0 \longrightarrow$ {isometry classes of ideal lattices of covolume $\sqrt{\Delta}$}
the class of $D = (I, u) \longmapsto$ the isometry class of (I, q_u)
is a bijection. Moreover, the following sequence is exact.

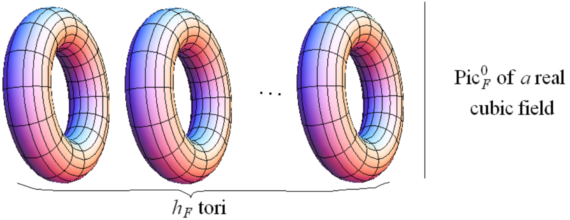
$$0 \longrightarrow \mathcal{H}/\Lambda_F \longrightarrow \mathrm{Pic}_F^0 \longrightarrow Cl_F \longrightarrow 0.$$

The Arakelov class group Pic_F^0

The following sequence is exact.

$$0 \longrightarrow \mathcal{H}/\Lambda_F \longrightarrow \mathrm{Pic}_F^0 \longrightarrow Cl_F \longrightarrow 0.$$





- If $[D]$ is on the principal (hyper)torus of Pic_F^0, then $\exists u$ such that $\log u = (\log(u_i))_i \in \mathcal{H}/\Lambda_F$ and $[D] = [(O_F, u)]$.
- h^0 is well defined on Pic_F^0.

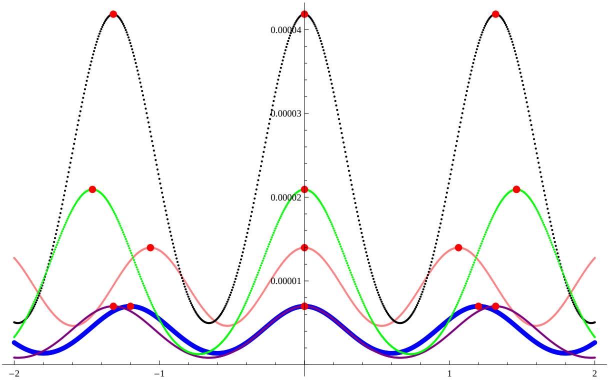
The conjecture of van der Geer and Schoof

At which class of ideal lattices in Pic_F^0 that h^0 attains its maximum?

The conjecture of van der Geer and Schoof

At which class of ideal lattices in Pic^0_F that h^0 attains its maximum?

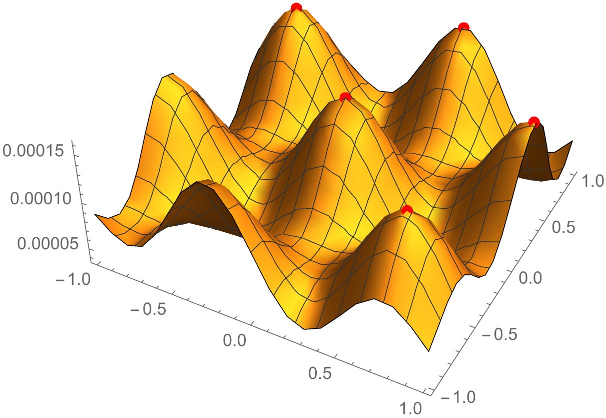
Let K be a real quadratic field (Galois over $\mathbb{Q}$) or quadratic extension of an imaginary quadratic field k (Galois/ k). The origin is the trivial ideal lattice $(O_K, 1)$.



The conjecture of van der Geer and Schoof

At which class of ideal lattices in Pic_F^0 that h^0 attains its maximum?

Let K be a cyclic cubic field or an imaginary cyclic sextic field (Galois over $\mathbb{Q}$).
The origin is the trivial ideal lattice $(O_K, 1)$.



The conjecture of van der Geer and Schoof

Conjecture. Let K be a number field that is Galois over $\mathbb{Q}$ or over an imaginary quadratic field. Then the function h^0 on Pic_K^0 assumes its maximum on the trivial class $(O_K, 1)$.

The conjecture of van der Geer and Schoof

Conjecture. Let K be a number field that is Galois over $\mathbb{Q}$ or over an imaginary quadratic field. Then the function h^0 on Pic_K^0 assumes its maximum on the trivial class $(O_K, 1)$.

Results. The conjecture was proved for number fields of degree n and unit rank r:

- $n = 2, r = 0, 1$: Francini (2001).
- $n = 4, r = 1$: (2014) quadratic extensions of imaginary quadratic fields.
- $n = 3, r = 2$: (2016) cyclic cubic fields.
- $n = 6, r = 2$: (2021) imaginary cyclic sextic fields (this talk).

Notations

- Let F be an imaginary cylic sextic field with discriminant Δ and the ring of integers O_F.
- $\sigma_1, \sigma_2, \sigma_3$: 3 complex embeddings of F (up to conjuate).
- $\Phi = (\sigma_1, \sigma_2, \sigma_3) : F \hookrightarrow \mathbb{C}^3$

$$x \in F \mapsto (\sigma_1(x), \sigma_2(x), \sigma_3(x)).$$

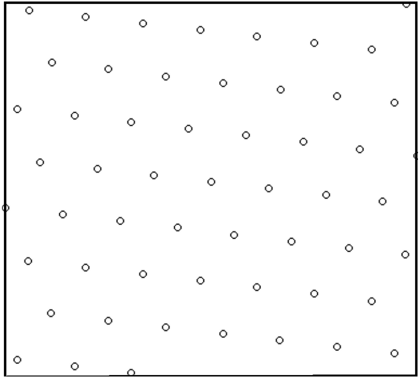
- Let $u = (u_1, u_2, u_3) \in (\mathbb{R}_{>0})^3$, for $x \in I$ an ideal of F. Then

$$\|x\|_u^2 := \|ux\|^2 = 2\sum_i u_i^2 |\sigma_i(x)|^2.$$

The log unit lattice of F

F: imaginary cyclic sextic field.

The log unit lattice Λ_F.



$\Lambda_F := \{(\log|\sigma_i(x)|)_i : x \in O_F^\times\}$ - the log unit lattice of F - is hexagonal.

Prove the conjecture: $[D]$ is not on the principal torus

$$k^0(D) = 1 + \Sigma_1(I,u) + \Sigma_2(I,u) + \Sigma_3(I,u), \text{ where}$$

$$\Sigma_1(I,u) = \sum_{f\in I,\ \|uf\|^2 < 6\cdot 2^{1/3}} e^{-\pi\|uf\|^2},$$
$$\Sigma_2(I,u) = \sum_{f\in I,\ 6\cdot 2^{1/3} \le \|uf\|^2 \le 6\cdot 3^{1/3}} e^{-\pi\|uf\|^2}$$
$$\Sigma_3(I,u) = \sum_{f\in I,\ \|uf\|^2 \ge 6\cdot 3^{1/3}} e^{-\pi\|uf\|^2}.$$

Prove the conjecture: $[D]$ is not on the principal torus

$$k^0(D) = 1 + \Sigma_1(I,u) + \Sigma_2(I,u) + \Sigma_3(I,u), \text{ where}$$

$$\Sigma_1(I,u) = \sum_{f\in I,\ \|uf\|^2<6\cdot 2^{1/3}} e^{-\pi\|uf\|^2},$$
$$\Sigma_2(I,u) = \sum_{f\in I,\ 6\cdot 2^{1/3}\le\|uf\|^2\le 6\cdot 3^{1/3}} e^{-\pi\|uf\|^2}$$
$$\Sigma_3(I,u) = \sum_{f\in I,\ \|uf\|^2\ge 6\cdot 3^{1/3}} e^{-\pi\|uf\|^2}.$$

Show that $h(O_F,1) > h(I,u)$ for all $[(I,u)] \neq [(O_F,1)]$.

Prove the conjecture: $[D]$ is not on the principal torus

$$k^0(D) = 1 + \Sigma_1(I,u) + \Sigma_2(I,u) + \Sigma_3(I,u), \text{ where}$$

$$\Sigma_1(I,u) = \sum_{f\in I,\ \|uf\|^2 < 6\cdot 2^{1/3}} e^{-\pi\|uf\|^2},$$
$$\Sigma_2(I,u) = \sum_{f\in I,\ 6\cdot 2^{1/3}\le \|uf\|^2 \le 6\cdot 3^{1/3}} e^{-\pi\|uf\|^2}$$
$$\Sigma_3(I,u) = \sum_{f\in I,\ \|uf\|^2 \ge 6\cdot 3^{1/3}} e^{-\pi\|uf\|^2}.$$

Show that $h(O_F,1) > h(I,u)$ for all $[(I,u)] \neq [(O_F,1)]$.

1) $[D]$ is not on the principal torus:

$\Sigma_1(I,u) = 0$ (since $\|uf\|^2 \ge 6\cdot 2^{1/3}, \forall f \in I\backslash\{0\}$).

$\Sigma_3(I,u) < 2.605\cdot 10^{-9}$ (bound for # vectors of bounded length in a rank 6 lattice).
$\Sigma_2(I,u) \le 6(\#\mu_F)e^{6\cdot 2^{1/3}}$ ($\le 6(\#\mu_F)$ elements in the sum).

$$k^0(I,u) \le 6(\#\mu_F)e^{-6\cdot 2^{1/3}\pi} + 2.605\cdot 10^{-9} < 1 + (\#\mu_F)e^{-6\pi} < k^0(O_F,1).$$

Prove the conjecture: 2) $[D]$ is on the principal torus

Assume that $[D]$ has the form $[(OF, u)]$, for some
$u = (u_1, u_2, u_3) \in (\mathbb{R}_+)^3$ and
$w = \log(u) = (\log u_1, \log u_2, \log u_3) \in \mathcal{H}/\Lambda_F$.
$\mathcal{F}$ is the fundamental domain of Λ_F.

1. Idea 1: Choose $w \in \mathcal{F}$.

Prove the conjecture: 2) $[D]$ is on the principal torus

Assume that $[D]$ has the form $[(OF, u)]$, for some
$u = (u_1, u_2, u_3) \in (\mathbb{R}_+)^3$ and
$w = \log(u) = (\log u_1, \log u_2, \log u_3) \in \mathcal{H}/\Lambda_F$.
$\mathcal{F}$ is the fundamental domain of Λ_F.

1. Idea 1: Choose $w \in \mathcal{F}$.

We divide into 2 cases:

Prove the conjecture: 2) $[D]$ is on the principal torus

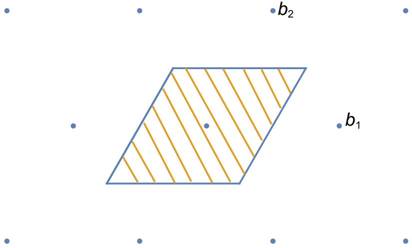
Assume that $[D]$ has the form $[(OF, u)]$, for some $u = (u_1, u_2, u_3) \in (\mathbb{R}_+)^3$ and $w = \log(u) = (\log u_1, \log u_2, \log u_3) \in \mathcal{H}/\Lambda_F$.
$\mathcal{F}$ is the fundamental domain of Λ_F.

1. Idea 1: Choose $w \in \mathcal{F}$.

We divide into 2 cases:

2a) When $\|w\| \geq 0.24163$: find tight upper bounds for Σ_i and so for $k^0(O_F, u)$ similar to the non principal case.

2b) When $\|w\| < 0.24163$: the above bounds do not work.



Prove the conjecture: 2b) $[D]$ is on the principal torus and $0 < \|w\| < 0.24163$

Idea 2. "Amplify" the difference[1]: To prove that $k^0(O_F, u) - k^0(O_F, 1) < 0$, we prove

$$C = \frac{k^0(O_F, u) - k^0(O_F, 1)}{\|w\|^2} < 0.$$

[1] Schoof's idea

Prove the conjecture: 2b) $[D]$ is on the principal torus and $0 < \|w\| < 0.24163$

Idea 2. "Amplify" the difference[1]: To prove that $k^0(O_F, u) - k^0(O_F, 1) < 0$, we prove

$$C = \frac{k^0(O_F, u) - k^0(O_F, 1)}{\|w\|^2} < 0.$$

Write $C = \sum_{0 \neq f \in O_F} G(u, f) = T_1(u) + T_2(u) + T_3(u)$ where

$$T_1(u) = \sum_{f \in \mu_F} G(u, f), \qquad T_2(u) = \sum_{f \in O_F, \|f\|^2 \geq 22} G(u, f)$$

$$T_3(u) = \sum_{0 \neq f \in O_F \setminus \mu_F, \|f\|^2 < 22} G(u, f).$$

$T_1(u)$ is easy to bound.

[1] Schoof's idea

Prove the conjecture: 2b) $[D]$ is on the principal torus and $0 < \|w\| < 0.24163$

Idea 3. Using Maclaurin expansion of $G(u, f)$ and its the symmetry [2] to bound for $T_2(u) = \sum_{f \in O_F, \|f\| \geq 22} G(u, f)$.

For all $f \in O_F$:

$$G(u, f) \leq 4\pi^2 \|f^2\|^2 e^{-\pi \|f\|^2} \left(1 + \frac{1}{2} e^{2\pi \|w\| \|f^2\|}\right).$$

In particular, if $f \in O_F$ with $\|f\|^2 \geq 22$ then

$$G(u, f) \leq 2\pi^2 \left(e^{-(\pi - 2/7)\|f\|^2} + \frac{1}{2} e^{-(\pi - 2\pi\|w\| - 2/7)\|f\|^2}\right).$$

[2] This is symmetric since F is cyclic.

Prove the conjecture: 2b) $[D]$ is on the principal torus and $0 < \|w\| < 0.24163$

K: cyclic cubic subfield of F of conductor p.

Idea 4. Enumerate all possible F such that there exist short elements to bound for $T_3(u) = \sum_{0 \neq f \in O_F \setminus \mu_F, \|f\|^2 < 22} G(u, f)$.

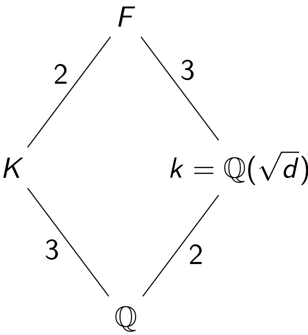
Prove the conjecture: 2b) $[D]$ is on the principal torus and $0 < \|w\| < 0.24163$

K: cyclic cubic subfield of F of conductor p.

Idea 4. Enumerate all possible F such that there exist short elements to bound for $T_3(u) = \sum_{0 \neq f \in O_F \setminus \mu_F, \|f\|^2 < 22} G(u, f)$.

New result: If $0 \neq f \in O_F \setminus \mu_F : \|f\|^2 < 22$, then

- $f \in O_K \cup O_k$, or
  (Enumerate all such K, k and then all f.)
- $f \in O_F \setminus (O_K \cup O_k \cup \mu_F)$, and $d \leq 22$ & $p \leq 61$.
  (Enumerate all such F and then all f.)



PS: if time permits

For any $f \in O_F$, we define $f_i = |\tau_i(f)|$, $i \in \{1, 2, 3\}$. Then

$$\|uf\|^2 = 2\left(e^{2x}|\tau_1(f)|^2 + e^{2y}|\tau_2(f)|^2 + e^{2z}|\tau_3(f)|^2\right)$$
$$= 2\left(f_1^2 e^{2x} + f_2^2 e^{2y} + f_3^2 e^{2z}\right).$$

For $f \in O_F$ we now define

$$G(u, f) = e^{-\pi\|f\|^2} G_2(f, u)/\|w\|^2,$$

where

$$\begin{aligned} G_1(u, f) &= e^{-\pi[\|uf\|^2 - \|f\|^2]} - 1 = e^{-2\pi[(e^{2x}-1)f_1^2 + (e^{2y}-1)f_2^2 + (e^{2z}-1)f_3^2]} - 1 \\ G_2(u, f) &= G_1(u, \tau_1(f)) + G_1(u, \tau_2(f)) + G_1(u, \tau_3(f)). \end{aligned}$$

PS2: if time still permits

Lemma

Let L be a lattice of rank 6 and λ be the length of its shortest vectors. Then for $M \geq \lambda^2 \geq a^2 > 0$ and $\xi > 0$, one has

$$\sum_{\substack{x \in L \\ \|x\|^2 \geq M}} e^{-\xi\|x\|^2} \leq \xi \int_M^\infty \left(\left(\frac{2\sqrt{t}}{a} + 1 \right)^6 - \left(\frac{2\sqrt{M}}{a} - 1 \right)^6 \right) e^{-\xi t} \, \mathrm{d}t.$$

Corollary

If $\lambda^2 \geq 6$, then

$$\sum_{\substack{x \in L \\ \|x\|^2 \geq 6 \cdot 3^{1/3}}} e^{-\pi\|x\|^2} < 2.6049 \cdot 10^{-9}, \qquad \sum_{\substack{x \in L \\ \|x\|^2 \geq 22}} e^{-(\pi - 2/7)\|x\|^2} < 10^{-23},$$

$$\sum_{x \in L, \|x\|^2 \geq 22} e^{-(\pi - 2\sqrt{2} \cdot 0.170856\, \pi - 2/7)\|x\|^2} < 2.19277 \cdot 10^{-9}.$$

Thank you!

Thank you so much for your attention!

References

Paolo Francini.
The size function h^0 for quadratic number fields.
J. Théor. Nombres Bordeaux, 13(1):125–135, 2001.

Gerard van der Geer and René Schoof.
Effectivity of Arakelov divisors and the theta divisor of a number field.
Selecta Math. (N.S.), 6(4):377–398, 2000.

Richard P. Groenewegen.
The size function for number fields.
Doctoraalscriptie, Universiteit van Amsterdam, 1999.

René Schoof.
Computing Arakelov class groups.
In *Algorithmic number theory: lattices, number fields, curves and cryptography*, volume 44 of *Math. Sci. Res. Inst. Publ.*, pages 447–495. Cambridge Univ. Press, Cambridge, 2008.

Ha T. N. Tran.
The size function of quadratic extensions of complex quadratic fields.
Journal de théorie des nombres de Bordeaux, 29 no. 1 (2017), p. 243-259.

Ha T. N. Tran and Peng Tian.
The size function for cyclic cubic fields.
Int. J. Number Theory, 14:399–415, 2018.

Ha T. N. Tran, Peng Tian and Amy Feaver.
The size function for imaginary cyclic sextic fields
to be appeared in *Journal de théorie des nombres de Bordeaux.*