# Hilbert Class Fields and Embedding Problems

(Lethbridge Number Theory and Combinatorics Seminar)

Abbas Maarefparvar

Department of Mathematics & Computer Science

University of Lethbridge

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# Outline

# Preliminaries

**Definition**

A number field is a finite extension $K$ of $\mathbb{Q}$, i.e., a $\mathbb{Q}$-vector space of finite dimension. We denote this dimension by $[K : \mathbb{Q}]$ and call it the degree of $K$ over $\mathbb{Q}$.

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Let $\zeta_n = \exp(\frac{2\pi i}{n})$ be a primitive $n^{\text{th}}$ root of unity. The number field

$$\mathbb{Q}(\zeta_n) = \{a_{m-1}\zeta_n^{m-1} + \cdots + a_1\zeta_n + a_0 \,:\, a_i \in \mathbb{Q}, \forall i\}$$

is a cyclotomic field of degree $m = [\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$.

**Definition**

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## Definition

The set of all algebraic integers of a number field $K$ is denoted by $\mathcal{O}_K$. In fact, $\mathcal{O}_K$ is a ring which is called the ring of integers of $K$.

## Example

Let $f(X) \in \mathbb{Z}[X]$ be a monic polynomial. By a theorem of Gauss,

$$\text{if} \quad \frac{a}{b} \in \mathbb{Q},\ f(\frac{a}{b}) = 0 \ \Rightarrow b = \pm 1.$$

Hence the ring of integers of $\mathbb{Q}$ is $\mathbb{Z}$.

## Example

Let $d$ be a square-free integer. Then

$$\mathcal{O}_{\mathbb{Q}(\sqrt{d})} = \begin{cases} \mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \,:\, a, b \in \mathbb{Z}\}, & d \equiv 2, 3 \pmod 4, \\ \mathbb{Z}[\frac{1+\sqrt{d}}{2}] = \{a + b(\frac{1+\sqrt{d}}{2}) \,:\, a, b \in \mathbb{Z}\}, & d \equiv 1 \pmod 4. \end{cases}$$

## Proposition

Let $K$ be a number field. Then every nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_K$ can be written uniquely in the form

$$\mathfrak{a} = \mathfrak{P}_1^{e_1} \dots \mathfrak{P}_g^{e_g},$$

where $\mathfrak{P}_i$'s are distinct prime ideals of $\mathcal{O}_K$ and $e_i$'s are positive integers.

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Let $K/F$ be a finite extension of number fields. A prime ideal $\mathfrak{p}$ of $F$ will factor in $\mathcal{O}_K$, say $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}_1^{e_1} \dots \mathfrak{P}_g^{e_g}$ ($e_i \geq 1$). The exponents $e_i$'s are called the ramificaton indices of $\mathfrak{p}$ in $K$.

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- If $e_i > 1$, for at least one $i$, then we say $\mathfrak{p}$ is ramified in $K$;
- If $e_1 = g = 1$, then $\mathfrak{p}$ is said to be inert in $K$;
- If $g > 1$, and $e_1 = \cdots = e_g = 1$, then $\mathfrak{p}$ is said to split in $K$. If also $f_i := [\frac{\mathcal{O}_K}{\mathfrak{P}_i} : \frac{\mathcal{O}_F}{\mathfrak{p}}] = 1$ for all $i$, $\mathfrak{p}$ is said to split completely in $K$.

### Example

Let $K = \mathbb{Q}(i)$, where $i^2 = -1$. Then $\mathcal{O}_K = \mathbb{Z}[i]$ (The Gaussian integers). We have

- $2\mathcal{O}_K = (1+i)^2$, so 2 ramifies in $\mathbb{Q}(i)$;

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## Remark

In fact, one can show that for an odd prime $p$:

$$p \text{ splits in } \mathbb{Z}[i] \iff p \equiv 1 \pmod 4 \iff p = a^2 + b^2, \text{ for some } a, b \in \mathbb{Z}$$

Let $K$ be a number field and denote its ring of integers by $\mathcal{O}_K$.

- A fractional ideal of $K$, is a non-zero $\mathcal{O}_K$-submodule $\mathfrak{a}$ of $K$ for which there exists an element $0 \neq d \in \mathcal{O}_K$ such that

$$d\mathfrak{a} = \{dx : x \in \mathfrak{a}\} \subseteq \mathcal{O}_K.$$

We denote by $I(K)$ the set of all the fractional ideals of $K$.

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  We denote by $I(K)$ the set of all the fractional ideals of $K$.

- A principal fractional ideal of $K$ is of the form

$$\langle b \rangle = b\mathcal{O}_K = \{bx : x \in \mathcal{O}\}$$

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- The ideal class group of $K$, denoted by $\mathrm{Cl}(K)$, is defined as

$$\mathrm{Cl}(K) = \frac{I(K)}{P(K)}.$$

### Theorem

Let $K$ be a number field. Then the ideal class group $\mathrm{Cl}(K)$ is a finite abelian group.

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## Remark

The structure of $\mathrm{Cl}(K)$ indicates how far $\mathcal{O}_K$ is from being a unique factorization domain:

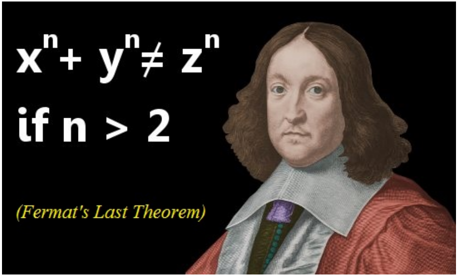
$$h_K = 1 \iff \mathcal{O}_K \text{ is PID} \iff \mathcal{O}_K \text{ is UFD}$$

## Example

The quadratic field $K = \mathbb{Q}(\sqrt{-5})$ has class number 2. Its ring of integers is $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$ in which we have

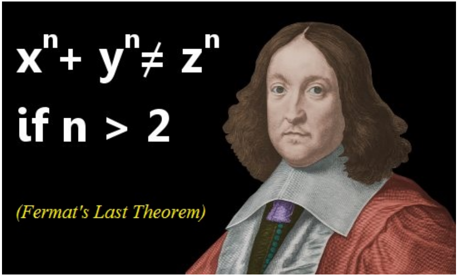
$$6 = 2.3 = (1+\sqrt{-5}).(1-\sqrt{-5}).$$

# The Classical Embedding Problem




### Lamé observation (1847)

Fermat's Last Theorem would be proven if the $p^{\text{th}}$ cyclotomic fields $\mathbb{Q}(\zeta_p)$ had class number 1 for odd primes $p$.

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However, Ernst Kummer had shown three years earlier that this is false for most primes $p$, with $p = 23$ being the famous first example.


## Gauss' class number one problems for quadratic fields (1801)

1. An imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ has class number one, if and only if $d = -1, -2, -3, -7, -11, -19, -43, -67, -163$.


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2. There are infinitely many real quadratic number fields with class number one
   - This is still an open problem! ▸ quadratic Pólya fields

## The classical embedding problem

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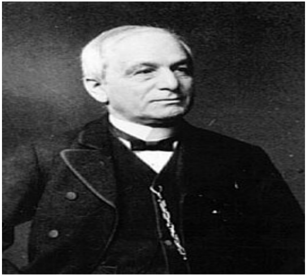
Kummer didn't have the tools to answer this embedding question; but his work has led to the foundation of class field theory (the study of abelian extensions of arbitrary number fields).

## Kronecker-Weber Theorem

Let $K/\mathbb{Q}$ be a finite abelian extension. Then $K \subseteq \mathbb{Q}(\zeta_n)$ for some positive integer $n$.

## Conjecture (Hilbert, 1902)

For any number field $K$, there exists a unique finite extension $H(K)$ of $K$ such that principal prime ideals $\mathfrak{p}$ of $K$ split completely in $H(K)$:

$$\mathfrak{p}\mathcal{O}_{H(K)} = \mathfrak{P}_1 \dots \mathfrak{P}_g,$$

where $\mathfrak{P}_i$'s are distinct prime ideals of $H(K)$ and $g = [H(K) : K]$.

**Theorem (Furtwängler, 1925)**

Let $K$ be an arbitrary number field. Then there exists a unique finite extension $H(K)$ of $K$ such that the extension $H(K)/K$ is

- unramified (for every prime ideal $\mathfrak{p}$ of $K$, the ideal $\mathfrak{p}\mathcal{O}_{H(K)}$ either remains prime or splits completely in $H(K)$);
- abelian (a finite Galois extension whose Galois group is abelian);
- maximal respect to the above properties.

### Definition

The Hilbert class field of a number field $K$, denoted by $H(K)$, is the maximal abelian unramified extension of $K$.

### Principal Ideal Theorem (Furtwängler, 1930)

Every fractional ideal $\mathfrak{a}$ of $K$ becomes principal in $H(K)$.

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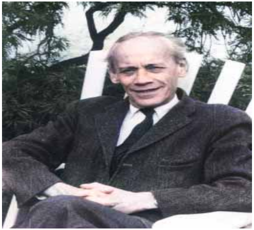

Artin's reciprocity law gives a canonical isomorphism $\mathrm{Gal}(H(K)/K) \simeq \mathrm{Cl}(K)$. In particular, $[H(K) : K] = h_K$.

## Example

Let $K = \mathbb{Q}(\sqrt{-5})$. Then $H(K) = \mathbb{Q}(\sqrt{-1}, \sqrt{-5})$. Also,

- $\mathrm{Gal}(H(K)/K) \simeq \mathrm{Cl}(K) =< (2, 1 + \sqrt{-5}) >\simeq \mathbb{Z}/2\mathbb{Z}$;
- $(2, 1 + \sqrt{-5})\mathcal{O}_{H(K)} = (1 + \sqrt{-1})$.

## Remark

The number field $K$ has class number 1 if and only if $H(K) = K$. In particular, if $h_K = 1$ then there exists no (non-trivial) abelian unramified extension of $K$.

## Class Field Tower Problem (Furtwängler, 1925)

Let $K = K_1$ be a number field. For every $n \geq 1$, let $K_{n+1}$ be the Hilbert class field of $K_n$. Decide whether the tower

$$K = K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots \qquad \text{(Class Field Tower)}$$

can be infinite, or must always terminate with a field of class number 1 after a finite number of steps.

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can be infinite, or must always terminate with a field of class number 1 after a finite number of steps.

## Remark

The Class Field Tower Problem is equivalent to the classical embedding problem.

## Example

The class field tower for $\mathbb{Q}(\sqrt{-5})$ is $\mathbb{Q}(\sqrt{-5}) \subseteq \mathbb{Q}(\sqrt{-1}, \sqrt{-5})$.

For nearly 40 years, no counterexamples emerged, leading many to suppose that class field towers always terminated!

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**A counterexample for Class Field Tower Problem (Golod and Shafarevich, 1964)**

The class field tower for $\mathbb{Q}(\sqrt{-2 \times 3 \times 5 \times 7 \times 11 \times 13})$ is infinite. Equivalently, the quadratic field $\mathbb{Q}(\sqrt{-2 \times 3 \times 5 \times 7 \times 11 \times 13})$ is not contained in any number filed with class number one.

# The New Embedding Problem

# On Pólya fields and Pólya groups

**Theorem (Pólya, 1919)**

A polynomial $f(X) \in \mathbb{Q}[X]$ maps $\mathbb{Z}$ to $\mathbb{Z}$ if and only of it can be written as a finite $\mathbb{Z}$-linear combination of the polynomials

$$\binom{X}{n} = \frac{X(X-1)(X-2)\cdots(X-n+1)}{n!} \quad : \quad n = 0, 1, 2, \cdots .$$

**Definition (Zantema, 1982)**

A number field $K$, with ring of integers $\mathcal{O}_K$, is called a Pólya field, if the $\mathcal{O}_K$-module

$$\mathrm{Int}(\mathcal{O}_K) = \{f \in K[X] \, : \, f(\mathcal{O}_K) \subseteq \mathcal{O}_K\}$$

has a regular basis. That is, an $\mathcal{O}_K$-basis $\{f_n\}_{n\geq 0}$ with $\deg(f_n) = n$.

### Theorem (Ostrowski, 1919)

A number field $K$ is a Pólya field if and only if for every $q$, a prime power, the ideal

$$\Pi_q(K) := \prod_{\substack{\mathfrak{p}\in\mathbb{P}_K \\ N_{K/\mathbb{Q}}(\mathfrak{p})=q}} \mathfrak{p} \qquad \text{(Ostrowski ideal)}$$

is principal (If $q$ is not the norm of any prime ideal of $\mathcal{O}_K$, set $\Pi_q(K) = \mathcal{O}_K$).

**Definition(Cahen-Chabert, 1997)**

The *Pólya group* of a number field $K$, denoted by $\mathrm{Po}(K)$, is the subgroup of $\mathrm{Cl}(K)$ defined as follows

$$\mathrm{Po}(K) = \langle [\Pi_q(K)] \,:\, q \text{ is a prime power} \rangle.$$

relative Pólya group

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▸ relative Pólya group

## Remark

The number field $K$ is Pólya if and only if $\mathrm{Po}(K) = 0$. In particular, if $h_K = 1$ then $K$ is a Pólya field.

### Theorem (Zantema, 1982)

A quadratic number field $K = \mathbb{Q}(\sqrt{d})$ is a Pólya field if and only if one of the following conditions holds:

- $d = -1, -2, -p$, where $p \equiv 3 \pmod 4$ is a prime number;
- $d = p$, where $p$ is a prime number;
- $d = 2p, pq$, where $p \equiv q \pmod 4$ are primes, and $x^2 - y^2 d = -1$ has no solution in $\mathcal{O}_K$.

◂ Gauss' conjecture

**Theorem (Zantema, 1982)**

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## The Kronecker-Weber Theorem

Every finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field.

## Corollary

The quadratic field $\mathbb{Q}(\sqrt{-2 \times 3 \times 5 \times 7 \times 11 \times 13})$ is contained in a Pólya field.

## The New Embedding Problem (Leriche, 2014)

Is a number field $K$ contained in a Pólya field?

## Theorem (Leriche, 2014)

Let $K$ be a number field. Then the Hilbert class field of $K$, i.e., $H(K)$, is Pólya field. In particular, $K$ is contained in a Pólya field, namely its Hilbert class field.

# The Relativized Version of New Embedding Problem

# Relative Pólya group

**Definition (M.-Rajaei, 2020 & Chabert 2019)**

Let $L/K$ be a finite extension of number fields. The *relative Pólya group* of $L/K$, denoted by $\mathrm{Po}(L/K)$, is defined as

$$\mathrm{Po}(L/K) = \left\langle \left[ \overbrace{\Pi_{\mathfrak{p}^f}(L/K)}^{\text{relative Ostrowski ideals}} = \prod_{\substack{\mathfrak{P}\in\mathbb{P}_L \\ N_{L/K}(\mathfrak{P})=\mathfrak{p}^f}} \mathfrak{P} \right] : \mathfrak{p}\in\mathbb{P}_K\,,\, f\in\mathbb{N} \right\rangle.$$

In particular, $\mathrm{Po}(L/\mathbb{Q}) = \mathrm{Po}(L)$ and $\mathrm{Po}(L/L) = \mathrm{Cl}(L)$. ◂ Pólya group

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In particular, $\mathrm{Po}(L/\mathbb{Q}) = \mathrm{Po}(L)$ and $\mathrm{Po}(L/L) = \mathrm{Cl}(L)$. ◂ Pólya group

**Theorem (M.-Rajaei, 2020)**

Let $F \subseteq K \subseteq L$ be a tower of finite extensions of number fields. If $L/K$ is Galois, then $\mathrm{Po}(L/F) \subseteq \mathrm{Po}(L/K)$. In particular,

$$\mathrm{Po}(L) = \mathrm{Po}(L/\mathbb{Q}) \subseteq \mathrm{Po}(L/K).$$

### The relativized version of new embedding problem

Is every number field $K$ contained in a number field $L$ with $\mathrm{Po}(L/K) = 0$?

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Is every number field $K$ contained in a number field $L$ with $\mathrm{Po}(L/K) = 0$?

### Theorem (M.-Rajaei, 2020)

Let $L/K$ be a finite Galois extension of number fields. Then there exists a surjective map

$$\psi : \bigoplus_{\mathfrak{p} \in \mathbb{P}_K} \frac{\mathbb{Z}}{e_{\mathfrak{p}(L/K)}\mathbb{Z}} \to \frac{\mathrm{Po}(L/K)}{\epsilon_{L/K}(\mathrm{Cl}(K))},$$

where $e_{\mathfrak{p}(L/K)}$ denotes the ramification index of $\mathfrak{p}$ in $L/K$, and $\epsilon_{L/K} : [\mathfrak{a}] \in \mathrm{Cl}(K) \to [\mathfrak{a}\mathcal{O}_L] \in \mathrm{Cl}(L)$ denotes the *capitulation map*.

### Theorem (M.-Rajaei, 2020)

Let $L/K$ a finite Galois extension of number fields. If $L/K$ is unramified at all prime ideals of $K$, then $\mathrm{Po}(L/K) = \epsilon_{L/K}(\mathrm{Cl}(K))$.

### Corollary (M.-Rajaei, 2020)

Let $K$ be a number field, and denote its Hilbert class field by $H(K)$. Then $\mathrm{Po}(H(K)/K) = 0$. In particular, $K$ is contained in a number field with trivial relative Pólya group (over $K$).

Proof. Since $H(K)/K$ is unramified, $\mathrm{Po}(H(K)/K) = \epsilon_{H(K)/K}(\mathrm{Cl}(K))$. By the Principal Ideal Theorem, $\epsilon_{H(K)/K}(\mathrm{Cl}(K)) = 0$.

- Since
$$\mathrm{Po}(H(K)) = \mathrm{Po}(H(K)/\mathbb{Q}) \subseteq \mathrm{Po}(H(K)/K) = 0,$$
we obtain Leriche's result on Pólya-ness of Hilbert Class Fields.

- Since
$$\mathrm{Po}(H(K)) = \mathrm{Po}(H(K)/\mathbb{Q}) \subseteq \mathrm{Po}(H(K)/K) = 0,$$
we obtain Leriche's result on Pólya-ness of Hilbert Class Fields.
- Let
$$K = K_1 \subseteq K_2 = H(K_1) \subseteq K_3 = H(K_2) \subseteq \dots,$$
be the class field tower of $K$. Then
$$\mathrm{Po}(K_i/K) = 0, \quad \forall i = 2, 3 \dots.$$

For instance, for $K = \mathbb{Q}(\sqrt{-2 \times 3 \times 5 \times 7 \times 11 \times 13})$, there are infinitely many number fields, containing $K$, whose relative Pólya groups over $K$ are trivial.


Happy
Valentine's
Day!